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Algebraic computation of some intersection D-modules

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Abstract

Let X be a complex analytic manifold, $D \subset X$ a locally quasi-homogeneous free divisor, $\mathcal{E}$ an integrable logarithmic connection with respect to D and $\mathcal{L}$ the local system of the horizontal sections of $\mathcal{E}$ on $X - D$. In this paper we give an algebraic description in terms of $\mathcal{E}$ of the regular holonomic $\mathcal{D}_X$-module whose de Rham complex is the intersection complex associated with $\mathcal{L}$. As an application, we perform some effective computations in the case of quasi-homogeneous plane curves.

Introduction

On a complex analytic manifold, intersection complexes associated with irreducible local systems on a dense open regular subset of a closed analytic subspace are the simple pieces which form any perverse sheaf. The Riemann-Hilbert correspondence allows us to consider the regular holonomic D-modules which correspond to these intersection complexes, that we call "intersection D-modules". They are the simple pieces which form any regular holonomic D-module. Whereas intersection complexes are topological objects, intersection D-modules are algebraic: they are given by a system of partial linear differential equations with holomorphic coefficients.

Intersection complexes can be constructed by an important operation: the intermediate direct image. Its description in terms of Verdier duality and usual derived direct images can be algebraically interpreted in the category of holonomic regular D-modules by using the deep properties of the de Rham functor. We need to compute localizations and D-duals.

This can be effectively done, in principle, by using the general available algorithms in [25, 27, 26], but in the case of integrable logarithmic connections along a locally quasi-homogeneous free divisor, we exploit the logarithmic point of view [2, 4, 5, 8, 9, 30, 31] to previously obtain a general algebraic description of their associated intersection D-modules, from which we can easily derive effective computations.

The main ingredients we use are the duality theorem proved in [5] and the logarithmic comparison theorem for arbitrary integrable logarithmic connections proved in [6], both with respect to locally quasi-homogeneous free divisors.

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The algorithmic treatment of the computations in this paper will be developed elsewhere.

Let us now comment on the content of this paper.

In section 1 we remind the reader of the basic notions and notations and we review our previous results on logarithmic $\mathcal{D}$-modules with respect to free divisors. We recall the logarithmic comparison theorem for arbitrary integrable logarithmic connections from [6], and we give the theorem describing the intersection D-module associated with an integrable logarithmic connection along a locally quasi-homogeneous free divisor.

In section 2, given a locally quasi-homogeneous free divisor D with a reduced local equation $f = 0$ and a cyclic integrable logarithmic connection $\mathcal{E}$ with respect to D, we explicitly describe a presentation of $\mathcal{D}[s]\cdot(\mathcal{E}f^s)$ over $\mathcal{D}[s]$ in terms of a presentation of $\mathcal{E}$ over the ring of logarithmic differential operators. This description will be useful in order to compute the Bernstein-Sato polynomials associated with $\mathcal{E}$.

In section 3, the general results of the previous section are explicitly written down in the case of a family of integrable logarithmic connections with respect to a quasi-homogeneous plane curves.

In section 4 we perform some explicit computations with respect to a cusp.

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1 Logarithmic connections with respect to a free divisor: theoretical set-up

Let X be a n-dimensional complex analytic manifold and $D \subset X$ a hypersurface, and let us denote by $j : U = X - D \hookrightarrow X$ the corresponding open inclusion.

We say that D is a *free divisor* [28] if the $\mathcal{O}_X$-module $\mathrm{Der}(\log D)$ of logarithmic vector fields with respect to D is locally free (of rank n), or equivalently if the $\mathcal{O}_X$-module $\Omega^1_X(\log D)$ of logarithmic 1-forms with respect to D is locally free (of rank n).

Normal crossing divisors, plane curves, free hyperplane arrangements (e.g. the union of reflecting hyperplanes of a complex reflection group), discriminant of stable mappings or bifurcation sets are examples of free divisors.

We say that D is quasi-homogeneous at $p \in D$ if there is a system of local coordinates $\underline{x}$ centered at p such that the germ (D, p) has a reduced weighted homogeneous defining equation (with strictly positive weights) with respect to $\underline{x}$. We say that D is locally quasi-homogeneous if it is so at each point $p \in D$.

Let us denote by $\mathcal{D}_X(\log D)$ the 0-term of the Malgrange-Kashiwara filtration with respect to D on the sheaf $\mathcal{D}_X$ of linear differential operators on X. When D is a free divisor, the first author has proved in [2] that $\mathcal{D}_X(\log D)$ is the universal enveloping algebra of the Lie algebroid $\mathrm{Der}(\log D)$, and then it is coherent and has noetherian stalks of finite global homological dimension. Locally, if $\{\delta_1, \dots, \delta_n\}$ is a local basis of the logarithmic vector fields on an open set V, any differential operator in $\Gamma(V, \mathcal{D}_X(\log D))$ can be written in a unique way as a finite sum

$$\sum_{\substack{\alpha\in\mathbb{N}^n\\|\alpha|\leq d}} a_\alpha \delta_1^{\alpha_1}\cdots\delta_n^{\alpha_n},$$

where the a_α are holomorphic functions on V.

From now on, let us assume that D is a free divisor.

We say that D is a *Koszul free* divisor [2] at a point $p\in D$ if the symbols of any (some) local basis $\{\delta_1,\dots,\delta_n\}$ of $\operatorname{Der}(\log D)_p$ form a regular sequence in $\operatorname{Gr}\mathcal{D}_{X,p}$. We say that D is a *Koszul free* divisor if it is so at any point $p\in D$. Actually, as M. Schulze pointed out, Koszul freeness is equivalent to holonomicity in the sense of [28].

Plane curves and locally quasi-homogeneous free divisors (e.g. free hyperplane arrangements or discriminant of stable mappings in Mather's "nice dimensions") are example of Koszul free divisors [3].

A *logarithmic connection* with respect to D is a locally free $\mathcal{O}_X$-module $\mathcal{E}$ endowed with:
-) a $\mathbb{C}$-linear morphism (connection) $\nabla' : \mathcal{E}\to\mathcal{E}\otimes_{\mathcal{O}_X}\Omega_X^1(\log D)$, satisfying $\nabla'(ae)=a\nabla'(e)+e\otimes da$, for any section a of $\mathcal{O}_X$ and any section e of $\mathcal{E}$,
or equivalently, with
-) a left $\mathcal{O}_X$-linear morphism $\nabla:\operatorname{Der}(\log D)\to\operatorname{End}_{\mathbb{C}_X}(\mathcal{E})$ satisfying the Leibniz rule $\nabla(\delta)(ae)=a\nabla(\delta)(e)+\delta(a)e$, for any logarithmic vector field δ, any section a of $\mathcal{O}_X$ and any section e of $\mathcal{E}$.

The integrability of ∇' is equivalent to the fact that ∇ preserve Lie brackets. Then, we know from [2] that giving an integrable logarithmic connection on a locally free $\mathcal{O}_X$-module $\mathcal{E}$ is equivalent to extending its original $\mathcal{O}_X$-module structure to a left $\mathcal{D}_X(\log D)$-module structure, and so integrable logarithmic connections are the same as left $\mathcal{D}_X(\log D)$-modules which are locally free of finite rank over $\mathcal{O}_X$.

Let us denote by $\mathcal{O}_X(\star D)$ the sheaf of meromorphic functions with poles along D. It is a holonomic left $\mathcal{D}_X$-module.

The first examples of integrable logarithmic connections (ILC for short) are the invertible $\mathcal{O}_X$-modules $\mathcal{O}_X(mD)\subset\mathcal{O}_X(\star D)$, $m\in\mathbb{Z}$, formed by the meromorphic functions h such that $\operatorname{div}(h)+mD\geq 0$.

If $f=0$ is a reduced local equation of D at $p\in D$ and $\delta_1,\dots,\delta_n$ is a local basis of $\operatorname{Der}(\log D)_p$ with $\delta_i(f)=\alpha_i f$, then f^{-m} is a local basis of $\mathcal{O}_{X,p}(mD)$ over $\mathcal{O}_{X,p}$ and we have the following local presentation over $\mathcal{D}_{X,p}(\log D)$ ([2], th. 2.1.4)

$$\mathcal{O}_{X,p}(mD)\simeq\mathcal{D}_{X,p}(\log D)/\mathcal{D}_{X,p}(\log D)(\delta_1+m\alpha_1,\dots,\delta_n+m\alpha_n). \qquad (1)$$

(1.1) For any ILC $\mathcal{E}$ and any integer m, the locally free $\mathcal{O}_X$-modules $\mathcal{E}(mD):=\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{O}_X(mD)$ and $\mathcal{E}^*:=\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{E},\mathcal{O}_X)$ are endowed with a natural structure of left $\mathcal{D}_X(\log D)$-module, where the action of logarithmic vector fields is given by

$$(\delta h)(e)=-h(\delta e)+\delta(h(e)),\quad \delta\,(e\otimes a)=(\delta e)\otimes a+e\otimes\delta(a) \qquad (2)$$

for any logarithmic vector field δ, any local section h of $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{E},\mathcal{O}_X)$, any local section e of $\mathcal{E}$ and any local section a of $\mathcal{O}_X(mD)$ (cf. [5], §2). Then $\mathcal{E}(mD)$ and $\mathcal{E}^*$ are ILC again, and the usual isomorphisms

$$\mathcal{E}(mD)(m'D)\simeq\mathcal{E}((m+m')D),\quad \mathcal{E}(mD)^*\simeq\mathcal{E}^*(-mD)$$

are $\mathcal{D}_X(\log D)$-linear.

(1.2) If D is Koszul free and $\mathcal{E}$ is an ILC, then the complex $\mathcal{D}_X \overset{L}{\otimes}_{\mathcal{D}_X(\log D)} \mathcal{E}$ is concentrated in degree 0 and its 0-cohomology $\mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} \mathcal{E}$ is a holonomic $\mathcal{D}_X$-module (see [5], prop. 1.2.3).

If $\mathcal{E}$ is an ILC, then $\mathcal{E}(\star D)$ is a meromorphic connection (locally free of finite rank over $\mathcal{O}_X(\star D)$) and then it is a holonomic $\mathcal{D}_X$-module (cf. [20], th. 4.1.3). Actually, $\mathcal{E}(\star D)$ has regular singularities on the smooth part of D (it has logarithmic poles! [10]) and then it is regular everywhere [19], cor. 4.3-14, which means that if $\mathcal{L}$ is the local system of horizontal sections of $\mathcal{E}$ on $U = X - D$, the canonical morphism

$$\Omega^\bullet_X(\mathcal{E}(\star D)) \to Rj_*\mathcal{L}$$

is an isomorphism in the derived category.

For any ILC $\mathcal{E}$, or even for any left $\mathcal{D}_X(\log D)$-module (without any finiteness property over $\mathcal{O}_X$), one can define its logarithmic de Rham complex $\Omega^\bullet_X(\log D)(\mathcal{E})$ in the classical way (cf. [10, def. I.2.15]), which is a subcomplex of $\Omega^\bullet_X(\mathcal{E}(\star D))$. It is clear that both complexes coincide on U.

For any ILC $\mathcal{E}$ and any integer m, $\mathcal{E}(mD)$ is a sub-$\mathcal{D}_X(\log D)$-module of the regular holonomic $\mathcal{D}_X$-module $\mathcal{E}(\star D)$, and then we have a canonical morphism in the derived category of left $\mathcal{D}_X$-modules

$$\rho_{\mathcal{E},m} : \mathcal{D}_X \overset{L}{\otimes}_{\mathcal{D}_X(\log D)} \mathcal{E}(mD) \to \mathcal{E}(\star D),$$

given by $\rho_{\mathcal{E},m}(P \otimes e') = Pe'$.

Since $\mathcal{E}(m'D)(mD) = \mathcal{E}((m+m')D)$ and $\mathcal{E}(m'D)(\star D) = \mathcal{E}(\star D)$, we can identify morphisms $\rho_{\mathcal{E}(m'D),m}$ and $\rho_{\mathcal{E},m+m'}$.

For any bounded complex $\mathcal{K}$ of sheaves of $\mathbb{C}$-vector spaces on X, let us denote by $\mathcal{K}^\vee = R\operatorname{Hom}_{\mathbb{C}_X}(\mathcal{K}, \mathbb{C}_X)$ its Verdier dual.

The dual local system $\mathcal{L}^\vee$ appears as the local system of the horizontal sections of the dual ILC $\mathcal{E}^*$.

We have the following theorem (see [5, th. 4.1] and [6, th. (2.1.1)]):

(1.3) Theorem. *Let $\mathcal{E}$ be an ILC (with respect to the divisor D) and let $\mathcal{L}$ be the local system of its horizontal sections on $U = X - D$. The following properties are equivalent:*

1) *The canonical morphism $\Omega^\bullet_X(\log D)(\mathcal{E}) \to Rj_*\mathcal{L}$ is an isomorphism in the derived category of complexes of sheaves of complex vector spaces.*

2) *The inclusion $\Omega^\bullet_X(\log D)(\mathcal{E}) \hookrightarrow \Omega^\bullet_X(\mathcal{E}(\star D))$ is a quasi-isomorphism.*

3) *The morphism $\rho_{\mathcal{E},1} : \mathcal{D}_X \overset{L}{\otimes}_{\mathcal{D}_X(\log D)} \mathcal{E}(D) \to \mathcal{E}(\star D)$ is an isomorphism in the derived category of left $\mathcal{D}_X$-modules.*

4) *The complex $\mathcal{D}_X \overset{L}{\otimes}_{\mathcal{D}_X(\log D)} \mathcal{E}(D)$ is concentrated in degree 0 and the $\mathcal{D}_X$-module $\mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} \mathcal{E}(D)$ is holonomic and isomorphic to its localization along D.*

Moreover, if D is a Koszul free divisor, the preceding properties are also equivalent to:

5) The canonical morphism $j_!\mathcal{L}^\vee \to \Omega_X^\bullet(\log D)(\mathcal{E}^*(-D))$ is an isomorphism in the derived category of complexes of sheaves of complex vector spaces.

For D a locally quasi-homogeneous free divisor and $\mathcal{E} = \mathcal{O}_X$, the equivalent properties in theorem (1.3) hold: this is the so called "logarithmic comparison theorem" [7] (see also [5, th. 4.4] and [6, cor. (2.1.3)] for other proofs based on D-module theory).

(1.4) Let $\mathcal{E}$ be an ILC (with respect to D) and p a point in D. Let $f \in \mathcal{O} = \mathcal{O}_{X,p}$ be a reduced local equation of D and let us write $\mathcal{D} = \mathcal{D}_{X,p}$, $\mathcal{V}_0 = \mathcal{D}_X(\log D)_p$ and $E = \mathcal{E}_p$. We know from [6, lemma (3.2.1)] that the ideal of polynomials $b(s) \in \mathbb{C}[s]$ such that

$$b(s)Ef^s \subset \mathcal{D}[s]\cdot\left(Ef^{s+1}\right)\left(\subset E[f^{-1},s]f^s\right)$$

is generated by a non constant polynomial $b_{\mathcal{E},p}(s)$. By the coherence of the involved objects we deduce that $b_{\mathcal{E},q}(s) \mid b_{\mathcal{E},p}(s)$ for $q \in D$ close to p.

If $b_{\mathcal{E},p}(s)$ has some integer root, let us call $\kappa(\mathcal{E},p)$ the minimum of those roots. If not, let us write $\kappa(\mathcal{E},p) = +\infty$.

Let us call

$$\kappa(\mathcal{E}) = \inf\{\kappa(\mathcal{E},p) \mid p \in D\} \in \mathbb{Z}\cup\{\pm\infty\}.$$

From now on let us suppose that D is a locally quasi-homogeneous free divisor.

(1.5) Theorem. *Under the above hypothesis, if $\kappa(\mathcal{E}) > -\infty$, then the morphism*

$$\rho_{\mathcal{E},k} : \mathcal{D}_X \overset{L}{\otimes}_{\mathcal{D}_X(\log D)} \mathcal{E}(kD) \to \mathcal{E}(\star D) \tag{3}$$

is an isomorphism in the derived category of left $\mathcal{D}_X$-modules, for all $k \geq -\kappa(\mathcal{E})$.

Proof. It is a straightforward consequence of [3], [4, th. 5.6] and theorem (3.2.6) of [6] and its proof. Q.E.D.

Let us note that the hypothesis $\kappa(\mathcal{E}) > -\infty$ in theorem (1.5) holds locally on X.

In the situation of theorem (1.5), if $\mathcal{L}$ is the local system of the horizontal sections of $\mathcal{E}$ on $U = X - D$, then the derived direct image $Rj_*\mathcal{L}$ is canonically isomorphic (in the derived category) to the de Rham complex of the holonomic $\mathcal{D}_X$-module $\mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} \mathcal{E}(kD)$:

$$\mathrm{DR}\left(\mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} \mathcal{E}(kD)\right) = \mathrm{DR}\left(\mathcal{D}_X \overset{L}{\otimes}_{\mathcal{D}_X(\log D)} \mathcal{E}(kD)\right) \simeq$$
$$\mathrm{DR}\,\mathcal{E}(\star D) \simeq \Omega_X^\bullet(\mathcal{E}(\star D)) \simeq Rj_*\mathcal{L}.$$

Proceeding as above for the dual ILC $\mathcal{E}^*$, we find that if $\kappa(\mathcal{E}^*) > -\infty$, then we have that the canonical morphism

$$\mathrm{DR}\left(\mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} \mathcal{E}^*(k'D)\right) \to Rj_*\mathcal{L}^\vee$$

is an isomorphism in the derived category for $k' \geq -\kappa(\mathcal{E}^*)$.

Let us denote by

$$\varrho_{\mathcal{E},k,k'} : \mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} \mathcal{E}((1-k')D) \to \mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} \mathcal{E}(kD), \qquad (4)$$

the $\mathcal{D}_X$-linear morphism induced by the inclusion $\mathcal{E}((1-k')D) \subset \mathcal{E}(kD)$, $1-k' \leq k$, and by $\mathrm{IC}_X(\mathcal{L})$ the intersection complex of Deligne-Goresky-MacPherson associated with $\mathcal{L}$, which is described as the intermediate direct image $j_{!*}\mathcal{L}$, i.e. the image of $j_!\mathcal{L} \to Rj_*\mathcal{L}$ in the category of perverse sheaves (cf. [1], def. 1.4.22).

The following theorem describes the "intersection $\mathcal{D}_X$-module" corresponding to $\mathrm{IC}_X(\mathcal{L})$ by the Riemann-Hilbert correspondence of Mebkhout-Kashiwara [13, 16, 17].

(1.6) THEOREM. *Under the above hypothesis, we have a canonical isomorphism in the category of perverse sheaves on X,*

$$\mathrm{IC}_X(\mathcal{L}) \simeq \mathrm{DR}\left(\mathrm{Im}\, \varrho_{\mathcal{E},k,k'}\right),$$

for $k \geq -\kappa(\mathcal{E})$, $k' \geq -\kappa(\mathcal{E}^*)$ and $1-k' \leq k$.

PROOF. Using our duality results in [5, §3], the Local Duality Theorem for holonomic $\mathcal{D}_X$-modules ([18], ch. I, th. (4.3.1); see also [22]) and theorem (1.5), we obtain

$$\mathrm{DR}\left(\mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} \mathcal{E}((1-k')D)\right) \simeq \mathrm{DR}\left(\mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} \mathcal{E}^*(k'D)^*(D)\right) \simeq$$
$$\mathrm{DR}\left(\mathbb{D}_{\mathcal{D}_X}\left(\mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} \mathcal{E}^*(k'D)\right)\right) \simeq \left[\mathrm{DR}\left(\mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} \mathcal{E}^*(k'D)\right)\right]^\vee \simeq$$
$$\left[Rj_*\mathcal{L}^\vee\right]^\vee \simeq j_!\mathcal{L}.$$

On the other hand, the canonical morphism $j_!\mathcal{L} \to Rj_*\mathcal{L}$ corresponds, through the de Rham functor, to the $\mathcal{D}_X$-linear morphism $\varrho_{\mathcal{E},k,k'}$, and the theorem is a consequence of the Riemann-Hilbert correspondence which says that the de Rham functor establishes an equivalence of abelian categories between the category of regular holonomic $\mathcal{D}_X$-modules and the category of perverse sheaves on X. Q.E.D.

(1.7) REMARK. For $\mathcal{E} = \mathcal{O}_X$, one has $\mathcal{E}^* = \mathcal{O}_X$ and there are examples where morphisms $\rho_{\mathcal{O}_X,k}$ in (3) are never isomorphisms ([5], ex. 5.3). Nevertheless, for $k = k' = 1$ the image of the morphism

$$\varrho_{\mathcal{O}_X,1,1} : \mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} \mathcal{O}_X \to \mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} \mathcal{O}_X(D)$$

is always (canonically isomorphic to) $\mathcal{O}_X$, which is the regular holonomic $\mathcal{D}_X$-module corresponding by the Riemann-Hilbert correspondence to $\mathrm{IC}_X(\mathbb{C}_U) = \mathbb{C}_X$, where $\mathbb{C}_U$ is the local system of horizontal sections of $\mathcal{O}_X$ on U. To see this, let us work locally as in (1). Then, morphism $\varrho_{\mathcal{O}_X,1,1}$ is given at point p by

$$\overline{P} \in \mathcal{D}_{X,p}/\mathcal{D}_{X,p}(\delta_1,\ldots,\delta_n) \mapsto \overline{Pf} \in \mathcal{D}_{X,p}/\mathcal{D}_{X,p}(\delta_1+\alpha_1,\ldots,\delta_n+\alpha_n)$$

and the stalk at p of $\mathrm{Im}\,\varrho_{\mathcal{O}_X,1,1}$ is given by $\mathcal{D}_{X,p}/J$ where J is the left ideal

$$J = \{P \in \mathcal{D}_{X,p} \mid Pf \in \mathcal{D}_{X,p}(\delta_1+\alpha_1,\ldots,\delta_n+\alpha_n)\}.$$

By Saito's criterion [28] we can suppose

$$\begin{pmatrix} \delta_1 \\ \vdots \\ \delta_n \end{pmatrix} = A \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}$$

where A is a $n \times n$ matrix with entries in $\mathcal{O}_{X,p}$ and $\det A = f$. Writing $B = \operatorname{adj}(A)^t$ we obtain

$$B \begin{pmatrix} \delta_1 \\ \vdots \\ \delta_n \end{pmatrix} = f \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} \quad \overset{\text{eval. on } f}{\rightsquigarrow} \quad B \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}.$$

Then

$$\begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} f = f \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} + \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} = \cdots = B \begin{pmatrix} \delta_1 + \alpha_1 \\ \vdots \\ \delta_n + \alpha_n \end{pmatrix}$$

and $\frac{\partial}{\partial x_i} \in J$ for $i = 1, \ldots, n$. Since J is is not the total ideal, we deduce by maximality that J is the ideal generated by the $\frac{\partial}{\partial x_i}$ and $\mathcal{D}_{X,p}/J \simeq \mathcal{O}_{X,p}$. To conclude, one easily sees, from the fact that morphism $\varrho_{\mathcal{O}_X,1,1}$ factors through

$$a \in \mathcal{O}_X \mapsto 1 \otimes a \in \mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} \mathcal{O}_X(D)$$

[it is $\mathcal{D}_X$-linear since, for any derivation δ and any holomorphic function a, $\delta(1 \otimes a) = \delta \otimes a = \delta \otimes (ff^{-1}a) = (\delta f) \otimes (f^{-1}a) = 1 \otimes (\delta f)(f^{-1}a) = 1 \otimes (\delta a)$] that the isomorphisms above at different p glue together and give a global isomorphism $\operatorname{Im} \varrho_{\mathcal{O}_X,1,1} \simeq \mathcal{O}_X$.

This example suggests studying the comparison between $\mathrm{DR}(\operatorname{Im} \varrho_{\mathcal{E},k,k'})$, $k, k' \gg 0$, and $\mathrm{IC}_X(\mathcal{L})$ in theorem (1.6), independent of the fact that $\rho_{\mathcal{E},k}$ and $\rho_{\mathcal{E}^*,k'}$ are isomorphisms or not.

2 Bernstein-Sato polynomials for cyclic integrable logarithmic connections

In the situation of (1.4), let us assume that E is a cyclic $\mathcal{V}_0$-module generated by an element $e \in E$. The following result is proved in [6, prop. (3.2.3)].

(2.1) Proposition. *Under the above conditions, the polynomial $b_{\mathcal{E},p}(s)$ coincides with the Bernstein-Sato polynomial $b_e(s)$ of e with respect to f, where e is considered to be an element of the holonomic $\mathcal{D}$-module $E[f^{-1}]$ (cf. [12]).*

(2.2) Let $\Theta_{f,s} \subset \mathcal{D}[s]$ be the set of operators in $\operatorname{ann}_{\mathcal{D}[s]} f^s$ of total order (in s and in the derivatives) ≤ 1. The elements of $\Theta_{f,s}$ are of the form $\delta - \alpha s$ with $\delta \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O})$, $\alpha \in \mathcal{O}$ and $\delta(f) = \alpha f$. In particular $\Theta_{f,s} \subset \mathcal{V}_0[s]$.

The $\mathcal{O}$-linear map

$$\delta \in \mathrm{Der}(\log D)_p \mapsto \delta - \frac{\delta(f)}{f} s \in \Theta_{f,s}$$

is an isomorphism of Lie-Rinehart algebras over $(\mathbb{C}, \mathcal{O})$ and extends to a unique ring isomorphism $\Phi : \mathcal{V}_0[s] \to \mathcal{V}_0[s]$ with $\Phi(s) = s$ and $\Phi(a) = a$ for all $a \in \mathcal{O}$. Let us note that $\Phi^{-1}(\delta) = \delta + \frac{\delta(f)}{f} s$ for each $\delta \in \operatorname{Der}(\log D)_p$.

It is clear that $E[s]f^s$ is a sub-$\mathcal{V}_0[s]$-module of $E[s, f^{-1}]f^s$ and that for any $P \in \mathcal{V}_0[s]$ and any $e' \in E[s]$, the following relation holds

$$(Pe')f^s = \Phi(P)(e'f^s). \tag{5}$$

(2.3) Proposition. *Under the above conditions, the following relation holds*

$$\operatorname{ann}_{\mathcal{V}_0[s]}(ef^s) = \mathcal{V}_0[s] \cdot \Phi\left(\operatorname{ann}_{\mathcal{V}_0} e\right).$$

Proof. The inclusion $\supset$ comes from (5). For the other inclusion, let $Q \in \operatorname{ann}_{\mathcal{V}_0[s]}(ef^s)$ and let us write $\Phi^{-1}(Q) = \sum_{i=1}^d P_i s^i$ with $P_i \in \mathcal{V}_0$. We have

$$0 = Q\,(ef^s) = \left(\Phi^{-1}(Q)e\right) f^s = \left(\sum_{i=1}^d (P_i e)s^i\right) f^s$$

and then $P_i \in \operatorname{ann}_{\mathcal{V}_0} e$. Therefore

$$Q = \Phi\left(\sum_{i=1}^d P_i s^i\right) = \sum_{i=1}^d \Phi(P_i)s^i \in \mathcal{V}_0[s] \cdot \Phi\left(\operatorname{ann}_{\mathcal{V}_0} e\right).$$

Q.E.D.

(2.4) Proposition. *Under the above conditions, if D is a locally quasi-homogeneous free divisor, then*

$$\operatorname{ann}_{\mathcal{D}[s]}(ef^s) = \mathcal{D}[s] \cdot \operatorname{ann}_{\mathcal{V}_0[s]}(ef^s).$$

Proof. From (5) we know that $E[s]f^s = \mathcal{V}_0[s] \cdot (ef^s)$, and from [6, cor. (3.1.2)] we know that the morphism

$$\rho_{E,s} : P \otimes (e'f^s) \in \mathcal{D}[s] \otimes_{\mathcal{V}_0[s]} E[s]f^s \mapsto P(e'f^s) \in \mathcal{D}[s] \cdot (E[s]f^s) = \mathcal{D}[s] \cdot (ef^s)$$

is an isomorphism of left $\mathcal{D}[s]$-modules. Therefore

$$\operatorname{ann}_{\mathcal{D}[s]}(ef^s) = \mathcal{D}[s] \cdot \operatorname{ann}_{\mathcal{V}_0[s]}(ef^s).$$

Q.E.D.

(2.5) Corollary. *Under the above conditions, if D is a locally quasi-homogeneous free divisor, then*

$$\operatorname{ann}_{\mathcal{D}[s]}(ef^s) = \mathcal{D}[s] \cdot \Phi\left(\operatorname{ann}_{\mathcal{V}_0} e\right).$$

Proof. It follows from propositions (2.3) and (2.4). Q.E.D.

(2.6) Remark. Theorems (1.5) and (1.6), proposition (2.4) and corollary (2.5) remain true if we only assume that our divisor D is Koszul free and of commutative linear type, i.e. its jacobian ideal is of linear type (see [6, §3]).

(2.7) Remark. As we shall see in sections 3 and 4, theorem (1.6), proposition (2.1) and corollary (2.5) provide an effective method of computing the intersection $\mathcal{D}_X$-module corresponding to $\mathrm{IC}_X(\mathcal{L})$ in terms of the ILC $\mathcal{E}$, at least if D is a locally quasi-homogeneous free divisor, or more generally, if D is Koszul free and of commutative linear type (see remark (2.6)).

(2.8) Remark. In the particular case of $\mathcal{E} = \mathcal{O}_X$ and $E = \mathcal{O}$, corollary (2.5) says that

$$\mathrm{ann}_{\mathcal{D}[s]}(f^s) = \mathcal{D}[s] \cdot (\delta_1 - \alpha_1 s, \ldots, \delta_n - \alpha_n s)\,,$$

where $\delta_1, \ldots, \delta_n$ is a local basis of $\mathrm{Der}(\log D)_p$ and $\delta_i(f) = \alpha_i f$ (see corollary 5.8, (b) in [4]).

(2.9) Example. Let us suppose that $D \subset X$ is a non-necessarily free divisor and let $f = 0$ be a reduced local equation of D at a point $p \in D$. Let $\{\delta_1, \ldots, \delta_m\}$ a system of generators of $\mathrm{Der}(\log D)_p$ and let us write $\delta_i(f) = \alpha_i f$.

Let us call $\mathrm{ann}^{(1)}_{\mathcal{D}[s]}(f^s)$ the ideal of $\mathcal{D}[s]$ generated by $\Theta_{f,s}$ (see (2.2)):

$$\mathrm{ann}^{(1)}_{\mathcal{D}[s]}(f^s) = \mathcal{D}[s] \cdot (\delta_1 - \alpha_1 s, \ldots, \delta_m - \alpha_m s) \subset \mathrm{ann}_{\mathcal{D}[s]}(f^s).$$

The Bernstein functional equation for f

$$b(s) f^s = P(s) f^{s+1}$$

means that the operator $b(s) - P(s)f$ belongs to the annihilator of f^s over $\mathcal{D}[s]$. Then, an explicit knowledge of the ideal $\mathrm{ann}_{\mathcal{D}[s]}(f^s)$ allows us to find $b(s)$ by computing the ideal

$$\mathbb{C}[s] \cap \left(\mathcal{D}[s] \cdot f + \mathrm{ann}_{\mathcal{D}[s]}(f^s)\right),$$

(see [25]). However, the ideal $\mathrm{ann}_{\mathcal{D}[s]}(f^s)$ is in general difficult to compute.

When D is a locally quasi-homogeneous free divisor, or more generally, a divisor of differential linear type ([6], def. (1.4.5)), $\mathrm{ann}_{\mathcal{D}[s]}(f^s) = \mathrm{ann}^{(1)}_{\mathcal{D}[s]}(f^s)$ and the computation of $b(s)$ is in principle easier.

But there are other examples where the Bernstein polynomial $b(s)$ belongs to

$$\mathbb{C}[s] \cap \left(\mathcal{D}[s] \cdot f + \mathrm{ann}^{(1)}_{\mathcal{D}[s]}(f^s)\right)$$

even if $\mathrm{ann}_{\mathcal{D}[s]}(f^s) \neq \mathrm{ann}^{(1)}_{\mathcal{D}[s]}(f^s)$. For instance, when $X = \mathbb{C}^3$ and $f = x_1 x_2 (x_1 + x_2)(x_1 + x_2 x_3)$ (see example 6.2 in [4]) or in any of the examples in page 445 of [9]. In all this examples the divisor is free and satisfies the logarithmic comparison theorem.

3 Integrable logarithmic connections along quasi-homogeneous plane curves

Let $D \subset X = \mathbb{C}^2$ be a divisor defined by a reduced polynomial equation $h(x_1,x_2)$, which is quasi-homogeneous with respect to the strictly positive integer weights ω_1,ω_2 of the variables x_1,x_2. We denote by $\omega(f)$ the weight of a quasi-homogeneous polynomial $f(x_1,x_2)$. The divisor D is free, a global basis of $\mathrm{Der}(\log D)$ is $\{\delta_1,\delta_2\}$, where

$$\begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \begin{pmatrix} \omega_1 x_1 & \omega_2 x_2 \\ -h_{x_2} & h_{x_1} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix}.$$

We have:

-) $\delta_1(h) = \omega(h)h, \quad \delta_2(h) = 0$,
-) the determinant of the coefficient matrix is equal to $\omega(h)h$,
-) $[\delta_1,\delta_2] = c\delta_2$, with $c = \omega(h) - \omega_1 - \omega_2$.

We consider a logarithmic connection $\mathcal{E} = \oplus_{i=1}^n \mathcal{O}_X e_i$ given by actions:

$$\delta_1 \cdot \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} = A_1 \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}, \quad \delta_2 \cdot \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} = A_2 \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}.$$

For $\mathcal{E}$ to be integrable, the following integrability condition

$$\delta_1(A_2) - \delta_2(A_1) + [A_2,A_1] = cA_2 \tag{6}$$

must hold.

(3.1) We shall focus on the case where A_1, A_2 are $n\times n$ matrices satisfying (6) and of the form:

$$A_1 = \begin{pmatrix} -a & 0 & 0 & \cdots & 0 & 0 \\ -\delta_2(a) & -a+c & 0 & \cdots & 0 & 0 \\ -\delta_2^2(a) & -\binom{2}{1}\delta_2(a) & -a+2c & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\delta_2^{n-2}(a) & -\binom{n-2}{1}\delta_2^{n-3}(a) & -\binom{n-2}{2}\delta_2^{n-4}(a) & \cdots & -a+(n-2)c & 0 \\ -\delta_2^{n-1}(a) & -\binom{n-1}{1}\delta_2^{n-2}(a) & -\binom{n-1}{2}\delta_2^{n-3}(a) & \cdots & -\binom{n-1}{n-2}\delta_2(a) & -a+(n-1)c \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ -b_0 & -b_1 & -b_2 & \cdots & -b_{n-1} \end{pmatrix}.$$

with $a, b_0, \ldots, b_{n-1}$ polynomials. Let us call $\mathcal{E}_{a,\underline{b}}$ the corresponding ILC.

(3.2) Lemma. *The $\mathcal{D}_X(\log D)$-module $\mathcal{E}_{a,\underline{b}}$ is generated by e_1 (so it is cyclic) and the $\mathcal{D}_X(\log D)$-annihilator of e_1 is the left ideal $J_{a,\underline{b}}$ generated by $\delta_1 + a$ and*

$\delta_2^n + b_{n-1}\delta_2^{n-1} + \cdots + b_1\delta_2 + b_0$. *So, the* $\mathcal{D}_X(\log D)$*-module* $\mathcal{E}_{a,\underline{b}}$ *is isomorphic to* $\mathcal{D}_X(\log D)/J_{a,\underline{b}}$.

PROOF. The first part is clear since $\delta_2 \cdot e_i = e_{i+1}$ for $i = 1, \ldots, n-1$. For the second part, the inclusion $J_{a,\underline{b}} \subset \mathrm{ann}_{\mathcal{D}_X(\log D)}(e_1)$ is also clear. To prove the opposite inclusion, we use the fact that any germ of logarithmic differential operator P has a unique expression as a sum $P = \sum_{i,j} a_{i,j}\delta_1^i\delta_2^j$, where the $a_{i,j}$ are germs of holomorphic functions ([2], th. 2.1.4) and a division argument. Q.E.D.

(3.3) REMARK. Theorem 2.1.4 in [2] says that $\mathcal{D}_X(\log D) = \mathcal{O}_X[\delta_1, \delta_2]$ with relations:

$$[\delta_1, f] = \delta_1(f), [\delta_2, f] = \delta_2(f), [\delta_1, \delta_2] = c\delta_2, \quad f \in \mathcal{O}_X.$$

In particular, we can define the *support* and the *exponent* of any germ of logarithmic differential operator P (or of any polynomial logarithmic differential operator in the Weyl algebra) by using the (unique) expression $P = \sum_{i,j} a_{i,j}\delta_1^i\delta_2^j$, and we obtain a division theorem and a notion of *Gröbner basis* for ideals. Under this scope, the integrability condition (6) reads out as the fact that the generators

$$g_1 = \delta_1 + a, \quad g_2 = \delta_2^n + b_{n-1}\delta_2^{n-1} + \cdots + b_0$$

of $J_{a,\underline{b}}$ satisfy Buchberger's criterion, i.e. that $\delta_2^n g_1 - \delta_1 g_2$ has a vanishing remainder with respect to the division by g_1, g_2, and then they form a Gröbner basis of $J_{a,\underline{b}}$.

(3.4) COROLLARY. *The* $\mathcal{D}_X$*-module* $\mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} \mathcal{E}_{a,\underline{b}}$ *is isomorphic to* $\mathcal{D}_X/I_{a,\underline{b}}$*, where* $I_{a,\underline{b}} = \mathcal{D}_X(\delta_1 + a, \delta_2^n + b_{n-1}\delta_2^{n-1} + \cdots + b_0)$.

For any integer k, we can consider the logarithmic connections $\mathcal{E}_{a,\underline{b}}(kD)$ and $\mathcal{E}^*_{a,\underline{b}}$ (see section (1.1)).

(3.5) LEMMA. *With the above notations, the ILC* $\mathcal{E}_{a,\underline{b}}(kD)$ *and* $\mathcal{E}_{a+\omega(h)k,\underline{b}}$ *are isomorphic.*

PROOF. An $\mathcal{O}_X$-basis of $\mathcal{E}_{a,\underline{b}}(kD)$ is $\{e_i^k = e_i \otimes h^{-k}\}_{i=1}^n$ and the action of $\mathrm{Der}(\log D)$ over this basis is given by (see (2)):

$$\delta_1 \cdot e_i^k = (\delta_1 \cdot e_i) \otimes h^{-k} + e_i \otimes (-\omega(h)kh^{-k}), \quad \delta_2 \cdot e_i^k = (\delta_2 \cdot e_i) \otimes h^{-k}.$$

Then, the isomorphism of $\mathcal{O}_X$-modules

$$\sum_{i=1}^n b_i e_i \in \mathcal{E}_{a+\omega(h)k,\underline{b}} \mapsto \sum_{i=1}^n b_i e_i^k \in \mathcal{E}_{a,\underline{b}}(kD)$$

is clearly $\mathcal{D}_X(\log D)$-linear. Q.E.D.

The proof of the following proposition is clear.

(3.6) PROPOSITION. *The morphism*

$$\varrho_{\mathcal{E}_{a,\underline{b}},k,k'} : \mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} \mathcal{E}_{a,\underline{b}}((1-k')D) \to \mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} \mathcal{E}_{a,\underline{b}}(kD),$$

defined in (4), corresponds, through the isomorphisms in corollary (3.4) and lemma (3.5), to the morphism

$$\varrho'_{\mathcal{E}_{a,\underline{b}},k,k'} : \overline{P} \in \mathcal{D}_X/I_{a+\omega(h)(1-k'),\underline{b}} \mapsto \overline{Ph^{k+k'-1}} \in \mathcal{D}_X/I_{a+\omega(h)k}.$$

For the dual connection $\mathcal{E}^*_{a,\underline{b}}$, in order to simplify, let us concentrate on case $n = 2$, where the integrability condition (6) reduces to:

$$(\delta_1 - c)(b_1) = 2\delta_2(a), \quad (\delta_1 - 2c)(b_0) = \delta_2^2(a) + b_1\delta_2(a). \tag{7}$$

(3.7) LEMMA. *With the above notations, the ILC $\mathcal{E}^*_{a,\underline{b}}$ and $\mathcal{E}_{c-a,\underline{b}^*}$, with $\underline{b} = (b_1, b_0)$ and $\underline{b}^* = (-b_1, b_0 - \delta_2(b_1))$, are isomorphic.*

PROOF. The action of $\mathrm{Der}(\log D)$ over the dual basis $\{e_1^*, e_2^*\}$ in $\mathcal{E}^*_{a,\underline{b}}$ is given by:

$$(\delta_i \cdot e_j^*)(e_k) = \delta_i(e_j^*(e_k)) - e_j^*(\delta_i e_k) = -e_j^*(\delta_i e_k),$$

for $i = 1, 2$ and $j, k = 1, 2$ (see (2)). Then

$$\delta_1 \begin{pmatrix} e_1^* \\ e_2^* \end{pmatrix} = -A_1^t \begin{pmatrix} e_1^* \\ e_2^* \end{pmatrix}, \quad \delta_2 \begin{pmatrix} e_1^* \\ e_2^* \end{pmatrix} = -A_2^t \begin{pmatrix} e_1^* \\ e_2^* \end{pmatrix}.$$

Choosing the new basis $\{w_1 = e_2^*, w_2 = -e_1^* + b_1 e_2^*\}$ of $\mathcal{E}^*_{a,\underline{b}}$, we obtain

$$\delta_1 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \cdots = \begin{pmatrix} a-c & 0 \\ \delta_2(a) & a \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

$$\delta_2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \cdots = \begin{pmatrix} 0 & 1 \\ \delta_2(b_1) - b_0 & b_1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

and the isomorphism of $\mathcal{O}_X$-modules

$$\sum_{i=1}^{2} b_i w_i \in \mathcal{E}^*_{a,\underline{b}} \mapsto \sum_{i=1}^{2} b_i e_i \in \mathcal{E}_{c-a,\underline{b}^*}$$

is clearly $\mathcal{D}_X(\log D)$-linear. Q.E.D.

4 Some explicit examples

In this section we consider the case where $D \subset X = \mathbb{C}^2$ is defined by the reduced equation $h = x_1^2 - x_2^3$, and then $\omega(x_1) = 3$, $\omega(x_2) = 2$, $\omega(h) = 6$ and the basis of $\mathrm{Der}(\log D)$ is $\{\delta_1, \delta_2\}$, with

$$\begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \begin{pmatrix} 3x_1 & 2x_2 \\ 3x_2^2 & 2x_1 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix},$$

-) $\delta_1(h) = 6h$, $\quad \delta_2(h) = 0$,
-) the determinant of the coefficient matrix is equal to $6h$,
-) $[\delta_1, \delta_2] = \delta_2$ $(c = 1)$.

(4.1) Since the ILC $\mathcal{E}_{a,\underline{b}}$ and the ideals $I_{a,\underline{b}}$ in corollary (3.4) are defined globally by differential operators with polynomial coefficients and D has a global polynomial equation, the study of morphism

$$\rho_{\mathcal{E}_{a,\underline{b}},k} : \mathcal{D}_X \overset{L}{\otimes}_{\mathcal{D}_X(\log D)} \mathcal{E}_{a,\underline{b}}(kD) \to \mathcal{E}_{a,\underline{b}}(\star D)$$

can be done globally at the level of the Weyl algebra $\mathbb{W}_2 = \mathbb{C}[x_1, x_2, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}]$.

The integrability conditions in (7) (for $n = 2$) become in our case

$$(\delta_1 - 1)(b_1) = 2\delta_2(a), \quad (\delta_1 - 2)(b_0) = \delta_2^2(a) + b_1\delta_2(a). \tag{8}$$

Once a is fixed, it allows us to determine, uniquely, b_1 (the operator $\delta_1 - 1$ is injective), and to also determine b_0 up to a term ex_2, $e \in \mathbb{C}$ (the kernel of the operator $\delta_1 - 2$ is generated by x_2). In order to simplify, let us take

$$a = \lambda + mx_1 + nx_2,$$

where $\underline{\mu} = (\lambda, m, n)$ are complex parameters, and then

$$b_1 = 2mx_2^2 + 2nx_1$$

and

$$b_0 = ex_2 + 3nx_2^2 + 4mx_1x_2 + n^2x_1^2 + 2mnx_1x_2^2 + m^2x_2^4,$$

with e another complex parameter. For convenience (see the rational factorization of $B(s)$ below), let us consider another complex parameter ν and make $e = \nu - \nu^2$.

Let us define the family of ILC of rank two, $\mathcal{F}_{\nu,\underline{\mu}} := \mathcal{E}_{a,\underline{b}}$ (see (3.1)), with a, b_0, b_1 as above. We have $\mathcal{F}_{\nu,\underline{\mu}} = \mathcal{D}_X(\log D) \cdot e_1$ and $\operatorname{ann}_{\mathcal{D}_X(\log D)} e_1 = \mathcal{D}_X(\log D)(g_1, g_2)$, with $g_1 = \delta_1 + a$ and $g_2 = \delta_2^2 + b_1\delta_2 + b_0$ (see lemma (3.2)). It is clear that $\mathcal{F}_{\nu,\underline{\mu}} = \mathcal{F}_{1-\nu,\underline{\mu}}$.

The conclusion of corollary (2.5) can be globalized and we obtain

$$\operatorname{ann}_{\mathcal{D}_X[s]}(e_1h^s) = \mathcal{D}_X[s](\Phi(g_1), \Phi(g_2)) = \mathcal{D}_X[s](\delta_1 + a - 6s, g_2)$$

and

$$\operatorname{ann}_{\mathbb{W}_2[s]}(e_1h^s) = \mathbb{W}_2[s](\delta_1 + a - 6s, g_2).$$

Let us consider the Weyl algebra with parameters

$$\mathbb{W}' = \mathbb{C}\left[\lambda, m, n, \nu, x_1, x_2, \tfrac{\partial}{\partial x_1}, \tfrac{\partial}{\partial x_2}\right][s]$$

and the left ideal I generated by

$$h, \quad \delta_1 + a - 6s, \quad \delta_2^2 + b_1\delta_2 + b_0.$$

By a Gröbner basis computation with an elimination order, for example, with the help of [14], we compute the generator $B(s)$ of the ideal $I \cap \mathbb{C}[s]$ and operators $P(s), C(s), D(s) \in \mathbb{W}'$ such that

$$B(s) = P(s)h + C(s)(\delta_1 + a - 6s) + D(s)(\delta_2^2 + b_1\delta_2 + b_0).$$

We find

$$B(s) = \left(s - \frac{\lambda-5}{6}\right)\left(s - \frac{\lambda-8}{6}\right)\left(s - \frac{\lambda-\nu-6}{6}\right)\left(s - \frac{\lambda+\nu-7}{6}\right).$$

For $\lambda,\nu \in \mathbb{C}$, let us call $B_{\lambda,\nu}(s) \in \mathbb{C}[s]$ the polynomial obtained from $B(s)$ in the obvious way. We obtain then for each $\nu, \lambda, m, n \in \mathbb{C}$ the global Bernstein-Sato functional equation

$$B_{\lambda,\nu}(s) e_1 h^s = P(s)\left(e_1 h^{s+1}\right) \tag{9}$$

in $\mathcal{F}_{\nu,\underline{\mu}}[h^{-1}, s]h^s$. Therefore, $b_{\mathcal{F}_{\nu,\underline{\mu}},p}(s) \mid B_{\lambda,\nu}(s)$ (see prop. (2.1)) for any $p \in D$[1] and

$$\kappa(\mathcal{F}_{\nu,\underline{\mu}}) \geq \tau(\lambda,\nu) := \min\{\text{integer roots of } B_{\lambda,\nu}(s)\} \in \mathbb{Z} \cup \{+\infty\}.$$

We can apply theorem (1.5) to deduce that morphism

$$\rho_{\mathcal{F}_{\nu,\underline{\mu}},k} : \mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} \mathcal{F}_{\nu,\underline{\mu}}(kD) \to \mathcal{F}_{\nu,\underline{\mu}}(\star D)$$

is an isomorphism for all $k \geq -\tau(\lambda,\nu)$. On the other hand, from lemma (3.7) we know that $(\mathcal{F}_{\nu,\lambda,m,n})^* = \mathcal{F}_{\nu,1-\lambda,-m,-n}$ and then morphism

$$\rho_{\mathcal{F}^*_{\nu,\underline{\mu}},k'} : \mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} \mathcal{F}^*_{\nu,\underline{\mu}}(k'D) \to \mathcal{F}^*_{\nu,\underline{\mu}}(\star D)$$

is an isomorphism for all $k' \geq -\tau(1-\lambda,\nu)$.

The above results can be rephrased in the following way:

1) Morphism

$$\rho_{\mathcal{F}_{\nu,\underline{\mu}},k} : \mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} \mathcal{F}_{\nu,\underline{\mu}}(kD) \to \mathcal{F}_{\nu,\underline{\mu}}(\star D)$$

is an isomorphism if the four following conditions hold:

$\lambda + 6k \neq -1, -7, -13, -19, \ldots$
$\lambda + 6k \neq 2, -4, -10, -16, \ldots$
$\lambda + 6k - \nu \neq 0, -6, -12, -18, \ldots$
$\lambda + 6k + \nu \neq 1, -5, -11, -17, \ldots$

2) Morphism

$$\rho_{\mathcal{F}^*_{\nu,\underline{\mu}},k'} : \mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} \mathcal{F}^*_{\nu,\underline{\mu}}(k'D) \to \mathcal{F}^*_{\nu,\underline{\mu}}(\star D)$$

is an isomorphism if the four following conditions hold:

$1 - \lambda + 6k' \neq -1, -7, -13, -19, \ldots$
$1 - \lambda + 6k' \neq 2, -4, -10, -16, \ldots$
$1 - \lambda + 6k' - \nu \neq 0, -6, -12, -18, \ldots$
$1 - \lambda + 6k' + \nu \neq 1, -5, -11, -17, \ldots$

or equivalently, if the four following conditions hold:

$\lambda - 6k' \neq 2, 8, 14, 20, \ldots$
$\lambda - 6k' \neq -1, 5, 11, 17, \ldots$
$\lambda + \nu - 6k' \neq 1, 7, 13, 19, \ldots$
$\lambda - \nu - 6k' \neq 1, -5, -11, -17, \ldots$

In particular, if the four following conditions:

[1] In fact it is possible to show that $b_{\mathcal{F}_{\nu,\underline{\mu}},0}(s) = B_{\lambda,\nu}(s)$.

(i) $\lambda \not\equiv 2 \pmod 6$ or $\lambda = 2$

(ii) $\lambda \not\equiv 5 \pmod 6$ or $\lambda = -1$

(iii) $\lambda + \nu \not\equiv 1 \pmod 6$ or $\lambda + \nu = 1$

(iv) $\lambda - \nu \not\equiv 0 \pmod 6$ or $\lambda - \nu = 0$

hold, both morphisms

$$\rho_{\mathcal{F}_{\nu,\underline{\mu}},1} : \mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} \mathcal{F}_{\nu,\underline{\mu}}(D) \to \mathcal{F}_{\nu,\underline{\mu}}(\star D),$$

$$\rho_{\mathcal{F}^*_{\nu,\underline{\mu}},1} : \mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} \mathcal{F}^*_{\nu,\underline{\mu}}(D) \to \mathcal{F}^*_{\nu,\underline{\mu}}(\star D)$$

are isomorphisms.

Let us denote by $\mathcal{L}_{\nu,\underline{\mu}}$ the local system over $X - D$ of the horizontal sections of $\mathcal{F}_{\nu,\underline{\mu}}$. By theorem (1.6), we have

$$\mathrm{IC}_X(\mathcal{L}_{\nu,\underline{\mu}}) \simeq \mathrm{DR}(\operatorname{Im} \varrho_{\mathcal{F}_{\nu,\underline{\mu}},1,1}),$$

provided that conditions (i)-(iv) are satisfied.

Proposition (3.6) and (4.1) reduce the computation of $\operatorname{Im} \varrho_{\mathcal{F}_{\nu,\underline{\mu}},1,1}$ to the computation of the image of the map

$$\theta_{\nu,\underline{\mu}} : \overline{L} \in \mathbb{W}_2/\mathbb{W}_2(g_1, g_2) \mapsto \overline{Lh} \in \mathbb{W}_2/\mathbb{W}_2(g_1 + 6, g_2),$$

but $\operatorname{Im} \theta_{\nu,\underline{\mu}} = \mathbb{W}_2/K_{\nu,\underline{\mu}}$ where

$$K_{\nu,\underline{\mu}} = \{R \in \mathbb{W}_2 \mid Rh \in \mathbb{W}_2(g_1 + 6, g_2)\}.$$

Now, in order to compute generators of $K_{\nu,\underline{\mu}}$, we proceed as follows. Since $[g_1, g_2] = 2g_2$ (for any $\nu, \underline{\mu}$) and the symbols $\sigma(g_1) = \sigma(\delta_1)$, $\sigma(g_2) = \sigma(\delta_2)^2$ form a regular sequence (D is Koszul free!), we deduce that

$$\sigma\left(\mathbb{W}_2(g_1 + 6, g_2)\right) = \left(\sigma(\delta_1), \sigma(\delta_2)^2\right)$$

and consequently $\sigma\left(K_{\nu,\underline{\mu}}\right) \subset (\sigma(\delta_1), \sigma(\delta_2)^2) : h$. A straightforward (commutative) computation shows that

$$(\sigma(\delta_1), \sigma(\delta_2)^2) : h = (\sigma(\delta_1), \sigma(Q_0))$$

with $Q_0 = 9x_2 \frac{\partial^2}{\partial x_1^2} - 4\frac{\partial^2}{\partial x_2^2}$, and

$$\sigma(Q_0)h = x_2\sigma(\delta_1)^2 - \sigma(\delta_2)^2 = x_2\sigma(\delta_1)\sigma(g_1 + 6) - \sigma(g_2). \tag{10}$$

Searching to lift the relation (10) to $\mathbb{W}_2$, we find

$$Qh = x_2(\delta_1 + mx_1 + nx_2 + 7 - \lambda)(g_1 + 6) - g_2 + (\lambda^2 - \lambda + \nu - \nu^2)x_2,$$

with $Q = Q_0 + 6mx_2\frac{\partial}{\partial x_1} - 4n\frac{\partial}{\partial x_2} + m^2x_2 - n^2$. In particular, if condition

$$\lambda^2 - \lambda + \nu - \nu^2 = 0 \quad (\Leftrightarrow \lambda - \nu = 0 \ \text{ or } \ \lambda + \nu = 1) \tag{11}$$

holds, then $Q \in K_{\nu,\underline{\mu}}$.

Actually, by using the equality $[Q, g_1] = 4Q$ and the fact that $\sigma(Q) = \sigma(Q_0)$ and $\sigma(g_1) = \sigma(\delta_1)$ also form a regular sequence in $\mathrm{Gr}\,\mathbb{W}_2$, condition (11) implies that

$$K_{\nu,\underline{\mu}} = \mathbb{W}_2(g_1, Q), \quad \sigma\left(K_{\nu,\underline{\mu}}\right) = (\sigma(\delta_1), \sigma(Q_0)).$$

On the other hand, since $\sigma(Q_0)$ is not contained in the ideal (x_1, x_2), we finally deduce the following result:

If parameters $\nu, \underline{\mu} = (\lambda, m, n)$ satisfy conditons (i)-(iv) and (11), then the conormal of the origin $T^*_0(X)$ does not appear as an irreducible component of the characteristic variety of $\mathrm{Im}\,\theta_{\nu,\underline{\mu}} = \mathbb{W}_2/K_{\nu,\underline{\mu}}$, and consequently

$$\mathrm{Ch}(\mathrm{IC}_X(\mathcal{L}_{\nu,\underline{\mu}})) = \mathrm{Ch}\left(\mathbb{W}_2/K_{\nu,\underline{\mu}}\right) = \{\sigma(\delta_1) = \sigma(Q_0) = 0\} = T^*_X(X) \cup T^*_D(X).$$

The existence of such an example has been suggested by [21], example (3.4), but the question on the values of the parameters $\nu, \underline{\mu}$ for which the local system $\mathcal{L}_{\nu,\underline{\mu}}$ is irreducible will be treated elsewhere.

If condition (11) does not hold, it is not clear that there exists a general expression for a system of generators of $K_{\nu,\underline{\mu}}$ as before.

(4.2) Remark. The relationship between the preceding results and examples and the hypergeometric local systems (cf. [23, 24, 29]) is interesting and possibly deserves further work.

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