Mutually nearest and farthest points of sets and the Drop Theorem in geodesic spaces

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1 Introduction

Let A and X be nonempty, bounded and closed subsets of a metric space (E, d). The *minimization* (resp. *maximization*) *problem* denoted by $\min(A, X)$ (resp. $\max(A, X)$) consists in finding $(a_0, x_0) \in A \times X$ such that $d(a_0, x_0) = \inf \{d(a, x) : a \in A, x \in X\}$ (resp. $d(a_0, x_0) = \sup \{d(a, x) : a \in A, x \in X\}$). Here we give generic results on the well-posedness of these problems in different geodesic spaces and under different conditions considering the set A fixed. Besides, we analyze the situations when one set or both sets are compact and give some specific results for CAT(0) spaces. We also state a variant of the Drop Theorem in Busemann convex geodesic spaces and apply it to obtain an optimization result for convex functions. The proofs of these results can be found in [11, 19].

For $A \in \mathcal{P}_{cl}(E)$ (resp. $A \in \mathcal{P}_{b,cl}(E)$) and $x \in E \setminus A$, the *nearest point problem* (resp. *farthest point problem*) of x to A consists in finding a point $a_0 \in A$ (the solution of the problem) such that $d(x, a_0) = \mathrm{dist}(x, A)$ (resp. $d(x, a_0) = \mathrm{Dist}(x, A)$). Stečkin [23] was one of the first who realized that in case E is a Banach space, geometric properties like strict convexity, uniform convexity, reflexivity and others play an important role in the study of nearest and farthest point problems. His work triggered a series of results so-called "in the spirit of Stečkin" because the ideas he used were adapted again and again by different authors to various contexts (see, for example, [7, 8, 18]). In [23], Stečkin proved, in particular, that for each nonempty and closed subset A of a uniformly convex Banach space, the complement of the set of all points $x \in E$ for which the nearest point problem of x to A has a unique solution is of first Baire category. One of the results also given in [23] and later improved by De Blasi, Myjak and Papini in [7] was going to become a key tool in proving best approximation results and was called Stečkin's Lens Lemma.

In [8], De Blasi, Myjak and Papini studied more general problems than the ones of nearest and farthest points. Namely, they considered the problem of finding two points which minimize (resp. maximize) the distance between two subsets of a Banach space. They focused on the well-posedness of the problem which consists in showing the uniqueness of the solution and that any approximating sequence of the problem must actually converge to the solution (see Section 4 for details). The authors proved that if A is a nonempty, bounded and closed subset of a uniformly convex Banach space E, the family of sets in $\mathcal{P}_{b,cl,cv}(E)$ for which the maximization problem, $\max(A, X)$, is well-posed is a dense G_δ-set in the family $\mathcal{P}_{b,cl,cv}(E)$ endowed with the Pompeiu-Hausdorff distance. For the minimization problem, $\min(A, X)$, a similar result is proved where X belongs to a particular subspace of $\mathcal{P}_{b,cl,cv}(E)$. A nice synthesis of issues concerning nearest and farthest point problems in connection with geometric properties of Banach spaces and some extensions of these problems can be found in [4].

Zamfirescu initiated in [24] the investigation of this kind of problems in the context of geodesic spaces. Later on, researchers have focused on adapting the ideas of Stečkin [23] into the geodesic setting. In particular, Zamfirescu [25] proved that, in a complete geodesic space E without bifurcating geodesics, having a fixed compact set A, the set of points $x \in E$ for which the nearest point problem of x to A has a single solution is a set of second Baire category. Motivated by this result, Kaewcharoen and Kirk [16] showed that if E is a complete CAT(0) space with the geodesic extension property and with curvature bounded below globally, for any fixed closed set A, the set of points $x \in E$ for which the nearest point problem of x to A has a unique solution is a set of second Baire category. A similar result was proved for the farthest point problem. Very recent results in the context of spaces with curvature bounded below globally were obtained in [10] where the authors proved a variant of Stečkin's Lemma that allowed them to give some porosity theorems which are stronger results than the ones in [16].

Here we are also concerned with the geometric result known as the Drop Theorem. The original version of this theorem was proved by Daneš [5] and is a very useful tool in nonlinear analysis. Moreover, it is equivalent to the Ekeland Variational Principle and the Flower Petal Theorem [20]. In [12], generalized versions of the Drop Theorem are proved and afterwards used in the proofs of various minimization problems. For more details see also [15].

We study in the context of geodesic metric spaces the problem of minimizing (resp. maximizing) the distance between two sets, originally considered by De Blasi, Myjak and Papini in [8] for uniformly convex Banach spaces. The given results rely on a property of the convex hull of the union of a convex set with a point in Busemann convex spaces, Lemma 4.4, which is given at the beginning of Section 4.1. We show that if E is a Busemann convex geodesic space with curvature bounded below and the geodesic extension property, the family of sets in $\mathcal{P}_{b,cl,cv}(E)$ for which $\max(A, X)$ is well-posed is a dense G_δ-set in $\mathcal{P}_{b,cl,cv}(E)$. A similar result is given for the minimizing problem, $\min(A, X)$, with no need of the geodesic extension property. These results give natural counterparts to those obtained by De Blasi *et al.* in [8]. After this we focus on the case of CAT(0) spaces, where the rich geometry of these spaces will be used to relax certain conditions in relation to the well-posedness problem. Then, in Section 4.2, we show that the boundedness condition on the curvature of the space is no longer needed if we impose compactness conditions on the sets. Both minimization and maximization problems are discussed in this context where we replace the condition on the curvature by that of not having bifurcating geodesics introduced by Zamfirescu in [25]. Finally, in our last section, we consider the Drop Theorem in geodesic spaces. With the aid of the Strong Flower Petal Theorem we derive a version of the Drop Theorem in our context which is used to study an optimization problem for convex and continuous real-valued functions defined on geodesic spaces.

2 Preliminaries

Let (E, d) be a metric space. A *geodesic* in E is an isometry from $\mathbb{R}$ into E (we may also refer to the image of this isometry as a geodesic). A *geodesic path* from x to y is a mapping $c : [0, l] \to E$, where $[0, l] \subseteq \mathbb{R}$, such that $c(0) = x, c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for every $t, t' \in [0, l]$. The image $c([0, l])$ of c forms a *geodesic segment* which joins x and y. Note that the geodesic segment from x to y is not necessarily unique. If no confusion arises, we will use $[x, y]$ to denote a geodesic segment joining x and y. (E, d) is a *geodesic space* if every two points $x, y \in E$ can be joined by a geodesic path. A point $z \in E$ belongs to the geodesic segment $[x, y]$ if and only if there exists

$t \in [0,1]$ such that $d(z,x) = td(x,y)$ and $d(z,y) = (1-t)d(x,y)$, and we will write $z = (1-t)x + ty$ for simplicity. This, too, may not be unique. (E,d) has the *geodesic extension property* if each geodesic segment is contained in a geodesic. For a very comprehensive treatment of geodesic metric spaces the reader may check [1].

The geodesic space (E,d) is *Busemann convex* if given any pair of geodesic paths $c_1 : [0,l_1] \to E$ and $c_2 : [0,l_2] \to E$ with $c_1(0) = c_2(0)$ one has

$$d(c_1(tl_1), c_2(tl_2)) \leq td(c_1(l_1), c_2(l_2)) \text{ for all } t \in [0,1].$$

A *subset X of E is convex* if any geodesic segment that joins every two points of X is contained in X. Let $G_1(X)$ denote the union of all geodesics segments with endpoints in X. Notice that X is convex if and only if $G_1(X) = X$. Recursively, for $n \geq 2$ we set $G_n(X) = G_1(G_{n-1}(X))$. Then the *convex hull* of X will be

$$\text{co}(X) = \bigcup_{n \in \mathbb{N}} G_n(X).$$

By $\overline{\text{co}}(X)$ we shall denote the closure of the convex hull. It is easy to see that in a Busemann convex geodesic space, the closure of the convex hull will be convex and hence it is the smallest closed convex set containing X.

Let $\kappa \in \mathbb{R}$ and $n \in \mathbb{N}$. The classical model spaces M_κ^n are defined in the following way: if $\kappa > 0$, M_κ^n is obtained from the spherical space $\mathbb{S}^n$ by multiplying the spherical distance with $1/\sqrt{\kappa}$; if $\kappa = 0$, M_0^n is the n-dimensional Euclidean space $\mathbb{R}^n$; and if $\kappa < 0$, M_κ^n is obtained from the hyperbolic space $\mathbb{H}^n$ by multiplying the hyperbolic distance with $1/\sqrt{-\kappa}$. For more details about these spaces and related topics one can consult [1, 13].

A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ consists of three points x_1, x_2 and x_3 in X (the *vertices* of the triangle) and three geodesic segments corresponding to each pair of points (the *edges* of the triangle). For the geodesic triangle $\Delta{=}\Delta(x_1, x_2, x_3)$, a *$\kappa$-comparison triangle* is a triangle $\bar{\Delta} = \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in M_κ^2 such that $d(x_i, x_j) = d_{M_\kappa^2}(\bar{x}_i, \bar{x}_j)$ for $i, j \in \{1, 2, 3\}$. For κ fixed, κ-comparison triangles of geodesic triangles (having perimeter less than $2\pi/\sqrt{\kappa}$ if $\kappa > 0$) always exist and are unique up to isometry (see [1, Lemma 2.14]).

A geodesic triangle Δ satisfies the *CAT(κ)* (resp. *reversed CAT(κ)*) *inequality* if for every κ-comparison triangle $\bar{\Delta}$ of Δ and for every $x, y \in \Delta$ we have

$$d(x,y) \leq d_{M_\kappa^2}(\bar{x}, \bar{y}) \text{ (resp. } d(x,y) \geq d_{M_\kappa^2}(\bar{x}, \bar{y})),$$

where $\bar{x}, \bar{y} \in \bar{\Delta}$ are the corresponding points of x and y, i.e., if $x = (1-t)x_i + tx_j$ then $\bar{x} = (1-t)\bar{x}_i + t\bar{x}_j$.

If $\kappa \leq 0$, a *CAT(κ) space* (also known as a space of bounded curvature in the sense of Gromov) is a geodesic space for which every geodesic triangle satisfies the CAT(κ) inequality. If $\kappa > 0$, a metric space is called a CAT(κ) space if every two points at distance less than $\pi/\sqrt{\kappa}$ can be joined by a geodesic path and every geodesic triangle having perimeter less then $2\pi/\sqrt{\kappa}$ satisfies the CAT(κ) inequality.

A geodesic metric space is said to have *curvature bounded below* if there exists $\kappa < 0$ such that every geodesic triangle satisfies the reversed CAT(κ) inequality. Other properties of spaces with curvature bounded below and equivalent definitions can be found in [3].

CAT(0) spaces are a particular class of CAT(κ) spaces which has called the attention of a large number of researchers in the last decades due to its rich geometry and relevance in different problems. The fact that a CAT(0) space is Busemann convex has a great impact on the geometry of the space, but we must mention that being Busemann convex is a weaker property than being CAT(0).

We say that the geodesic space (E, d) is *reflexive* if every descending sequence of nonempty, bounded, closed and convex subsets of E has nonempty intersection. A simple example of a reflexive metric space is a reflexive Banach space. Other examples include complete CAT(0) spaces, complete uniformly convex metric spaces with a monotone or a lower semi-continuous from the right modulus of uniform convexity, and others.

Let (E, d) be a metric space. Taking $z \in E$ and $r > 0$ we denote the open (resp. closed) ball centered at z with radius r by $B(z, r)$ (resp. $\widetilde{B}(z, r)$). Given X a nonempty subset of E, we define the *distance of a point* $z \in E$ to X by $\operatorname{dist}(z, X) = \inf\{d(z, x) : x \in X\}$. The *metric projection* (or *nearest point mapping*) P_X onto X is the mapping

$$P_X(y) = \{x \in X : d(x, y) = \operatorname{dist}(y, X)\}, \text{ for every } y \in E.$$

The closure of the set X will be denoted as $\overline{X}$.
If X is additionally bounded, the *diameter* of X is given by $\operatorname{diam} X = \sup\{d(x, y) : x, y \in X\}$ and the *remotal distance of a point* $z \in E$ to X is defined by $\operatorname{Dist}(z, X) = \sup\{d(z, x) : x \in X\}$. The *farthest point mapping* F_X onto X is given by

$$F_X(y) = \{x \in X : d(x, y) = \operatorname{Dist}(y, X)\}, \text{ for every } y \in E.$$

From now on, if nothing else is mentioned, E will stand for a geodesic metric space. We consider the following families of sets

$$\mathcal{P}_{cl}(E) = \{X \subseteq E : X \text{ is nonempty and closed}\},$$

$$\mathcal{P}_{b,cl}(E) = \{X \subseteq E : X \text{ is nonempty, bounded and closed}\},$$

$$\mathcal{P}_{b,cl,cv}(E) = \{X \subseteq E : X \text{ is nonempty, bounded, closed and convex}\},$$

$$\mathcal{P}_{cp}(E) = \{X \subseteq E : X \text{ is nonempty and compact}\},$$

$$\mathcal{P}_{cp,cv}(E) = \{X \subseteq E : X \text{ is nonempty, compact and convex}\}.$$

If E is complete, then $\mathcal{P}_{b,cl}(E)$ and $\mathcal{P}_{cp}(E)$ are complete under the Pompeiu-Hausdorff distance. If, additionally, E is Busemann convex, then, by an easy adaptation of the argument in the Banach space context, one can prove that $\mathcal{P}_{b,cl,cv}(E)$ and $\mathcal{P}_{cp,cv}(E)$ are also complete with respect to the Pompeiu-Hausdorff distance.

3 Nearest and farthest point problems

We give in the sequel some existence and well-posedness results for nearest and farthest point problems. We recall first the notion of well-posedness for such problems. The results we include in this section hold in the framework of Banach or complete geodesic spaces. This is why we consider next E to be a complete geodesic space although the definitions given below can be stated in the general setting of metric spaces.

For $A \in \mathcal{P}_{cl}(E)$ (resp. $A \in \mathcal{P}_{b,cl}(E)$) and $x \in E$, denote by $\min(x, A)$ (resp. $\max(x, A)$) the nearest point problem of x to A. A sequence $(a_n) \subseteq A$ is called a *minimizing* (resp. *maximizing*) *sequence* if $\lim_{n\to\infty} d(x, a_n) = \operatorname{dist}(x, A)$ (resp. $\lim_{n\to\infty} d(x, a_n) = \operatorname{Dist}(x, A)$). The problem $\min(x, A)$ (resp. $\max(x, A)$) is said to be *well-posed* if it has a unique solution $a_0 \in A$ and every minimizing (resp. maximizing) sequence converges to a_0.

We introduce next two sets that are used below to characterize the well-posedness of nearest and farthest point problems:

$$L_A(\sigma) = \{a \in A : d(x, a) \leq \operatorname{dist}(x, A) + \sigma\},$$

$$M_A(\sigma) = \{a \in A : d(x, a) \geq \operatorname{Dist}(x, A) - \sigma\}.$$

Using the definitions one can check that for A closed, the problem $\min(x, A)$ is well-posed if and only if $\lim_{\sigma\searrow 0} \operatorname{diam}(L_A(\sigma)) = 0$. Likewise, if A is bounded and closed, then $\max(x, A)$ is well-posed if and only if $\lim_{\sigma\searrow 0} \operatorname{diam}(M_A(\sigma)) = 0$

Let us consider the following special set. For

$$x \in E, \quad r > 0, \quad y \in B(x, r/2) \setminus \{x\} \quad \text{and} \quad 0 \leq \sigma \leq 2d(x, y),$$

set

$$D(x, y; r, \sigma) = \widetilde{B}(y, r - d(x, y) + \sigma) \setminus B(x, r).$$

Because of its shape in $\mathbb{R}^3$ this set was called a *lens* in [7].

Stečkin [23] studied nearest point problems in normed spaces and proved results that inspired many mathematicians to consider nearest and farthest point problems in various settings. One of the key tools proved by Stečkin [23] states that in uniformly convex Banach spaces, $\operatorname{diam}(D(x, y; r, \sigma))$ converges to 0 as $\sigma \searrow 0$ uniformly with respect to $y \in B(x, r/2)$, $y \neq x$ such that $d(x, y)$ is constant. Using this property, Stečkin [23] proved the following result in uniformly convex Banach spaces.

Theorem 3.1 (Stečkin [23]). *Let E be a uniformly convex Banach space and $A \in \mathcal{P}_{cl}(E)$. Then the set*

$$E \setminus \{x \in E : \min(x, A) \text{ is well-posed}\}$$

is of first Baire category.

De Blasi, Myjak and Papini gave in [7] an estimation of the diameter of the lens and called this result Stečkin's Lens Lemma. To this end, for E a uniformly convex Banach space with modulus of convexity δ_E and for $0 < \sigma \leq 1$, set

$$\delta^*(\sigma) = \sup\{\epsilon : 0 < \epsilon \leq 2 \text{ and } \delta_E(\epsilon) \leq \sigma\}.$$

Lemma 3.2 (De Blasi, Myjak, Papini [7]). *Let E be a uniformly convex Banach space, $x \in E$, $r > 0$, $y \in B(x, r/2)$, $y \neq x$. Then, for every $0 < \sigma \leq 2\|y - x\|$,*

$$\operatorname{diam}(D(x, y; r, \sigma)) \leq 2\sigma + 2(r - \|y - x\|)\delta^*\left(\frac{\sigma}{2\|y - x\|}\right).$$

Using the above lemma, the same authors proved in [8] the following uniform version of a result given by Stečkin [23].

Proposition 3.3 (De Blasi, Myjak, Papini [8]). *Let E be a uniformly convex Banach space. Suppose $\epsilon > 0$, $r, r_0 > 0$ with $r < r_0$. Then there exists $0 < \sigma_0 < r$ such that for every $x, y \in E$ with $\|x - y\| = r$ and for every $r < r' \leq r_0$ and $0 < \sigma \leq \sigma_0$,*

$$\operatorname{diam}(D(x, y; r', \sigma)) < \epsilon.$$

In the context of geodesic metric spaces, the property of not having bifurcating geodesics defined in [25] has been proved to play a significant role in the study of nearest and farthest point problems. A geodesic metric space is said to be *without bifurcating geodesics* if for any two geodesic segments with the same initial point and having another common point (different from the initial one), this second point is a common endpoint of both or one segment contains the other. From the definitions it can be seen that a space with curvature bounded below globally cannot have bifurcating geodesics. Using this property, Zamfirescu [25] proved the next result.

Theorem 3.4 (Zamfirescu [25]). *Let E be a complete geodesic space without bifurcating geodesics and $A \in \mathcal{P}_{cp}(E)$. Then P_A is singlevalued on a set of second Baire category.*

Kaewcharoen and Kirk [16] studied nearest and farthest point problems in complete CAT(0) spaces with curvature bounded below globally by $\kappa \leq 0$ and with the geodesic extension property. The proofs of these results rely closely on the uniform convexity of CAT(0) spaces.

Theorem 3.5 (Kaewcharoen, Kirk [16]). *Let E be a complete $CAT(0)$ space with curvature bounded below globally by $\kappa \leq 0$ and with the geodesic extension property. Suppose $A \in \mathcal{P}_{cl}(E)$. Then P_A is well-defined and singlevalued on a set of second Baire category.*

Theorem 3.6 (Kaewcharoen, Kirk [16]). *Let E be a complete $CAT(0)$ space with curvature bounded below globally by $\kappa \leq 0$ and with the geodesic extension property. Suppose $A \in \mathcal{P}_{b,cl}(E)$. Then F_A is well-defined and singlevalued on a dense subset of E.*

In the same paper it is shown (see [16, Example 3.9]) that the condition that the curvature is bounded below globally cannot be dropped in the above results.

Example 3.7 (Kaewcharoen, Kirk [16]). *Consider in ℓ_2 the sets*

$$L_i = \{te_i : t \in \mathbb{R}\},$$

where e_i is the standard i^{th} unit basis vector. Let $E = \bigcup_{i \in \mathbb{N}} L_i$ with the shortest path metric. This is a complete $\mathbb{R}$-tree. Take the closed sets

$$A_1 = \left\{te_i : t \geq 1 + \frac{1}{i}, i \in \mathbb{N}\right\},$$

$$A_2 = \left\{te_i : 0 \leq t \leq 1 - \frac{1}{i}, i \in \mathbb{N}\right\}.$$

Set $U = \{(u, 0, 0, \ldots) : u < 0\}$. Then any point in U has neither a nearest point in A_1, nor a farthest point in A_2.

Recent results in the context of spaces with curvature bounded below globally were obtained in [10] where the authors proved a variant of Stečkin's Lemma and extended the results given in [16]. We include below some results obtained in [10] that constitute key tools in proving our results. Following [10], for $\kappa \in (-\infty, 0)$, define the real function F_κ on $\mathbb{R}^3_+$ by

$$F_\kappa(d,r,\sigma) = \frac{2}{\sqrt{-\kappa}} \operatorname{arccosh}\Bigg(\cosh^2(\sqrt{-\kappa}\,(r-d+\sigma)) - \frac{\sinh(\sqrt{-\kappa}\,(r-d+\sigma))}{\sinh(\sqrt{-\kappa}\,d)} \cdot \left[\cosh(\sqrt{-\kappa}\,r) - \cosh(\sqrt{-\kappa}\,d)\cosh(\sqrt{-\kappa}\,(r-d+\sigma))\right] \Bigg)$$

for each $(d,r,\sigma) \in \mathbb{R}^3_+$.

In [10], the authors proved the following properties of the function F_κ and gave an estimation of the diameter of the lens $D(x,y;r,\sigma)$. This estimation yielded a variant of Stečkin's Lemma for spaces of curvature bounded below globally.

Proposition 3.8 (Espínola, Li, López [10]). *The function F_κ is continuous on $\mathbb{R}^3_+$ and for any $d \geq 0$ and $r \geq 0$, we have that $F_\kappa(d,r,0) = 0$.*

Proposition 3.9 (Espínola, Li, López [10]). *Let E be a geodesic space of curvature bounded below globally by κ and let $x \in E$, $r > 0$, $y \in B(x, r/2) \setminus \{x\}$ and $0 \leq \sigma \leq 2d(x,y)$. Suppose there exists $u \in E$ in a geodesic passing through x and y such that $d(x,u) = r$ and $d(y,u) = r - d(x,y)$. Then,*

$$\operatorname{diam}(D(x,y;r,\sigma)) \leq F_\kappa(d(x,y), r, \sigma) + 2\sigma.$$

We include below some of the results proved in [10] with the remark that the main results are much stronger and involve porosity concepts which we do not discuss in this work.

Theorem 3.10 (Espínola, Li, López [10]). *Let E be a complete geodesic metric space with curvature bounded below globally by $\kappa < 0$ and $A \in \mathcal{P}_{cl}(E)$. Then,*

$$\{x \in E \setminus A : \min(x,A) \text{ is well-posed}\}$$

is a dense G_δ-set in $E \setminus A$.

Theorem 3.11 (Espínola, Li, López [10]). *Let E be a compete geodesic metric space with curvature bounded below globally by $\kappa < 0$ and with the geodesic extension property. Suppose $A \in \mathcal{P}_{b,cl}(E)$. Then,*

$$\{x \in E : \max(x,A) \text{ is well-posed}\}$$

is a dense G_δ-set in E.

4 Minimization and maximization problems between two sets

In [8], De Blasi, Myjak and Papini studied the problem of finding two points which minimize (resp. maximize) the distance between two subsets of a Banach space. Although the next notions and the proposition below were originally given in the setting of Banach spaces they can be also introduced in the framework of geodesic metric spaces (or even general metric spaces). In this section, if nothing else is mentioned, E denotes a complete geodesic metric space. Following [8], for $X, Y \in \mathcal{P}_{b,cl}(E)$ and $\sigma > 0$, we set

$$\lambda_{XY} = \inf \{d(x,y) : x \in X, y \in Y\}, \quad \mu_{XY} = \sup \{d(x,y) : x \in X, y \in Y\},$$

$$L_{XY}(\sigma) = \{x \in X : \mathrm{dist}(x, Y) \le \lambda_{XY} + \sigma\},$$

$$M_{XY}(\sigma) = \{x \in X : \mathrm{Dist}(x, Y) \ge \mu_{XY} - \sigma\}.$$

The *minimization* (resp. *maximization*) *problem* denoted by $\min(X,Y)$ (resp. $\max(X,Y)$) consists in finding $(x_0, y_0) \in X \times Y$ (the *solution* of the problem) such that $d(x_0, y_0) = \lambda_{XY}$ (resp. $d(x_0, y_0) = \mu_{XY}$). A sequence (x_n, y_n) in $X \times Y$ such that $d(x_n, y_n) \to \lambda_{XY}$ (resp. $d(x_n, y_n) \to \mu_{XY}$) is called a *minimizing* (resp. *maximizing*) *sequence*. The problem $\min(X,Y)$ (resp. $\max(X,Y)$) is said to be *well-posed* if it has a unique solution $(x_0, y_0) \in X \times Y$ and for every minimizing (resp. maximizing) sequence (x_n, y_n) we have $x_n \to x_0$ and $y_n \to y_0$. In the following we give a characterization of the well-posedness of $\min(X,Y)$ (resp. $\max(X,Y)$) which can be proved by a straightforward verification of the above definitions.

Proposition 4.1 (De Blasi, Myjak, Papini [8])**.** *Let (E, d) be a complete geodesic metric space and $X, Y \in \mathcal{P}_{b,cl}(E)$. The problem $\min(X,Y)$ (resp. $\max(X,Y)$) is well-posed if and only if*

$$\inf_{\sigma>0} \mathrm{diam}\,(L_{XY}(\sigma)) = 0 \quad \textit{and} \quad \inf_{\sigma>0} \mathrm{diam}\,(L_{YX}(\sigma)) = 0,$$

$$(\textit{resp.} \inf_{\sigma>0} \mathrm{diam}\,(M_{XY}(\sigma)) = 0 \quad \textit{and} \quad \inf_{\sigma>0} \mathrm{diam}\,(M_{YX}(\sigma)) = 0).$$

In order to state the results proved in [8], consider E a uniformly convex Banach space, $A \in \mathcal{P}_{b,cl}(E)$ and denote

$$\mathcal{P}^A_{b,cl,cv}(E) = \overline{\{X \in \mathcal{P}_{b,cl,cv}(E) : \lambda_{AX} > 0\}}.$$

Then, $(\mathcal{P}^A_{b,cl,cv}(E), H)$ is a complete metric space. The following minimization and maximization results are given in [8].

Theorem 4.2 (De Blasi, Myjak, Papini [8])**.** *Let E be a uniformly convex Banach space and $A \in \mathcal{P}_{b,cl}(E)$. Then,*

$$\{X \in \mathcal{P}^A_{b,cl,cv}(E) : \min(A, X) \textit{ is well-posed}\}$$

is a dense G_δ-set in $\mathcal{P}^A_{b,cl,cv}(E)$.

Theorem 4.3 (De Blasi, Myjak, Papini [8])**.** *Let E be a uniformly convex Banach space and $A \in \mathcal{P}_{b,cl}(E)$. Then,*

$$\{X \in \mathcal{P}_{b,cl,cv}(E) : \max(A, X) \textit{ is well-posed}\}$$

is a dense G_δ-set in $\mathcal{P}_{b,cl,cv}(E)$.

In the next two subsections we study minimization and maximization problems between sets in particular geodesic metric spaces.

4.1 Results in Busemann convex spaces with curvature bounded below globally

We begin this subsection by giving an estimation for $\mathrm{dist}(y, X)$, where $X \in \mathcal{P}_{b,cv}(E)$, $x' \in E$ such that $\mathrm{dist}(x', X) > 0$ and $y \in \overline{\mathrm{co}}\,(X \cup \{x'\})$. It is easy to see that in a Busemann convex geodesic metric space, $\mathrm{dist}(y, X) < \mathrm{dist}(x', X)$ for every $y \in \mathrm{co}\,(X \cup \{x'\})$ with $y \neq x'$. We sharpen this upper bound in the following way.

Lemma 4.4. *Let E be a Busemann convex metric space and $X \in \mathcal{P}_{b,cv}(E)$. Suppose $x' \in E$ such that $\mathrm{dist}(x', X) > 0$. Then, for every $y \in \overline{\mathrm{co}}\,(X \cup \{x'\})$,*

$$\mathrm{dist}(y, X) \leq \mathrm{dist}(x', X) - \frac{\mathrm{dist}(x', X)}{\mathrm{dist}(x', X) + \mathrm{diam}(X)} d(x', y). \tag{1}$$

We give next a property of Banach spaces which was used in [8] to prove minimization and maximization problems between two sets in Banach spaces.

Proposition 4.5 (De Blasi, Myjak, Papini [8])**.** *Let E be a Banach space, $X \in \mathcal{P}_{b,cl,cv}(E)$ and $\epsilon, r > 0$. Then there exists $0 < \tau_0 < r$ such that for every $u \in E$ with $\mathrm{dist}(u, X) \geq r$ and for every $0 < \tau \leq \tau_0$ we have*

$$\mathrm{diam}(C_{X,u}(\tau)) < \epsilon,$$

where

$$C_{X,u}(\tau) = [\overline{\mathrm{co}}(X \cup \{u\})] \setminus [X + (\mathrm{dist}(u, X) - \tau)B(0, 1)].$$

The following lemma is an analogue in the metric setting of the above proposition. Its proof uses Lemma 4.4.

Lemma 4.6. *Let E be a Busemann convex metric space and $X \in \mathcal{P}_{b,cv}(E)$. For $r > 0$, $x' \in E$ with $\mathrm{dist}(x', X) \geq r$ and $n \in \mathbb{N}$ with $1/n < r$ define*

$$C_n = \overline{\mathrm{co}}\left(X \cup \{x'\}\right) \setminus \bigcup_{x \in X} B(x, \mathrm{dist}(x', X) - 1/n).$$

Then, the sequence $(\mathrm{diam}(C_n))$ converges to 0 uniformly with respect to $x' \in E$ such that $\mathrm{dist}(x', X) \geq r$.

In order to state our main results, we introduce the following notations. Let $A \in \mathcal{P}_{b,cl}(E)$ be fixed. Then, we denote $\lambda_X = \lambda_{XA}$ and $\mu_X = \mu_{XA}$ for $X \in \mathcal{P}_{b,cl}(E)$. Following [8], set

$$\mathcal{P}^A_{b,cl,cv}(E) = \overline{\{X \in \mathcal{P}_{b,cl,cv}(E) : \lambda_X > 0\}}.$$

Endowed with the Pompeiu-Hausdorff distance, $\mathcal{P}^A_{b,cl,cv}(E)$ is a complete metric space if E is Busemann convex.
For $p \in \mathbb{N}$ define

$$\mathcal{L}_p = \left\{X \in \mathcal{P}^A_{b,cl,cv}(E) : \inf_{\sigma > 0} \mathrm{diam}\left(L_{XA}(\sigma)\right) < \frac{1}{p} \text{ and } \inf_{\sigma > 0} \mathrm{diam}\left(L_{AX}(\sigma)\right) < \frac{1}{p}\right\}$$

and

$$\mathcal{M}_p = \left\{X \in \mathcal{P}_{b,cl,cv}(E) : \inf_{\sigma > 0} \mathrm{diam}\left(M_{XA}(\sigma)\right) < \frac{1}{p} \text{ and } \inf_{\sigma > 0} \mathrm{diam}\left(M_{AX}(\sigma)\right) < \frac{1}{p}\right\}.$$

We state next the two main results of this subsection, which are counterparts in the geodesic case of Theorems 4.2 and 4.3 respectively. The proofs of these results rely on Lemma 4.6.

Theorem 4.7. *Let E be a complete Busemann convex metric space with curvature bounded below globally by $\kappa < 0$. Suppose $A \in \mathcal{P}_{b,cl}(E)$. Then,*

$$\mathcal{W}_{min} = \left\{X \in \mathcal{P}^A_{b,cl,cv}(E) : \min(A, X) \textit{ is well-posed}\right\}$$

is a dense G_δ-set in $\mathcal{P}^A_{b,cl,cv}(E)$.

Theorem 4.8. *Let E be a complete Busemann convex metric space with the geodesic extension property and curvature bounded below globally by $\kappa < 0$. Suppose $A \in \mathcal{P}_{b,cl}(E)$. Then,*

$$\mathcal{W}_{max} = \{X \in \mathcal{P}_{b,cl,cv}(E) : \max(A, X) \textit{ is well-posed}\}$$

is a dense G_δ-set in $\mathcal{P}_{b,cl,cv}(E)$.

We conclude this subsection by giving a characterization of the well-posedness of the minimization problem $\min(X, Y)$ in complete CAT(0) spaces which shows that in the following particular context, the conditions in Proposition 4.1 can be relaxed.

Proposition 4.9. *Let E be a complete $CAT(0)$ space, $X \in \mathcal{P}_{b,cl,cv}(E)$ and $Y \in \mathcal{P}_{b,cl}(E)$. The problem $\min(X, Y)$ is well-posed if and only if*

$$\inf_{\sigma>0} \operatorname{diam}\left(L_{YX}(\sigma)\right) = 0.$$

4.2 Results involving compactness

In this subsection we study the same problems but we modify conditions we imposed in our results. More particularly, we focus on the situation in which the set A is compact. We show that under this stronger assumption on the set we can weaken the condition on the geodesic space from being of curvature bounded below globally to not having bifurcating geodesics. However, in the first theorem we need to add the reflexivity condition on the space. Before stating the theorem we give the following property of reflexive Busemann convex geodesic spaces.

Lemma 4.10. *Let (E, d) be a reflexive Busemann convex metric space. Then E is complete.*

Theorem 4.11. *Let E be a reflexive Busemann convex metric space with no bifurcating geodesics. Suppose $A \in \mathcal{P}_{cp}(E)$. Then,*

$$\mathcal{W}_{min} = \left\{X \in \mathcal{P}^A_{b,cl,cv}(E) : \min(A, X) \textit{ is well-posed}\right\}$$

is a dense G_δ-set in $\mathcal{P}^A_{b,cl,cv}(E)$.

The following is a particular case of the above result.

Corollary 4.12. *Let E be a complete $CAT(0)$ space with no bifurcating geodesics. Suppose $A \in \mathcal{P}_{cp}(E)$. Then,*

$$\mathcal{W}_{min} = \left\{X \in \mathcal{P}^A_{b,cl,cv}(E) : \min(A, X) \textit{ is well-posed}\right\}$$

is a dense G_δ-set in $\mathcal{P}^A_{b,cl,cv}(E)$.

Remark 4.13. The proof of Theorem 4.11 relies on the fact that $\min(A, X)$ always has a solution. In fact, the reflexivity of the space is mainly used to ensure this condition. Therefore, it is natural to ask whether it is possible to drop the condition that the problem has a solution.

Next we focus on the maximization problem for A compact. In order to follow the same line of argument as in the previous result we need the fact that the problem $\max(A, X)$ has a solution. However, in [22], it is proved that in a reflexive Banach space, the remotal distance from a point to a bounded, closed and convex set is guaranteed to be reached if and only if the space is finite dimensional. This is why it is natural to impose the compactness condition on the set X in our next result.

Theorem 4.14. *Let E be a complete geodesic space with no bifurcating geodesics and the geodesic extension property. Suppose $A \in \mathcal{P}_{cp}(E)$. Then,*

$$\mathcal{W}_{max} = \{X \in \mathcal{P}_{cp}(E) : \max(A, X) \textit{ is well-posed}\}$$

is a dense G_δ-set in $\mathcal{P}_{cp}(E)$.

Remark 4.15. Regarding the problem $\max(A, X)$, where the fixed set A is compact, we raise the following question: is

$$\mathcal{W}_{max} = \{X \in \mathcal{P}_{cp,cv}(E) : \max(A, X) \text{ is well-posed}\}$$

a dense G_δ-set in $\mathcal{P}_{cp,cv}(E)$? The Hopf-Rinow Theorem (see [1, Chapter I.3, Proposition 3.7]) states that if E is complete and locally compact, then it is proper. Hence, if the space is additionally locally compact and Busemann convex then we can answer the question in the positive by taking in the above proof the set $Y = \overline{\text{co}}\,(X \cup \{x'\})$, which is a compact and convex set.

5 The Drop Theorem in Busemann convex spaces

In [5], Daneš proved the following geometric result known as the Drop Theorem.

Theorem 5.1 (Drop Theorem)**.** *Let $(E, \|\cdot\|)$ be a Banach space and $A \in \mathcal{P}_{cl}(E)$ be such that* $\inf\{\|x\| : x \in A\} > 1$. *Then there exists $a \in A$ such that*

$$\text{co}\,(B(0,1) \cup \{a\}) \cap A = \{a\}.$$

The name of this theorem has its origin in the fact that the set $\text{co}\,(B(0,1) \cup \{a\})$ was called a *drop*. Equivalences of this result or of its generalized versions with other fundamental theorems in nonlinear analysis and various areas of their applications are discussed, for instance, in [12, 20].

In this section we give a variant of the Drop Theorem in the setting of Busemann convex metric spaces. We derive this result from the following theorem called the Strong Flower Petal Theorem. For a proof of this theorem see [12, Proposition 2.5]. This result uses the following extension of the definition of a petal given in [20]: having a metric space (E, d) and a function $f : E \to \mathbb{R}$, we say that the set

$$P_{\alpha,\delta}(x_0, f) = \{x \in E : f(x) \le f(x_0) - \alpha d(x, x_0) + \delta\}$$

is the *petal associated to* $\delta \ge 0$, $\alpha > 0$, $x_0 \in E$ and f.

Theorem 5.2 (Strong Flower Petal Theorem)**.** *Let (E,d) be a complete metric space, $A \in \mathcal{P}_{cl}(E)$ and $f : E \to \mathbb{R}$ a Lipschitz function bounded below on A. Suppose $\delta > 0$, $\alpha > 0$ and $x_0 \in A$. Then there exists a point $a \in A \cap P_{\alpha,\delta}(x_0, f)$ such that*

(i) $P_{\alpha,0}(a, f) \cap A = \{a\}$;

(ii) $x_n \to a$ *for every sequence* (x_n) *in* $P_{\alpha,0}(a, f)$ *with* $\operatorname{dist}(x_n, A) \to 0$.

The following is a variant of the Drop Theorem in Busemann convex geodesic spaces.

Theorem 5.3. *Let (E,d) be a complete Busemann convex metric space and let $A \in \mathcal{P}_{cl}(E)$ and $B \in \mathcal{P}_{b,cl,cv}(E)$ be such that $\lambda_{AB} > 0$. Suppose $\epsilon > 0$. Then there exists $a \in A$ such that*

(i) $\operatorname{dist}(a, B) < \lambda_{AB} + \epsilon$;

(ii) $\overline{\mathrm{co}}\,(B \cup \{a\}) \cap A = \{a\}$;

(iii) $x_n \to a$ *for every sequence* (x_n) *in* $\overline{\mathrm{co}}\,(B \cup \{a\})$ *with* $\operatorname{dist}(x_n, A) \to 0$.

As an application of this version of the Drop Theorem we obtain an analogue of an optimization result proved by Georgiev [12, Theorem 4.2] in the context of Banach spaces. In order to state this result we need to briefly introduce some notions which can also be found in [12].

Let (E,d) be a complete metric space, $f : E \to \mathbb{R}$ a lower semi-continuous function which is bounded below, and $A \in \mathcal{P}_{b,cl}(E)$. The *minimization problem* denoted by $\min(A, f)$ consists in finding $x_0 \in A$ (the solution of the problem) such that $f(x_0) = \inf\{f(x) : x \in A\}$.
For $\sigma > 0$, let

$$L_{A,f}(\sigma) = \left\{ x \in E : f(x) \leq \inf_{y \in A} f(y) + \sigma \text{ and } \operatorname{dist}(x, A) \leq \sigma \right\}.$$

The problem $\min(A, f)$ is *well-posed in the sense of Levitin-Polyak* (see [9, 17, 21]) if

$$\inf_{\sigma > 0} \operatorname{diam}\left(L_{A,f}(\sigma)\right) = 0.$$

This is equivalent to requesting that it has a unique solution $x_0 \in A$ and every sequence (x_n) in E converges to x_0 provided $f(x_n) \to f(x_0)$ and $\operatorname{dist}(x_n, A) \to 0$.

The following lemma is the counterpart of [12, Lemma 4.1] for geodesic metric spaces.

Lemma 5.4. *Let E be a geodesic space, $X \in \mathcal{P}_b(E)$ and $f : E \to \mathbb{R}$ continuous and convex. For $c \in \mathbb{R}$, let $A = \{x \in E : f(x) \leq c\}$. Suppose there exists $z \in E$ such that $f(z) < c$. Then for every $\epsilon > 0$ there exists $\delta > 0$ such that* $\operatorname{dist}(x, A) < \epsilon$ *for each $x \in X$ with $f(x) < c + \delta$.*

Before stating the optimization result we define, for $p \in \mathbb{N}$ and E a geodesic space, the set

$$\mathcal{L}_p = \left\{ X \in \mathcal{P}_{b,cl,cv}(E) : \inf_{\sigma > 0} \operatorname{diam}\left(L_{X,f}(\sigma)\right) < \frac{1}{p} \right\}.$$

Theorem 5.5. *Let E be a complete Busemann convex metric space and let $f : E \to \mathbb{R}$ be continuous, convex, bounded below on bounded sets and satisfying one of the following conditions:*

(i) $\inf_{x \in E} f(x) = -\infty$;

(ii) *there exists* $z_0 \in E$ *such that* $f(z_0) = \inf_{x\in E} f(x)$ *and every sequence* (x_n) *in* E *converges to* z_0 *if* $f(x_n) \to f(z_0)$.

Then,

$$\mathcal{W}_{min} = \{X \in \mathcal{P}_{b,cl,cv}(E) : \min(X, f) \text{ is well-posed in the sense of Levitin-Polyak}\}$$

is a dense G_δ*-set in* $\mathcal{P}_{b,cl,cv}(E)$.

Remark 5.6. If f is a continuous function, then the problem $\min(A, f)$ is well-posed in the sense of Levitin-Polyak if and only if it is well-posed in the sense of Hadamard (see [21] for definition and proof). Hence, in the above result we can substitute the well-posedness in the sense of Levitin-Polyak by the one in the sense of Hadamard.

Theorem 5.5 is not only interesting by itself, but it is also important because several best approximation results follow as simple consequences thereof. We finish our exposition by deriving such a consequence which is, in fact, an extension of a result proved in [6].

Corollary 5.7. *Let* E *be a complete Busemann convex metric space and suppose* $y \in E$. *Then,*

$$\mathcal{W}_{min} = \{X \in \mathcal{P}_{b,cl,cv}(E) : \min(y, X) \text{ is well-posed}\}$$

is a dense G_δ*-set in* $\mathcal{P}_{b,cl,cv}(E)$.

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