The Regularity of a Toric Variety.

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We give a method computing the degrees of the minimal syzygies of a toric variety by means of combinatorial techniques. Indeed, we complete the explicit description of the minimal free resolution of the semigroup algebra associated, using the simplicial representation of Koszul homology appeared in [8]. As an application, we obtain an algorithm computing the Castelnuovo-Mumford regularity of a projective toric variety. This regularity is explicity bounded by means of the semigroup generators which parametrize the variety.

Key Words: Toric varieties, syzygies, simplicial complexes, regularity.

INTRODUCTION

Let $S \subset \mathbf{Z}^h$ be a finitely generated commutative semigroup with zero element, such that $S \cap (-S) = \{0\}$. Let $\{n_1, \ldots, n_r\} \subset S$ be a set of generators for S. Let k be a field, let $k[S]$ be the semigroup $k-$algebra associated to S, and let $R = k[X_1, \ldots, X_r]$ be the polynomial ring in r variables. $k[S]$ is obviously an $S-$graded ring, and R is $S-$graded assigning the degree $\{n_i\}$ to X_i. Let $\mathbf{m}$ the irrelevant ideal of R. The $k-$algebra morphism,

$$\varphi : R \longrightarrow k[S],$$

defined by $\varphi(X_i) = \{n_i\}$, is an $S-$graded morphism of degree zero. Thus, the ideal $I_S = ker(\varphi)$ is an $S-$homogeneous ideal that it is called a Semigroup Ideal or a Toric Ideal because it defines a toric variety.

The condition $S \cap (-S) = \{0\}$ guarantees the $S-$graded Nakayama's lemma (Proposition 1.4 in [4]). Then, there exists a minimal $S-$graded free resolution of $k[S]$.

Let N_i be the corresponding i-syzygy module ($N_0 = I$) and consider the $k-$vector spaces

$$V_i(m) := \frac{(N_i)_m}{(\mathbf{m}N_i)_m}, m \in S.$$

By Nakayama's lemma:

• A minimal generating set of N_i consists exactly of $\dim_k(V_i(m))$ elements of degree m, for each m.

• The elements of degree m in a minimal generating set of N_i, correspond with a basis of $V_i(m)$.

In particular, since R is noetherian, one has $V_i(m) = 0$ for all m but finite many of values. It is well-known that there exist methods using Gröbner Bases (Schreiyer's Theorem) computing a minimal generating set of N_i (see for example [11]).

However, we are interested in to understand combinatorially these generating sets, and therefore the minimal $S-$graded free resolution of $k[S]$.

We introduced the following notation: Let $\Lambda := \{1, \dots, r\}$, and if $F \subset \Lambda$, $n_F = \sum_{i \in F} n_i$ ($n_\emptyset = 0$). If $m \in S$, the set $\Delta_m = \{F \subset \Lambda \mid m - n_F \in S\}$ is an abstract simplicial complex.

These simplicial complexes appear for the first time in the literature in [8]. They are a generalization of some graphs defined in [20]. Although, both works are inside the context of numerical semigroups, actually Δ_m appears in a lot of good papers inside more extensive contexts. For example, consider $\tilde{H}_i(\Delta_m)$ the $k-$vector space of the i-reduced homology. There exists an isomorphism

$$(*) \quad \tilde{H}_i(\Delta_m) \simeq V_i(m).$$

The existence of this isomorphism is proved in [8]. The generalization to a semigroup S with our initial conditions appears in [10]. A such isomorphism is explicity constructed in [5]. Other very interesting papers inside this line are [1], [7],[15]. The connection between free resolutions and simplicial complexes comes from [13] (see also [21] and [6]).

Continuing with our problem. Notice that the previous isomorphism provides the following construction to obtain a minimal generating set for N_i.

CONSTRUCTION:
STEP 1: Find the set $C_i := \{m \in S \mid \tilde{H}_i(\Delta_m) \neq 0\}$ of $S-$degrees for the minimal i-syzygies.
STEP 2: For any $m \in C_i$, take the images of the elements in a basis for the i-reduced homology space $\tilde{H}_i(\Delta_m)$ by the isomorphism.

Step 2 is solved with an algorithmic method in [5] (Remark 3.6). Thus, Construction will be an algorithm when one knows an effective way of computing the set C_i.

In particular, since the set C_0 is arithmetically explicited in [10], this construction provides an algorithm for $i = 0$ (Toric/Semigroup ideals) by means of Integer Programming, [4]. Using a degree bound for the elements in C_0, other alternative algorithm appears in [12].

In [9] new simplicial complexes associated to a partition of Λ allows the authors to deduce applications to concrete situations. However, they cannot compute the finite sets C_i in the general case. Indeed, they establish as the main problem in the computation of the minimal resolution to find finite sets C'_i containing C_i.

A finite set containing C_1 is effectively computed in [17]. Therefore, a new method computing the first syzygy module N_1 is obtained. Moreover, an explicit bound for the degree of the minimal first syzygies is given. This bound is described by means of the generators of the semigroups S.

In this paper, we generalize the results in [17] to all of higher order syzygies (section 1). Therefore, we obtain a new method computing the minimal generating set of the syzygy modules because we solve effectively Step 1 in Construction. The results in this paper allow us to say that Construction is an algorithm. In this way, we complete the combinatorial description of the minimal $S-$graded free resolution of $k[S]$ initiated in [8].

As an application, we consider the particular case of a projective toric variety (section 4). We obtain an algorithm computing its Castelnuovo-Mumford regularity. Without using this computation, we describe an explicit bound of the regularity by means of the semigroup generators which parametrize the variety.

The key idea to find finite sets C'_i containing C_i is the following:
For any $F \subset \Lambda$ and for any τ triangulation of F into i-dimensional faces, we consider a diophantine linear system. The set C'_i is obtained as a union

of subsets of the Hilbert bases associated to all of these systems. C'_i is finite because the Hilbert bases are finite (Dickson's lemma). C'_i can be computed in an effective way because there exist algorithms computing the Hilbert basis (see, for example, [16]).

We can conclude that: the noetherian property of R as well as Integer Programming guarantee the set C_i is finite, but Integer Programming provides also the way computing this finite set.

1. FINDING THE SET C_I

We consider, as in the introduction, $S =< n_1, \dots, n_r >\subset \mathbf{Z}^h$ a finitely generated semigroup with zero element such that $S \cap (-S) = \{0\}$, the set $\Lambda = \{1, \dots, r\}$, and for any $m \in S$ the simplicial complex

$$\Delta_m = \{F \subset \Lambda \ \mid \ m - n_F \in S\},$$

where $n_F = \sum_{i \in F} n_i$. Given a field k, $\tilde{H}_i(\Delta_m)$ is the $k-$vector space of the i-reduced homology. We are looking for a finite set C'_i containing the set

$$C_i := \{m \in S \ \mid \ \tilde{H}_i(\Delta_m) \neq 0\}.$$

Recall that from the isomorphism $(*)$ in the introduction, C_i consists exactly of the $S-$graded minimal i-syzygies for $k[S]$, the semigroup $k-$algebra associated of S.

Fix $m \in S$, choose an orientation on each face of Δ_m, and consider the augmented chain complex with values in the field k. Let $\tilde{C}_i(\Delta_m)$ be the $k-$vector space generated freely by the i-dimensional faces of Δ_m, where $\dim F = \sharp F - 1$ ($\dim \emptyset = -1$), and let $\delta_i : \tilde{C}_i(\Delta_m) \to \tilde{C}_{i-1}(\Delta_m)$ the $k-$linear mapping given by

$$\delta_i(F) = \sum_{F' \in \Delta_m, \dim F' = i-1} \epsilon_{FF'} F',$$

where $\epsilon_{FF'} = 0$ if $F' \not\subset F$, and $\epsilon_{FF'} = \pm 1$ if $F' \subset F$, $\epsilon_{FF'} = 1$ if the orientation induced by F on F' is equal to the orientation chosen on F', and $\epsilon_{FF'} = -1$ otherwise. We are interested in the link

$$\tilde{C}_{i+1}(\Delta_m) \stackrel{\delta_{i+1}}{\to} \tilde{C}_i(\Delta_m) \stackrel{\delta_i}{\to} \tilde{C}_{i-1}(\Delta_m)$$

with $i \geq 2$. Indeed, the tildes can be omitted. However we keep them because our results are true even if $i \geq 0$. Let $\tilde{Z}_i(\Delta_m) = ker(\delta_i)$ and $\tilde{B}_i(\Delta_m) = Im(\delta_{i+1})$ be the spaces of cycles and borders. Then, we have that

$$\tilde{H}_i(\Delta_m) = \tilde{Z}_i(\Delta_m)/\tilde{B}_i(\Delta_m).$$

LEMMA 1.1. *Suppose that* $\tilde{H}_i(\Delta_m) \neq 0$, *and let* $c \in \tilde{Z}_i(\Delta_m) - \tilde{B}_i(\Delta_m)$, $c = \sum_{j=1}^{t} \lambda_j F_j$, $\lambda_j \in k - \{0\}$ *for any* $j = 1, \ldots, t$, $F_j \neq F_l$ *if* $j \neq l$. *Then, if* $F = \bigcup_{j=1}^{t} F_j$ *one has that*

$$\forall p \in F \quad \exists q, 1 \leq q \leq t \ \mid \ p \notin F_q \ \textit{ and } \ F_q \cup \{p\} \notin \Delta_m.$$

Proof.

We begin proving that $\forall p \in F \quad \exists j, 1 \leq j \leq t$, such that $p \notin F_j$. Suppose that $p \in F$ and $p \in F_j$ for any $j, 1 \leq j \leq t$. Then, $c' = \sum_{j=1}^{t} \lambda_j \epsilon_{F_j F_j - \{p\}} (F_j - \{p\}) \in \tilde{C}_{i-1}(\Delta_m)$.

Notice that $c' = 0$ because $\delta_i(c) = 0$, and therefore $\lambda_j = 0$, $\forall j = 1, \ldots, t$. But it is not possible because $c \neq 0$.

Fix $p \in F$. We can suppose that $p \notin F_1$. If $F_1 \cup \{p\} \notin \Delta_m$ we have finished. Suppose then that $F_1 \cup \{p\} \in \Delta_m$. Set $l := \{j \mid p \notin F_j\}$ and $F_1' = F_1 \cup \{p\}$.

We consider $c_1 := c - \lambda_1 \epsilon_{F_1' F_1} \delta_{i+1}(F_1')$. Notice that $c_1 \in \tilde{Z}_i(\Delta_m) - \tilde{B}_i(\Delta_m)$. Moreover,

$$c_1 = \sum_{j=2}^{t} \lambda_j F_j + \sum_{dim(F')=i, F' \neq F_1} \epsilon_{F_1 F'} F'.$$

Set $c_1 = \sum_{j=1}^{t^{(1)}} \lambda_j^{(1)} F_j^{(1)}$, where $\lambda_j^{(1)} \in k - \{0\}$ for any $j = 1, \ldots, t^{(1)}$, $F_j^{(1)} \neq F_l^{(1)}$ if $j \neq l$, $l^{(1)} := \sharp\{j \mid p \notin F_j^{(1)}\}$, and $F^{(1)} := \bigcup_{j=1}^{t^{(1)}} F_j^{(1)}$. The following properties are satisfied

$(*)_1$: $\sharp F^{(1)} \leq F$.

$(**)_1$: If $p \notin F_j^{(1)}$, then there exists q, $2 \leq q \leq t$ such that $F_j^{(1)} = F_q$.

$(***)_1$: $l^{(1)} < l$.

Indeed, we replace the face F_1 in c with $p \notin F_1$, by several faces $F' \neq F_1$ with $p \in F'$ to construct c_1.

We proceed by induction on $\sharp F$.

If $\sharp F$ is minimal, in $(*)_1$ the equality holds. Therefore, $p \in F^{(1)}$ and $l^{(1)} \neq 0$. Using the similar arguments with c_1, there exists j, $1 \leq j \leq t^{(1)}$, such that $p \notin F_j^{(1)}$. By $(**)_1$, $F_j^{(1)} = F_q$ for some q, $2 \leq q \leq t$. If $F_j^{(1)} \cup \{p\} \notin \Delta_m$, we have finished. But, if $F_j^{(1)} \cup \{p\} \in \Delta_m$, we can obtain $c_2 \in \tilde{Z}_i(\Delta_m) - \tilde{B}_i(\Delta_m)$ from c_1, similarly to the construction of c_1 from c. Set $c_2 = \sum_{j=1}^{t^{(2)}} \lambda_j^{(2)} F_j^{(2)}$ where $\lambda_j^{(2)} \in k - \{0\}$ for any $j = 1, \ldots, t^{(2)}$, $F_j^{(2)} \neq F_l^{(2)}$ if $j \neq l$. Then, if $l^{(2)} := \sharp\{j \mid p \notin F_j^{(2)}\}$, and $F^{(2)} := \bigcup_{j=1}^{t^{(2)}} F_j^{(2)}$ we obtain the properties:

$(*)_2$: $\sharp F^{(2)} \leq F^{(1)}$.

$(**)_2$: If $p \notin F_j^{(2)}$, then there exists q, $2 \leq q \leq t^{(1)}$ such that $F_j^{(2)} = F_q^{(1)}$.

$(***)_2$: $l^{(2)} < l^{(1)}$.

Again, in $(*)_2$ the equality holds because $\sharp F$ is minimal. Therefore $p \in F^{(2)}$ and $l^{(2)} \neq 0$. Now, our result follows by recurrence because the properties $(***)$ guarantees that this process must finish in a finite number of steps.

Suppose then our result is true for any $c' \in \tilde{Z}_i(\Delta_m) - \tilde{B}_i(\Delta_m)$, $c' = \sum_{j=1}^{t'} \lambda'_j F'_j$ where $\lambda'_j \in k - \{0\}$ for any $j = 1, \ldots, t'$, $F'_j \neq F'_l$ if $j \neq l$, with $\sharp F' < \sharp F$ where $F' = \bigcup_{j=1}^{t'} F'_j$. Then:

If in $(*)_1$ the equality does not hold, it is enough to apply the induction hypothesis to c_1. We obtain that there exists $F_j^{(1)}$ such that $p \notin F_j^{(1)}$ and $F_j^{(1)} \cup \{p\} \notin \Delta_m$. By $(**)_1$, $F_j^{(1)} = F_q$ with $2 \leq q \leq t$, and we have finished.

If in $(*)_1$ the equality holds, we repeat the same argument that in the case $\sharp F$ minimal. If we don't finish, we construct c_2 as before. Again, we obtain the result by induction if in $(*)_2$ the equality does not hold. Otherwise, we begin the process. We finish in a finite number of steps by the properties $(***)$ which are obtained in the recurrence. Now, our result is proved. ∎

Lemma 1.1 associates to $m \in S$ with $\tilde{H}_i(\Delta_m) \neq 0$ sets $F \subset \Lambda$ and $F_1, \ldots, F_t \in \Delta_m$ such that:

1. $F = \bigcup_{j=1}^{t} F_j$
2. $\dim(F_j) = i$, $\forall j = 1, \ldots, t$
3. $F \notin \Delta_m$

Notice that $\tau = \{F_1, \ldots, F_t\}$ is a triangulation of F into i-dimensional faces of Δ_m. This suggests the following definition.

Definition 1.1. Let $F \subset \Lambda$. We say that $\tau = \{F_1, \ldots, F_t\}$ is an i-triangulation of F if the following properties are satisfies:

1. $\dim(F_j) = i$, $\forall j = 1, \ldots, t$
2. $F = \bigcup_{j=1}^{t} F_j$

We say that τ is an i-triangulation of F in Δ_m, with $m \in S$, if $F_j \in \Delta_m$, $\forall j = 1, \dots, t$, and $F \notin \Delta_m$.

Lemma 1.2. *If $\tilde{H}_i(\Delta_m) \neq 0$, there exists $F \subset \Lambda$ and τ i-triangulation of F in Δ_m.*

Proof.

It is enough to take $c \in \tilde{Z}_i(\Delta_m) - \tilde{B}_i(\Delta_m)$ and $F, F_1, \dots, F_t$ as in Lemma 1.1. ∎

Remark 1. 1.

It is clear that given $m \in S$ with $\tilde{H}_i(\Delta_m) \neq 0$, F is not unique in general. Moreover, for a fixed F, the i-triangulation τ is not unique in general.

Suppose that $\tau = \{F_1, \dots, F_t\}$ is an i-triangulation of F in Δ_m. Notice that the property $F_1 \in \Delta_m$ is equivalent to $m - n_{F_1} \in S$. This means that there exists $\alpha_j \in \mathbf{N}$ such that $m = \sum_{j=1}^r \alpha_j n_j$, with $\alpha_j \geq 1$ for any $j \in F_1$.

Set $\mathcal{A}$ the matrix whose column vectors are the generators of S, $\mathcal{A} := (n_1 | \dots | n_r) \in \mathcal{M}_{h \times r}(\mathbf{Z})$.

Set $e_{F_1} \in \mathbf{N}^r$ the vector with coordinates equal to zero, excepting the jth one which is equal to one, for any $j \in F_1$.

Then, $F_1 \in \Delta_m$ if and only if there exists $\alpha^{(1)} \in \mathbf{N}^r$ such that $\mathcal{A}\alpha^{(1)} = m$, and $\alpha^{(1)} \gg e_{F_1}$, where the symbol $\gg$ stands for the natural partial order in $\mathbf{N}^r$.

With similar notations we can obtain an analogous condition for $F_j \in \Delta_m$. This is: for any $j, 1 \leq j \leq t$, $F_j \in \Delta_m$ if and only if there exists $\alpha^{(j)} \in \mathbf{N}^r$ such that $\mathcal{A}\alpha^{(j)} = m$, and $\alpha^{(j)} \gg e_{F_j}$.

Set

$$\mathcal{A}(t) := \begin{pmatrix} \mathcal{A} & -\mathcal{A} & 0 & 0 & 0 & & 0 & 0 \\ 0 & \mathcal{A} & -\mathcal{A} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \mathcal{A} & -\mathcal{A} & 0 & & 0 & 0 \\ & & \ddots & \ddots & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & & \mathcal{A} & -\mathcal{A} \end{pmatrix} \in \mathcal{M}_{h(t-1) \times rt}(\mathbf{Z})$$

and $e_\tau := (e_{F_1}, \dots, e_{F_t}) \in \mathbf{N}^{rt}$.

Then, from τ, an i-triangulation of F in Δ_m, we can obtain a vector $\alpha = (\alpha^{(1)}, \dots, \alpha^{(t)}) \in \mathbf{N}^{rt}$ which satisfies:

1. $\mathcal{A}(t)\alpha = 0$.
2. $\alpha \gg e_\tau$

3. $m = \mathcal{A}\alpha^{(1)} = \ldots = \mathcal{A}\alpha^{(t)}$.

But we cannot forget that we are looking for the $m \in S$ such that $\tilde{H}_i(\Delta_m) \neq 0$. Then, we must proceed on the contrary.

Let $F \subset \Lambda$ and $\tau = \{F_1, \ldots, F_t\}$ an i-triangulation of F. Now, m is not fixed. We are going to look for the elements $m \in S$ such that τ is an i-triangulation of F in Δ_m.

As before, we consider $e_\tau := (e_{F_1}, \ldots, e_{F_t}) \in \mathbf{N}^{rt}$. Set

$$R_\tau := \{\alpha = (\alpha^{(1)}, \ldots, \alpha^{(t)}) \in \mathbf{N}^{rt} \ \mid \ \mathcal{A}(t)\alpha = 0 \ \ , \alpha \gg e_\tau\}.$$

Notice that if $\alpha \in R_\tau$ and it is written $\alpha = (\alpha^{(1)}, \ldots, \alpha^{(t)})$ with $\alpha^{(j)} \in \mathbf{N}^r$, for any $j, 1 \leq j \leq t$, it is obtained $\mathcal{A}\alpha^{(1)} = \ldots = \mathcal{A}\alpha^{(t)} = m \in S$.

Set

$$\Sigma R_\tau := \{m \in S \ \mid \ m = \mathcal{A}\alpha^{(1)}, \alpha = (\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(t)}) \in R_\tau\}.$$

At the moment it is also known that if $m \in S$ is such that $\tilde{H}_i(\Delta_m) \neq 0$, using Lemma 1.1 we find F and τ as before such that $m \in \Sigma R_\tau$. Then, we obtain that

$$C_i = \bigcup_F \bigcup_\tau \Sigma R_\tau,$$

where the union is over all of τ i-triangulation of F, and $F \subset \Lambda$ with $\sharp F \geq i + 2$. (Notice that F comes from Lemma 1.1 and then $\sharp F \geq i + 2$).

The set ΣR_τ is not finite in general. Therefore, we have yet not found our set C_i'. However, R_τ, $\mathcal{H}R_\tau := \{\alpha \in R_\tau \ \mid \ \alpha \text{ is minimal for } \ll\}$ is finite. We are going to prove that C_i is contained in a union as before, but of subsets of

$$\Sigma\mathcal{H}R_\tau := \{m \in S \ \mid \ m = \mathcal{A}\alpha^{(1)}, \alpha = (\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(t)}) \in \mathcal{H}R_\tau\}.$$

By this way, we construct our desired finite set C_i'. We need to introduce some new notation. On the elements of S, we define a partial order $\geq_S$

$$m \geq_S m' \Leftrightarrow m - m' \in S$$

If $H \subset S$, we shall say that $m \in H$ is $S-$minimal in H if $m \geq_S m'$ with $m' \in H$, implies that $m = m'$. Set

$$C_\tau := \{m \in S \ \mid \ m \text{ is } S\text{−minimal in } \Sigma R_\tau\}.$$

Lemma 1.3. *In the conditions as above, for any $m \in \Sigma R_\tau$, there exists $m' \in C_\tau$ and $m'' \in S$ such that $m = m' + m''$.*

Proof.
If $m \in C_\tau$, it is enough to take $m' = m$ and $m'' = 0$. Otherwise, there exists $m'_1 \in \Sigma R_\tau$ such that $m \geq_S m'_1$. Then, $m = m'_1 + m''_1$, with $m''_1 \in S$. If $m'_1 \in C_\tau$ we have finished. Otherwise, we begin the reasoning with m'_1. By recurrence we construct a sequence of elements $m'_{j-1} = m'_j + m''_j$, where $m'_j \in \Sigma R_\tau$ and $m''_j \in S$.

The condition $S \cap (-S) = \{0\}$ guarantees that the number of different expressions of m as sum of non null elements in S is finite (Proposition 1.2 in [4]). Then, our process must finish.

Suppose that it finishes in the jth step. Then, $m'_j \in C_\tau$. Now $m = m'_j + m''_1 + \ldots + m''_j$, hence it is enough take $m' = m'_j$ and $m'' = m''_1 + \ldots + m''_j$. ∎

Lemma 1.4. *In the conditions as above, $C_\tau \subset \Sigma\mathcal{H}R_\tau$. Therefore, the set C_τ is finite.*

Proof.
Let $m \in C_\tau$. Then, $m = \mathcal{A}\alpha^{(1)}$ with $\alpha = (\alpha^{(1)}, \ldots, \alpha^{(t)}) \in R_\tau$. If $\alpha \in \mathcal{H}R_\tau$, we have finished. Suppose that $\alpha \notin \mathcal{H}R_\tau$. Then $\alpha = \alpha' + \alpha''$, with $\alpha' \in \mathcal{H}R_\tau$ and $\alpha'' \in \mathbf{N}^{rt}$. Moreover, $\alpha'' = \alpha - \alpha'$ satisfies that $\mathcal{A}(t)\alpha'' = 0$. Then, if $m' = \mathcal{A}\alpha'^{(1)}, m' \in \Sigma\mathcal{H}R_\tau$. By $m - m' = m'' = \mathcal{A}\alpha''^{(1)} \in S$ and m is $S-$minimal, we obtain that $m = m' \in \Sigma\mathcal{H}R_\tau$. ∎

Proposition 1.1. *If $m \in S$ and $\tilde{H}_i(\Delta_m) \neq 0$, then there exists $F \subset \Lambda$ with $\sharp F \geq i+2$, and there exists τ i-triangulation of F in Δ_m such that $m \in C_\tau$.*

Proof.
If $\tilde{H}_i(\Delta_m) \neq 0$, let $c \in \tilde{Z}_i(\Delta_m) - \tilde{B}_i(\Delta_m)$, $c = \sum_{j=1}^t \lambda_j F_j$, $\lambda_j \in k - \{0\}$ for any $j = 1, \ldots, t$, $F_j \neq F_l$ if $j \neq l$. Then, if $F = \cup_{j=1}^t F_j$ and $\tau = \{F_1, \ldots, F_t\}$, we have that τ is an i-triangulation of F in Δ_m and $m \in \Sigma R_\tau$.

By Lemma 1.3 $m = m' + m''$ with $m' \in C_\tau$ and $m'' \in S$. If $m'' = 0$, then $m = m' \in C_\tau$ and we have finished.

Suppose that $m'' \neq 0$. Let $m'' = \sum_{j=1}^r \beta_j n_j$ with $\beta_j \in \mathbf{N}$ for any $j, 1 \leq j \leq r$. We can suppose that $\beta_1 \neq 0$. If $1 \in F$, we apply lemma 1.1 for $p = 1$. Then, there exists $q, 1 \leq q \leq t$, such that $1 \notin F_q$ and $F_q \cup \{1\} \notin \Delta_m$. However, $m - n_{F_q} - n_1 \in S$ because $F_q \in \Delta_m$, $\beta_1 \neq 0$ and $1 \notin F_q$. This is a contradiction with $F_q \cup \{1\} \notin \Delta_m$. Therefore, $1 \notin F$.

By $1 \notin F$ we obtain that $m - n_{F_j} - n_1 \in S$, for any j. But it is not possible because then,

$$c' = \sum_{j=1}^{t} \lambda_j (F_j \cup \{1\}) \in \tilde{C}_{i+1}(\Delta_m)$$

satisfies that $\delta_{i+1} c' = c$, but $c \notin \tilde{B}_i(\Delta_m)$. Therefore $m'' = 0$, and our result is proved. ▮

By Proposition 1.1 $C_i \subset \bigcup_F \bigcup_\tau C_\tau$, where $F \subset \Lambda$, $\sharp F \geq i+2$, and τ is an i-triangulation of F. Then, we have found a finite set containing C_i because C_τ is finite (Lemma 1.4). We could call this set C'_i, however we can obtain a smaller subset of this one containing C_i. Notice that if $m \in C_\tau$ with $\tau = \{F_1, \ldots, F_t\}$ i-triangulation of F, it is guaranteed that $F_j \in \Delta_m$, for any $j, 1 \leq j \leq t$, but it is not that $F \notin \Delta_m$. Set

$$C'_\tau := \{m \in C_\tau \mid F \notin \Delta_m\},$$

and

$$C_i(F) := \bigcup_\tau C'_\tau,$$

where the union is over the i-triangualtion of F. Then, we obtain the following theorem.

THEOREM 1.1. *The set of the minimal i-syzygy $S-$degrees for $k[S]$, C_i, is contained in the finite set*

$$C'_i := \bigcup_{F \subset \Lambda, \sharp F \geq i+2} C_i(F).$$

Remark 1. 2.
1) The set C'_i can be obtained in an effective way by Integer Programming methods. Therefore, Theorem 1.1 solves the Step 1 of Construction in the introduction. Moreover, Construction is now an algorithm computing a minimal generating set of N_i, the i-syzygy module of $k[S]$.
2) Although we are considering $S \subset \mathbf{Z}^h$ all our previous results are true even if

$$S \subset \mathbf{Z}^h \oplus \mathbf{Z}/a_1\mathbf{Z} \oplus \ldots \oplus \mathbf{Z}/a_s\mathbf{Z}$$

with $a_j \in \mathbf{Z}$, $1 \leq j \leq s$.

This is equivalent to consider a finitely generated commutative cancellative semigroup. The new in this case is that the diophantine linear systems that appear, have congruences. In section 1 of [17] the necessary background about this type of systems is considered.

3) Our set C_i' is different of C_i in general.

2. COMPARISON WITH FORMER RESULTS

Now, we are going to compare Theorem 1.1 with the former results in the literature.

The case $i = 0$ is the more extensively developed. Now, $N_0 = I$ a Toric/Semigroup Ideal, and $C_0 = \{m \in S \ \mid \ \Delta_m$ is non-connected $\}$, because $\tilde{H}_0(\Delta_m)$ is the reduced homology. Indeed, the set C_0 is arithmetically explicited in [10]. Here, the elements in C_0 is characterized by three aritmethical conditions (Theorem 1 in [10]). This characterization is reformulated in [4] with the introduction of new combinatorial elements, ladders associated to some diophantine liner systems. By this way, Integer Programming solves the computation of C_0 and, at the same time, a minimal generating set of I in constructed. (Algorithm 5.1 in [4]). A nice combinatorial description of these generating sets is obtained. However, as algorithm, this method needs to solve diophantine linear systems for any $C \subset \Lambda$ with $\sharp C \leq r/2$ (notice that if $\tilde{H}_0(\Delta_m) \neq 0$ there exists a connected component $C \in \Delta_m$ with $\sharp C \leq r/2$). Hence, the algorithm is not faster than the methods using Gröbner Bases given in [3], and in [14], or if the semigroup has torsion non trivial in [23].

Theorem 1.1 in the case $i = 0$ can be improved. In fact, we can reduced the union to the sets $F \subset \Lambda$ with $\sharp F = 2$. Notice that if $\tilde{H}_0(\Delta_m) \neq 0$, it is enough to take F with two points in Δ_m in different connected components. Moreover, there exists an unique 0-triangulation of F in Δ_m. But even with this reduction, the subset C_0' obtained can be bigger than C_0.

In [9] a method computing an alternative C_0' appears. However this method can only be applied to a subclass of semigroup S. This subclass includes the simplicial semigroups, but is strictly contained in our class.

The case $i = 1$ is solved in [17]. Indeed, the techniques and the reasoning scheme used in this paper are the same ones employed in [17]. However, there is a crucial difference. We are going to explain where this difference is.

Suppose $m \in S$, $F = \{i_1, \ldots, i_t\} \subset \Lambda$, $t \geq 3$. In [17] a polygon σ whose vertex set is F is called an $F-$cavity of Δ_m if the following conditions are satisfied:

1. $F_j \in \Delta_m$, $\forall j = 1, \dots, t$, where $F_j := \{i_j, i_{j+1}\}$, $\forall j = 1, \dots, t-1$, and $F_t := \{i_t, i_1\}$, are the faces of σ.
2. If $F_j \neq F' \subset F$, $\sharp F' \geq 2$, then $F' \notin \Delta_m$.

Notice that an $F-$cavity defines a special type of 1-triangulation of F in Δ_m. By the condition 1, $\{F_1, \dots, F_t\}$ is an 1-triangulation of F, with $F_j \in \Delta_m$, $\forall j$, and by condition 2, $F \notin \Delta_m$.

In [17] it is proved that if $\tilde{H}_1(\Delta_m) \neq 0$, then there exists an $F-$cavity of Δ_m satisfying

$$c := \sum_{j=1}^{t} \epsilon_j F_j \in \tilde{Z}_1(\Delta_m) - \tilde{B}_1(\Delta_m),$$

for some $\epsilon_j = \pm 1$, $\forall j = 1, \dots, t$. Therefore, following our reasoning, we can only consider τ 1-triangulations of F which correspond to $F-$cavities. These 1-triangulations have all the same shape, a polygon with $\sharp F$ vertices. Moreover, the condition 2 allows to us to consider a smaller matrix than $\mathcal{A}(t)$. In fact, set $\mathcal{A}_{F_j} \in \mathcal{M}_{h \times (r-t+2)}(\mathbf{Z})$ the matrix whose columns are the semigroup generators corresponding to $(\Lambda - F) \cup F_j$, $\forall j = 1, \dots, t$. The matrix $\mathcal{A}(t)$ can be replaced by the matrix

$$\begin{pmatrix} \mathcal{A}_{F_1} & -\mathcal{A}_{F_2} & 0 & 0 & 0 & & 0 & 0 \\ 0 & \mathcal{A}_{F_2} & -\mathcal{A}_{F_3} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \mathcal{A}_{F_3} & -\mathcal{A}_{F_4} & 0 & & 0 & 0 \\ & & \ddots & \ddots & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & & \mathcal{A}_{F_{t-1}} & -\mathcal{A}_{F_t} \end{pmatrix} \in \mathcal{M}_{h(t-1) \times (r-t+2)t}(\mathbf{Z}).$$

Also the condition 2 yields another reduction. The set $C'_\tau = \{m \in C_\tau \mid F \notin \Delta_m\}$ can be replaced by the smaller subset

$$\{m \in C_\tau \mid F' \notin \Delta_m, \text{ for any } F' \neq F_j, F' \subset F, \sharp F' \geq 2\}.$$

With these changes, our method provides a finite subset containing C_1 which is the same one giving in [17].

There is not any former result for $i \geq 2$. In the case $i = 2$, looking for a generalization of an $F-$cavity in Δ_m the following difficulties appear:
<u>Difficulty 1.</u> The shape of an $F-$cavity cannot be the same. Fix, for example, $F = \{1, \dots, 9\}$ and consider two 2-triangulations whose shape are different (Figure 1 and Figure 2)
<u>Difficulty 2.</u> The condition 2 for $F-$cavity does not hold if $i = 2$. It is possible that $\exists F' \subset F$, $F' \neq F_j$, $\sharp F' < t$ such that $F' \in \Delta_m$. This situation is verified by Figure 3 and Figure 4.

$$\begin{array}{cc} F_1 = \{1,2,8\} & F_2 = \{1,2,9\} \\ F_3 = \{2,3,8\} & F_4 = \{2,3,9\} \\ F_5 = \{3,4,8\} & F_6 = \{3,4,9\} \\ F_7 = \{4,5,8\} & F_8 = \{4,5,9\} \\ F_9 = \{5,6,8\} & F_{10} = \{5,6,9\} \\ F_{11} = \{6,7,8\} & F_{12} = \{6,7,9\} \\ F_{13} = \{7,1,8\} & F_{14} = \{7,1,9\} \\ \multicolumn{2}{c}{\tau = \{F_1, \cdots, F_{14}\}} \end{array}$$

FIG. 1.

$$\begin{array}{ccc} F_1 = \{1,3,4\} & F_2 = \{3,4,6\} & F_3 = \{3,6,2\} \\ F_4 = \{2,6,8\} & F_5 = \{2,8,1\} & F_6 = \{1,8,4\} \\ F_7 = \{4,6,5\} & F_8 = \{6,5,7\} & F_9 = \{6,7,8\} \\ F_{10} = \{7,8,9\} & F_{11} = \{8,9,4\} & F_{12} = \{9,4,5\} \\ F_{13} = \{5,1,7\} & F_{14} = \{1,7,3\} & F_{15} = \{7,3,9\} \\ F_{16} = \{3,9,2\} & F_{17} = \{9,2,5\} & F_{18} = \{2,5,1\} \end{array}$$

$$\tau = \{F_1, \cdots, F_{18}\}$$

FIG. 2. A Torus

FIG. 3. $F' = \{1, 2, 4\} \in \Delta_m$

FIG. 4. $F' = \{2, 3, 5, 6\} \in \Delta_m$

For these reasons, we have generalized the concept of $F-$cavity for $i = 1$ to the concept of i-triangulation of F in Δ_m. Lemma 1.1 is the key to apply the same reasoning scheme used in [17] to solve the case $i = 2$ as well as the case $i \geq 2$.

We can conclude then that Theorem 1.1 is a new result for $i \geq 2$ and a generalization of the former results for $i = 0, 1$.

Now, we are going to analyze some applications and consequences of Theorem 1.1.

Begin analyzing Remark 1.2.1. Integer Programming methods compute the finite set C'_i in Theorem 1.1 and the simplicial complexes Δ_m with $m \in C'_i$. Then, it is possible to check the elements $m \in C'_i$ such that

$\tilde{H}_i(\Delta_m) \neq 0$ using Linear Algebra. By this way, the set C_i is effectively obtained and a basis for the $k-$vector space $\tilde{H}_i(\Delta_m)$ is computed, for any $m \in C_i$. Now, applying the isomorphism $\tilde{H}_i(\Delta_m) \simeq V_i(m)$ explicited in Remark 3.6 of [5], a basis for $V_i(m)$ is obtained. By Nakayama's Lemma, the union of these bases for any $m \in C_i$ yields a minimal generating set of N_i.

Then, we can say that Theorem 1.1 provides an algorithm computing a minimal generating set of N_i. Moreover, the minimal resolution of $k[S]$ is obtained by recursively applying this algorithm.

It is instructive to construct the minimal resolution of $k[S]$ using the simplicial complexes Δ_m. However, this method cannot compete in speed with the method using Gröbner Bases (Schreiyer's Theorem). This is not surprising. In the case $i = 0$ ([4]) and in the case $i = 1$ ([17]), the situation was analogous. The main drawback is that it is necessary to solve linear diophantine systems for any τ i-triangulation of F, for any $F \subset \Lambda$ with $\sharp F \geq i + 2$. In spite of this, our view point is interesting because it allows to obtain an explicit bound for the degree of the minimal i-syzygies. To find this bound we need to introduced new notation in the following section.

3. DEGREE BOUNDS

Notation 3.1.
- *If* $x \in \mathbf{Z}^r$*,* $||x||_1 := \sum_{j=1}^{r} |x_j|$.
- *If* $L \subset \mathbf{N}^r$ $\mathcal{H}L := \{x \in L - \{0\} \mid x$ *is minimal for* $\ll\}$ *if* $L \neq \{0\}$ *and* $\mathcal{H}\{0\} := \{0\}$.
- *If* A *is an integer matrix*

$$||A||_{1,\infty} := sup_l \sum_j |a_{lj}|$$

$$\mathcal{H}(A) := \mathcal{H}\{x \in \mathbf{N}^r \quad | \quad Ax = 0\} \textit{(Hilbert basis of the system).}$$

Remark 3. 1. It is known that if $x \in \mathcal{H}(A)$ then $||x||_1 \leq (1 + ||A||_{1,\infty})^s$, where $s = rank(A)$. (Theorem 1 in [19], see also [18]).

In order to obtain our bound using Remark 3.1, we need to associate to an element $m \in C_i$ a homogeneous linear diophantine system.

Fix $m \in C_i$. Applying Proposition 1.1, there exists $F \subset \Lambda$, with $\sharp F \geq i + 2$, and there exists τ i-triangulation of F in Δ_m, such that $m \in C_\tau$. By Lemma 1.4, $m \in \Sigma\mathcal{H}R_\tau$. Then, there exists $\alpha \in \mathcal{H}R_\tau$ such that $m = \mathcal{A}\alpha^{(1)}$, where $\alpha = (\alpha^{(1)}, \ldots, \alpha^{(t)})$.

Notice that $\alpha \in \mathcal{H}R_\tau$ if and only if

$$\alpha - e_\tau \in \mathcal{H}\{\beta \in \mathbf{N}^{rt} \mid \mathcal{A}(t)(\beta + e_\tau) = 0\}.$$

Set

$$\mathcal{A}_\tau := (\mathcal{A}(t)| - \mathcal{A}(t)e_\tau).$$

We are looking for the relation with Hilbert basis of the homogeneous systems

$$\mathcal{A}_\tau \beta' = 0, \beta' \in \mathbf{N}^{rt+1}.$$

It is easy to verify that

$$\mathcal{H}\{\beta \in \mathbf{N}^{rt} \mid \mathcal{A}(t)\beta = -\mathcal{A}(t)e_\tau\} = \{\beta \in \mathbf{N}^{rt} \mid (\beta, 1) \in \mathcal{H}(\mathcal{A}_\tau)\}.$$

LEMMA 3.1. *With the notation and conditions as before, the following inequality is satisfied*

$$||(\alpha - e_\tau, 1)||_1 \leq (1 + 4||\mathcal{A}||_{1,\infty})^{h(d_i - 1)},$$

being $d_i := \binom{r}{i+1}$.

Proof.
Using the above reasoning we obtain that $(\alpha - e_\tau, 1) \in \mathcal{H}(\mathcal{A}_\tau)$. Then, by Remark 3.1

$$||(\alpha - e_\tau, 1)||_1 \leq (1 + ||\mathcal{A}_\tau||_{1,\infty})^s,$$

where $s = rank(\mathcal{A}_\tau)$.
Recall that the matrix $\mathcal{A}_\tau$ has $h(t-1)$ columns, where $\sharp\tau = t$, $\tau = \{F_1, \ldots, F_t\}$, $F_j \subset F$, and $\sharp F_j = i+1$, for any j. Then $t = \sharp\tau \leq \binom{\sharp F}{i+1} \leq \binom{r}{i+1} = d_i$, and rank$\mathcal{A}_\tau \leq h(t-1) \leq h(d_i - 1)$.
Finally, let see us that $||\mathcal{A}_\tau||_{1,\infty} \leq 4||\mathcal{A}||_{1,\infty}$.
It is enough to notice that

$$||\mathcal{A}||_{1,\infty} = max_{1 \leq j \leq t-1}\{||(\mathcal{A}(2)| - \mathcal{A}(2)(e_{F_j}, e_{F_{j+1}}))||_{1,\infty}\}$$

and

$$||(\mathcal{A}(2)| - \mathcal{A}(2)(e_{F_j}, e_{F_{j+1}}))||_{1,\infty} =$$

$$max_{1\leq p\leq h}\{\sum_{q=1}^{r}|a_{pq}|+\sum_{q=1}^{r}|-a_{pq}|+\sum_{q\in F_j}|a_{pq}|+\sum_{q\in F_{j+1}}|-a_{pq}|\}\leq 4||\mathcal{A}||_{1,\infty}$$

▮

THEOREM 3.2. *If $m \in S$ is an $S-$degree of a minimal i-syzygy for $k[S]$, then $m = \mathcal{A}x$ with $x \in \mathbf{N}^r$ such that*

$$||x||_1 \leq (1+4||\mathcal{A}||_{1,\infty})^{h(d_i-1)}+(i+1)d_i-1,$$

where $d_i = \begin{pmatrix} r \\ i+1 \end{pmatrix}$.

Proof.
With the notation in Lemma 3.1, $m = \mathcal{A}\alpha^{(1)}$ with $\alpha = (\alpha^{(1)}, \ldots, \alpha^{(t)})$ satisfying this lemma. Now, it is enough to notice that

$$||\alpha^{(1)}||_1 \leq ||\alpha||_1 = ||\alpha-e_\tau+e_\tau||_1 \leq ||\alpha-e_\tau||_1+||e_\tau||_1 = ||(\alpha-e_\tau,1)||_1-1+(i+1)t.$$

By Lemma 3.1, $||\alpha^{(1)}||_1 \leq (1+4||\mathcal{A}||_{1,\infty})^{h(d_i-1)}+(i+1)d_i-1$. ▮

We are going to see how it is possible to improve Theorem 3.2 in the particular cases $i = 0, 1$.

Suppose $i = 0$. It is well-known that any minimal generating set for the ideal I (the 0-syzygies) is contained in the Graver basis of $\mathcal{A}$, $Gr_{\mathcal{A}}$

$$Gr_{\mathcal{A}} := \{X^\alpha - X^\beta \;\; | \;\; (\alpha,\beta) \in \mathcal{H}(\mathcal{A}(2))\}$$

(see [22] for details).

Then, if $m \in S$ is a minimal degree of I, we obtain that $m = \mathcal{A}\alpha = \mathcal{A}\beta$, with $(\alpha,\beta) \in \mathcal{H}(\mathcal{A}(2))$. Using again Remark 3.1 we have that

$$||\alpha||_1 \leq (1+2||\mathcal{A}||_{1,\infty})^h$$

because $||\mathcal{A}(2)||_{1,\infty} \leq 2||\mathcal{A}||_{1,\infty}$.

This is clearly an improvement of Theorem 3.2 for the case $i = 0$.

Suppose $i = 1$. As we have seen before, we can reduce to a special type of 1-triangulations of F in Δ_m, the F-cavities. All of them have the same shape, a polygon with $\sharp F$ vertices. Thus, following the steps in the proof of Theorem 3.2, we obtain that

$$||x||_1 \leq (1 + 4||\mathcal{A}||_{1,\infty})^{h(r-1)} + 2r - 1$$

because $\sharp F \leq r$. The improvement now is that $d_1 = \begin{pmatrix} r \\ 2 \end{pmatrix} > r$ because $r \geq 3$.

In [17] a similar result appears. Even forgetting the torsion, there exists a small difference. In [17] the number

$$D := max\{||\mathcal{A}_\tau||_{1,\infty} \ \mid \ \tau \text{ is an F-cavity, } \sharp F \geq 3\}$$

replaces to $4||\mathcal{A}||_{1,\infty}$. It is clear that $D \leq 4||\mathcal{A}||_{1,\infty}$ (the same arguments used in Theorem 3.2 are right). The advantage of our version is that the bound is straight forward obtained from $\mathcal{A}$, without being necessary to construct any matrix $\mathcal{A}_\tau$. It is also clear that in Theorem 3.2 the number

$$D_i := max\{||\mathcal{A}_\tau||_{1,\infty} \ \mid \ \tau \text{ is a i-triangulation of F}, \sharp F \geq i+2\}$$

can replace to $4||\mathcal{A}||_{1,\infty}$, but our version is more useful for the proof of Theorem 4.1.

There is no result similar to Theorem 3.2 in the literature, for $i \geq 2$.

In the following remark we consider the generalization to the case S with torsion non trivial.

Remark 3. 2.

$$S \subset \mathbf{Z}^h \oplus \mathbf{Z}/a_1\mathbf{Z} \oplus \ldots \oplus \mathbf{Z}/a_s\mathbf{Z}$$

with $a_j \in \mathbf{Z}$, $1 \leq j \leq s$.

Now, $\mathcal{A} \in \mathcal{M}_{(h+s)\times r}(\mathbf{Z})$ and $\mathcal{A}(t) \in \mathcal{M}_{(h+s)(t-1)\times rt}(\mathbf{Z})$.

To use the reasoning in the without torsion case, we need to remove the congruences in the systems of kind

$$\mathcal{A}(t)\beta = -\mathcal{A}(t)e_\tau.$$

Then, it is enough to consider the auxiliary matrices (see [17] for details)

$$T = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ & & \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_1 & -a_1 & 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & a_2 & -a_2 & 0 & \cdots & 0 & 0 \\ & & \ddots & \ddots & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & & a_s & -a_s \end{pmatrix} \in \mathcal{M}_{(h+s)\times 2s}(\mathbf{Z}),$$

and $\widetilde{\mathcal{A}(t)} = (\mathcal{A}(t)|\tilde{T})$, where

$$\tilde{T} = \begin{pmatrix} T & 0 & \ldots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \ldots & 0 & T \end{pmatrix} \in \mathcal{M}_{(t-1)(h+s)\times(t-1)2s}(\mathbf{Z}).$$

Using the system without congruences of kind

$$\widetilde{\mathcal{A}(t)}\beta = -\mathcal{A}(t)e_\tau,$$

the following result is obtained.

If $m \in S$ is an S-degree of a minimal i-syzygy for $k[S]$, then $m = \mathcal{A}x$ with $x \in \mathbf{N}^r$ such that

$$||x||_1 \leq (1 + 2a + 4||\mathcal{A}||_{1,\infty})^{(h+s)(d_i-1)} + (i+1)d_i - 1,$$

where $a = max_{1\leq j\leq s}|a_j|$ and $d_i = \begin{pmatrix} r \\ i+1 \end{pmatrix}$.

4. REGULARITY OF PROJECTIVE TORIC VARIETIES

In this section we suppose that I is homogeneous for the natural graduation. This is equivalent to there exists a vector $w \in \mathbf{Q}^h$ such that $n_i \cdot w = 1$, for any $i = 1, \ldots, r$. (Lemma 4.14 in [22]). Geometrically, I defines a projective variety in $\mathbf{P}^{r-1}(k)$. Notice that if $f \in I$ is S-homogeneous of S-degree $m \in S$, then $\deg(f) = ||\alpha||_1$, for any $\alpha \in \mathbf{N}^r$ with $\mathcal{A}\alpha = m$.

Suppose that $\{f_1, \ldots, f_{\beta_1}\}$ is a minimal generating set of I, S-$\deg(f_i) = p_i \in S$ and $\deg(f_i) = ||\alpha_i||_1$, where $p_i = \mathcal{A}\alpha_i$, $1 \leq i \leq \beta_1$. If $g = (g_1, \ldots, g_{\beta_1}) \in N_1$ is a 1-syzygy of S-degree $m \in S$, then S-$\deg(g_i) = m - p_i$ and $\deg(g_i) = ||\beta_i||_1$ where $\mathcal{A}\beta_i = m - p_i$. Thus, $m = \mathcal{A}(\alpha_i + \beta_i)$ and $\deg(g) = ||\alpha_i + \beta_i||_1$.

More generally, if $h = (h_1, \ldots, h_{\beta_i}) \in N_i$ is an i-syzygy of S-degree $m \in S$, then $\deg(h) = ||\alpha||_1$, for any $\alpha \in \mathbf{N}^r$ such that $m = \mathcal{A}\alpha$.

By this way, we obtain an explicit bound for the Castelnuovo-Mumford regularity of I. In the following theorem, the symbol $\lfloor\rfloor$ stands for integral part.

Theorem 4.1.
With assumptions and notations as above,

$$reg(I) \leq (1 + 4||\mathcal{A}||_{1,\infty})^{h(d-1)} + (r+1)(d-1)$$

where $d = \begin{pmatrix} r \\ \lfloor r/2 \rfloor \end{pmatrix}$.

Proof.
The regularity of I is $\mathrm{reg}(I) = max_{1\leq i\leq r}\{t_i - i\}$, where t_i is the maximum degree of the i-syzygies of I (see, for example, [2]).

By Theorem 3.2,

$$t_i \leq (1 + 4||\mathcal{A}||_{1,\infty})^{h(d_i-1)} + (i+1)d_i - 1,$$

with $d_i = \begin{pmatrix} r \\ i+1 \end{pmatrix}$. It is clear that $d_i \leq d$, for any i. Then

$$t_i - i \leq (1 + 4||\mathcal{A}||_{1,\infty})^{h(d-1)} + (r+1)(d-1).$$

It is enough to take maximum. ∎

Finally, notice that the effective computation of the regularity of I doesn't require the computation of the minimal resolution. It is sufficient to use the Step 1 of Construction in the introduction for any i. We describe this algorithm.

Algorithm 1.

Computing the regularity

Input: Set of generators $\{n_1, \ldots, n_r\}$ of S. (Recall that they must lie on a rational hiperplane.)
Output: The regularity of the ideal I of S.

1. For any i, $1 \leq i \leq r$
   - Compute the set C'_i (Theorem 1.1).
   - Check the element $m \in C'_i$ such that $\tilde{H}_i(\Delta_m) \neq 0$ and obtain C_i.

- Compute $t_i = \{||\alpha||_1 \ \mid \ m = \mathcal{A}\alpha \in C_i\}$.

2. Output $\mathrm{reg}(I) = max\{t_i - i \ \mid \ i = 1, \ldots, r\}$.

REFERENCES

1. A. ARAMOVA, J. HERZOG, Free resolution and Koszul homology. *J. of Pure and Applied Algebra,* **105** (1995), 1-16.
2. D. BAYER, M. STILLMAN, A criterion for detecting m-regularity. *Inventiones mathematicae,* **87** (1987), 1-11.
3. F. DI BIASE, R. URBANKE, An Algorithm to Calculate the Kernel of Certain Polynomial Ring Homomorphisms, *Experimental Mathematics*, Vol.4, No. 3 (1995), 227-234.
4. E. BRIALES, A. CAMPILLO, C. MARIJUÁN, P. PISÓN, Minimal Systems of Generators for Ideals of Semigroups, *J. of Pure and Applied Algebra,* **124** (1998), 7-30.
5. E. BRIALES, A. CAMPILLO, C. MARIJUÁN, P. PISÓN, Combinatorics of syzygies for semigroup algebra. *Collet. Math.* **49** (1998), 239-256.
6. W. BRUNS, J. HERZOG, Cohen-Macaulay rings. *Cambrigde University Press*, (1993).
7. W. BRUNS, J. HERZOG, Semigroup rings and simplicial complexes. *J. of Pure and Applied Algebra*, **122** (1997), 185-208.
8. A. CAMPILLO, C. MARIJUÁN, Higher relations for a numerical semigroup. *Sém. Théor. Nombres Bordeaux* **3** (1991), 249-260.
9. A. CAMPILLO, P. GIMÉNEZ, Syzygies of affine toric varieties. *Journal of Algebra* to appear.
10. CAMPILLO A.; PISÓN P. L'idéal d'un semi-grupe de type fini. *C. R. Acad. Sci. Paris, Série I*, **316** (1993), 1303-1306.
11. D. EISENBUD, *Introduction to Commutative Algebra with a View Towards Algebraic Geometry.* Graduate Texts in Mathematics **150** (1995), Springer, New York.
12. P.A. GARCÍA-SÁNCHEZ *Semigrupos afines.* Ph. D. Thesis, Universidad de Granada (1996).
13. M. HOCHSTER, Cohen-Macaulay rings, combinatorics, and simplicial complexes, In BR. McDonald, R.A. Morris editors, *Ring Theory II*, LNPAM **26**, Marcel-Dekker (1995), 171-223.
14. S. HOSTEN, B. STURMFELS, GRIN, An Implementation of Gröbner Bases for Integer Programming, In E. Balas and J. Clausen editors, *Integer Programming and Combinatorial Optimization*, LNCS **920**, Springer-Verlag (1995), 267-276.
15. I. PEEVA, B. STURMFELS, Syzygies of codimension 2 lattice ideals, *Math. Z.* **229** (1998), 163-194.
16. P. PISÓN-CASARES, A. VIGNERON-TENORIO, $\mathbf{N}-$solutions to linear systems over $\mathbf{Z}$. *Preprint of University of Sevilla* **43** (1998).
17. P. PISÓN-CASARES, A. VIGNERON-TENORIO, First Syzygies of Toric Varieties and Diophantine Equations in Congruence, *Preprint of University of Sevilla* **52** (1999).

18. L. POTTIER, Bornes et algorithmes de calcul des generateurs des solutions de systémes diophantines linéaries. *C.R. Acad. Sci. Paris,* Serie I **311**(1990), 813-816.

19. L. POTTIER, Minimal solutions of linear diophantine systems: bounds and algorithms. *Proceedings of the Fourth International Conference on Rewriting Techniques and Applications,* Italy 1991, 162-173.

20. J.C. ROSALES, *Semigrupos numéricos* . PhD Thesis, Universidad de Granada (1991).

21. R. STANLEY, *Combinatorics and commutative algebra 2nd ed.* Progress in Mathematics Vol.**41**, Boston Basel Berlin, Birkhäuser (1996).

22. B. STURMFELS, Gröbner bases of Toric Varieties. *Tôhoku Math. J.* **43** (1991), 249-261.

23. A. VIGNERON-TENORIO, Semigroup Ideals and Linear Diophantine Equations. *Linear Algebra and its Applications,* **295** (1999), 133-144.