HARDY-LITTLEWOOD INEQUALITIES FOR NORMS OF POSITIVE OPERATORS ON SEQUENCE SPACES

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ABSTRACT. We consider estimates of Hardy and Littlewood for norms of operators on sequence spaces, and we apply a factorization result of Maurey to obtain improved estimates and simplified proofs for the special case of a positive operator.

In 1934, Hardy and Littlewood [1], using powerful but technically difficult methods, extended results of Littlewood [2] and Toeplitz [4] to give lower bounds for norms of bilinear forms on sequence spaces. In 2001, Osikiewicz and Tonge [5], exploiting a deep interpolation theorem, obtained relatively simple proofs for the inequalities of Hardy and Littlewood. The aim of this paper is to provide improved estimates and simplified proofs for the special case of a matrix operator with non-negative entries (we will call such operators positive).

If $1 \le p < \infty$, we write ℓ_p for the complex vector space of all complex sequences $x = (x_k)$ such that

$$\|x\|_p \colon = \left(\sum_k |x_k|^p \right)^{1/p} < \infty.$$

We also write c_0 for the space of all null complex sequences $x = (x_k)$ with the norm

$$\|x\|_\infty \colon = \sup_k |x_k|.$$

It turns out that ℓ_p and c_0 are Banach spaces under the indicated norms. If X and Y are Banach spaces then we write $\mathcal{B}(X, Y)$ for the complex vector space of all bounded linear operators $A : X \to Y$. This is a Banach space under the operator norm

$$\|A\| \colon = \sup\{\|Ax\| \colon x \in X, \|x\| \le 1\}.$$

When $1 \le p, q \le \infty$, it is convenient to write $\|A\|_{p,q}$ for the norm of an operator $A \in \mathcal{B}(\ell_p, \ell_q)$. Every operator $A \in \mathcal{B}(\ell_p, \ell_q)$ has a matrix representation $A = (a_{j,k})$ with respect to the usual bases. The dual of a Banach space X is denoted by X^*. If $1 \le p < \infty$, it is a standard fact that $c_0^* = \ell_1$ and that $\ell_p^* = \ell_{p^*}$, where the conjugate index p^* is given by the expression

$$\frac{1}{p} + \frac{1}{p^*} = 1.$$

Now the theorems of Hardy and Littlewood can be stated as follows:

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Theorem 1 (Hardy and Littlewood, [1]). *Let $1 \leq q \leq 2 \leq p \leq \infty$ and set $1/r = 1/q - 1/p$. There is an absolute constant $M = M(p,q) > 0$ such that for every $A \in \mathcal{B}(\ell_p, \ell_q)$ we have*

$$\left[\sum_j \left(\sum_k |a_{j,k}|^2\right)^{r/2}\right]^{1/r} \leq M\|A\|_{p,q}, \qquad \text{and}$$

$$\left[\sum_k \left(\sum_j |a_{j,k}|^2\right)^{r/2}\right]^{1/r} \leq M\|A\|_{p,q}.$$

Theorem 2 (Hardy and Littlewood, [1]). *Let $1 \leq q \leq p \leq \infty$, and set $1/r = 1/q - 1/p$, $1/s = 1/(2r) + 1/4$. There is an absolute constant $M = M(p,q) > 0$ such that for every $A \in \mathcal{B}(\ell_p, \ell_q)$ we have*

(i) *If $r \geq 2$ then*

$$\left(\sum_{j,k} |a_{j,k}|^r\right)^{1/r} \leq M\|A\|_{p,q}.$$

(ii) *If $r \leq 2$ then*

$$\left(\sum_{j,k} |a_{j,k}|^s\right)^{1/s} \leq M\|A\|_{p,q}.$$

Now we prove Theorem 1 and Theorem 2 for the special case of an operator with non negative entries. The key for our approach is a factorization theorem of Maurey [3] that can be stated as follows:

Theorem 3 (Maurey, [3]). *Let $0 < q \leq p \leq \infty$ with $p \geq 1$, and set $1/r = 1/q - 1/p$. If $A \in \mathcal{B}(\ell_p, \ell_q)$ has non negative entries then there is a factorization $A = DB$, where $B \in \mathcal{B}(\ell_p, \ell_p)$ has non negative entries and $D \in \mathcal{B}(\ell_p, \ell_q)$ is a diagonal operator with non negative entries, say $D = \operatorname{diag}(d)$, with $d \in \ell_r$. Moreover,*

$$\|A\|_{p,q} = \inf \|d\|_r \cdot \|B\|_{p,p},$$

where the infimum is taken over all possible factorizations.

Proof of Theorem 1 for a positive operator. Let $A = (a_{j,k})$ with $a_{j,k} \geq 0$ for all j,k. Take a factorization $A = DB$, where $B \in \mathcal{B}(\ell_p, \ell_p)$ and $D = \operatorname{diag}(d)$, with $d \in \ell_r$. We have $a_{j,k} = d_j b_{j,k}$, so that

$$\begin{aligned}
\left[\sum_j \left(\sum_k a_{j,k}^2\right)^{r/2}\right]^{1/r} &= \left[\sum_j \left(\sum_k d_j^2 b_{j,k}^2\right)^{r/2}\right]^{1/r} \\
&= \left[\sum_j d_j^r \left(\sum_k b_{j,k}^2\right)^{r/2}\right]^{1/r} \\
&\leq \left(\sum_j d_j^r\right)^{1/r} \sup_j \left(\sum_k b_{j,k}^2\right)^{1/2} \\
&= \|d\|_r \cdot \sup_j \|B^* e_j\|_2 \\
&= \|d\|_r \cdot \|B^*\|_{1,2} \\
&= \|d\|_r \cdot \|B\|_{2,\infty} \\
&\leq \|d\|_r \cdot \|B\|_{p,p},
\end{aligned}$$

and taking the infimum over all possible factorizations, we get the first inequality with constant $M = 1$. The second inequality follows easily from the first one by using duality. Indeed, set $1/p^* = 1 - 1/p$ and $1/q^* = 1 - 1/q$, so that $p^* \leq 2 \leq q^*$. Let $A \in \mathcal{B}(\ell_p, \ell_q)$, say $A = (a_{j,k})$ with $a_{j,k} \geq 0$ for all j, k, and notice that the adjoint operator $A^* \in \mathcal{B}(\ell_{q^*}, \ell_{p^*})$ has the matrix representation $A^* = (a_{k,j})$. Hence,

$$\left[\sum_k \left(\sum_j a_{j,k}^2\right)^{r/2}\right]^{1/r} \leq \|A^*\|_{q^*,p^*} = \|A\|_{p,q},$$

as we wanted. □

Proof of Theorem 2 for a positive operator. Let $A = (a_{j,k})$ with $a_{j,k} \geq 0$ for all j, k. Take a factorization $A = DB$, where $B \in \mathcal{B}(\ell_p, \ell_p)$ and $D = \operatorname{diag}(d)$, with $d \in \ell_r$. Let $1/r^* = 1 - 1/r$. Since $q \geq 1$, we have $r^* \leq p$. We have $a_{j,k} = d_j b_{j,k}$, so that

$$\begin{aligned}
\left(\sum_j \sum_k a_{j,k}^r\right)^{1/r} &= \sum_j \left(d_j^r \sum_k b_{j,k}^r\right)^{1/r} \\
&\leq \left(\sum_j d_j^r\right)^{1/r} \sup_j \left(\sum_k b_{j,k}^r\right)^{1/r} \\
&= \|d\|_r \cdot \sup_j \|B^* e_j\|_r \\
&= \|d\|_r \cdot \|B^*\|_{1,r} \\
&= \|d\|_r \cdot \|B\|_{r^*,\infty} \\
&\leq \|d\|_r \cdot \|B\|_{p,p}
\end{aligned}$$

and taking the infimum over all possible factorizations, we get the inequality (i) with the constant $M = 1$. Notice that we have shown that inequality (i) holds regardless of whether $r \geq 2$ or $r \leq 2$. If $r \leq 2$ then $r \leq s \leq 2$, so that we have obtained an improvement on Theorem 2, namely, that the left hand side in inequality (ii) can be replaced by the left hand side in inequality (i). □

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