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Extended eigenvalues for Cesàro operators

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Abstract

A complex scalar λ is said to be an *extended eigenvalue* of a bounded linear operator T on a complex Banach space if there is a nonzero operator X such that $TX = \lambda XT$. Such an operator X is called an *extended eigenoperator* of T corresponding to the extended eigenvalue λ.

The purpose of this paper is to give a description of the extended eigenvalues for the discrete Cesàro operator C_0, the finite continuous Cesàro operator C_1 and the infinite continuous Cesàro operator C_∞ defined on the complex Banach spaces ℓ^p, $L^p[0,1]$ and $L^p[0,\infty)$ for $1 < p < \infty$ by the expressions

$$
\begin{aligned}
(C_0 f)(n) :&= \frac{1}{n+1}\sum_{k=0}^{n} f(k),\\
(C_1 f)(x) :&= \frac{1}{x}\int_0^x f(t)\,dt,\\
(C_\infty f)(x) :&= \frac{1}{x}\int_0^x f(t)\,dt.
\end{aligned}
$$

It is shown that the set of extended eigenvalues for C_0 is the interval $[1,\infty)$, for C_1 it is the interval $(0,1]$, and for C_∞ it reduces to the singleton $\{1\}$.

Keywords: Extended eigenvalue, Extended eigenoperator, Cesàro operator, Shift operator, Euler operator, Hausdorff operator, Rich point spectrum, Bilateral weighted shift, Analytic Toeplitz operator, Analytic kernel.
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Contents

1. Introduction

We shall represent by $\mathcal{B}(E)$ the algebra of all bounded linear operators on a complex Banach space E. A complex scalar λ is said to be an *extended eigenvalue* of an operator $T \in \mathcal{B}(E)$ provided that there is a nonzero operator $X \in \mathcal{B}(E)$ such that $TX = \lambda XT$, and in that case X called an *extended eigenoperator* of T corresponding to the extended eigenvalue λ. We shall represent by $\{T\}'$ the commutant of an operator T, i.e., the set of operators that commute with T, or in other words, the family of all the extended eigenoperators for T corresponding to the extended eigenvalue $\lambda = 1$.

Recently, the study of the extended eigenvalues for some classes of operators has received a considerable amount of attention [2, 3, 5, 13, 14, 17, 18, 22].

The purpose of this paper is to describe the set of the extended eigenvalues for the discrete Cesàro operator C_0, the finite continuous Cesàro operator C_1, and the infinite continuous Cesàro operator C_∞ defined on the complex Banach spaces ℓ^p, $L^p[0,1]$ and $L^p[0,\infty)$ for $1 < p < \infty$ by the expressions

$$(C_0 f)(n) \colon = \frac{1}{n+1} \sum_{k=0}^{n} f(k), \tag{1.1}$$

$$(C_1 f)(x) \colon = \frac{1}{x} \int_0^x f(t)\, dt, \tag{1.2}$$

$$(C_\infty f)(x) \colon = \frac{1}{x} \int_0^x f(t)\, dt. \tag{1.3}$$

It is shown that the set of extended eigenvalues for C_0 is the interval $[1,\infty)$, for C_1 is the interval $(0,1]$, and for C_∞ is the singleton $\{1\}$. The notion of an operator with rich point spectrum is introduced and it is shown that the geometry of the point spectrum for such an operator determines its extended eigenvalues. Then, it is shown that both C_1 and C_0^* have rich point spectrum. Further, it is shown that a bilateral weighted shift whose point spectrum has non empty interior and the adjoint of an analytic Toeplitz operator with non constant symbol are further examples of operators with rich point spectrum. Then, this result is applied to obtain information on the extended eigenvalues of those operators. Finally, a factorization is provided for the extended eigenoperators of a Hilbert space operator under certain conditions.

The paper is organized as follows.

In section 2 we show that every $\lambda \in (0,1]$ is an extended eigenvalue for C_1 on $L^2[0,1]$ and the Euler operator is a corresponding extended eigenoperator. Moreover, any extended eigenoperator for C_1 on $L^2[0,1]$ factors as the product of the Euler operator, a Toeplitz matrix, and a power of a backward unilateral shift of multiplicity one.

In section 3 we introduce the notion of an operator with rich point spectrum. We show that if λ is an extended eigenvalue of an operator T with rich point spectrum then λ multiplies $\operatorname{int}\sigma_p(T)$, the interior of the point spectrum of T, into $\operatorname{clos}\sigma_p(T)$, the closure of the point spectrum of T. We show that both C_1 and C_0^* have rich point spectrum and we apply this geometric result to prove that for every $1 < p < \infty$ we have

1. if λ is an extended eigenvalue for C_1 on $L^p[0,1]$ then $0 < \lambda \leq 1$,
2. if λ is an extended eigenvalue for C_0 on ℓ^p then $\lambda \geq 1$.

In section 4 we show that every $\lambda \in (0,1]$ is an extended eigenvalue for C_1 on $L^p[0,1]$ and that a certain weighted composition operator is a corresponding extended eigenoperator.

In section 5 we show when $p = 2$ that if λ is real with $\lambda \geq 1$ then λ is an extended eigenvalue for C_0.

In section 6 we show that if the point spectrum of a bilateral weighted shift W has non empty interior then W has rich point spectrum, and as a consequence, the set of the extended eigenvalues for W is the unit circle.

In section 7 we show that a result of Deddens [7] about extended eigenvalues of an analytic Toeplitz operators can be regarded as a special case of our main result in section 3.

In section 8 we show under certain conditions that if λ is an extended eigenvalue for an operator T on a Hilbert space then there is a particular extended eigenoperator X_0 corresponding to λ such that every extended eigenoperator X corresponding to λ factors as $X = X_0 R$ for some $R \in \{T\}'$.

In section 9 we show that the family of the extended eigenvalues for C_∞ on the complex Hilbert space $L^2[0,\infty)$ reduces to the singleton $\{1\}$.

In section 10 we show that the family of the extended eigenvalues for C_∞ on the complex Banach space $L^p[0,\infty)$, for $1 < p < \infty$, reduces to the singleton $\{1\}$.

2. The finite continuous Cesàro operator on Hilbert space

Brown, Halmos and Shields [6] proved in the Hilbertian case that C_1 is indeed a bounded linear operator, and they also proved that $I - C_1^*$ is unitarily equivalent to a unilateral shift of multiplicity one.

Recall that a bounded linear operator S on a complex Hilbert space H is a *unilateral shift of multiplicity one* provided that there is an orthonormal basis (e_n) of H such that $Se_n = e_{n+1}$ for all $n \in \mathbb{N}$. It is easy to see that the adjoint of a such a unilateral shift satisfies $S^* e_0 = 0$ and $S^* e_n = e_{n-1}$ for all $n \geq 1$.

Consider a unilateral shift of multiplicity one $S \in \mathcal{B}(L^2[0,1])$ and a unitary operator $U \in \mathcal{B}(L^2[0,1])$ such that $I - C_1^* = U^* S U$. We have $C_1 = U^*(I - S^*)U$, and since the extended eigenvalues are preserved under similarity in general, and under unitary equivalence in particular, it follows that the extended eigenvalues of C_1 are precisely the extended eigenvalues of $I - S^*$, and the extended eigenoperators of C_1 are in one to one correspondence with the extended eigenoperators of $I - S^*$ under conjugation with U.

We shall use repeatedly the following elementary, standard fact.

Lemma 2.1. *The point spectrum of S^* is the open unit disc $\mathbb{D}$. More precisely, every $\lambda \in \mathbb{D}$ is a simple eigenvalue of S^*, and a corresponding eigenvector f is given by the expression*

$$f = \sum_{n=0}^{\infty} \lambda^n e_n. \tag{2.1}$$

Now we are ready to describe the set of the extended eigenvalues for $I-S^*$. Our first goal is to show that the interval $(0,1]$ is contained in the set of the extended eigenvalues for $I-S^*$, and to exhibit a corresponding extended eigenoperator. We shall prove that a particular extended eigenoperator is the Euler operator.

It is convenient now to have a digression about the Euler operator and the discrete Cesàro operator. We follow the discussion in the paper of Rhoades [19]. Recall that the discrete Cesàro operator C_0 is defined on ℓ^2 by the sequence of arithmetic means (1.1).

Let $\lambda \in \mathbb{C}$. The *Euler operator* E_λ is defined on ℓ^2 by the binomial means

$$(E_\lambda f)(n) = \sum_{k=0}^{n} \binom{n}{k} \lambda^k (1-\lambda)^{n-k} f(k), \quad n \in \mathbb{N}. \tag{2.2}$$

Let (μ_k) be a sequence of complex scalars and let Δ denote the *forward difference operator* defined by

$$\Delta \mu_k = \mu_k - \mu_{k+1}. \tag{2.3}$$

A *Hausdorff matrix* is an infinite matrix $A=(a_{nk})$ whose entries are given by the expression

$$a_{nk} = \begin{cases} \binom{n}{k} \Delta^{n-k} \mu_k & \text{if } 0 \le k \le n, \\ 0 & \text{if } k > n. \end{cases} \tag{2.4}$$

The sequence (μ_k) is called the *generating sequence* for the Hausdorff matrix A and it is determined by the diagonal entries of A. The *Hausdorff operator* associated with a Hausdorff matrix $A=(a_{nk})$ is defined by the expression

$$(Af)(n) = \sum_{k=0}^{n} a_{nk} f(k). \tag{2.5}$$

The discrete Cesàro operator C_0, with generating sequence $\mu_n = (n+1)^{-1}$, and the Euler operator E_λ, with generating sequence $\mu_n = \lambda^n$, are two examples of Hausdorff operators. Rhoades [19] notes that E_λ is bounded for $1/2 < \lambda \le 1$. We show in Proposition 2.5 below that E_λ is bounded also for $0 < \lambda \le 1/2$.

There is a strong connection between Hausdorff operators and the discrete Cesàro operator. Hurwitz and Silvermann [10] showed that the commutant of C_0 is precisely the set of all Hausdorff operators, whereas Shields and Wallen [20] showed that the commutant of C_0 is the weakly closed algebra with identity generated by C_0.

Proposition 2.2. *If $0 < \lambda \le 1$ then λ is an extended eigenvalue for $I-S^*$, and moreover, the Euler operator E_λ is a corresponding extended eigenoperator.*

Proof. First of all, for $k=0$ we have

$$E_\lambda e_0 = \sum_{n=0}^{\infty} (1-\lambda)^n e_n.$$

Then, it follows from Lemma 2.1 that $(I-S^*)E_\lambda e_0 = \lambda E_\lambda (I-S^*) e_0$. Next, for $k \ge 1$ we have

$$\begin{aligned} S^* E_\lambda e_k &= \sum_{n=k}^{\infty} \binom{n}{k} \lambda^k (1-\lambda)^{n-k} e_{n-1} \\ &= \lambda^k e_{k-1} + \sum_{n=k}^{\infty} \binom{n+1}{k} \lambda^k (1-\lambda)^{n+1-k} e_n, \end{aligned}$$

so that

$$(I-S^*)E_\lambda e_k = -\lambda^k e_{k-1} + \sum_{n=k}^{\infty} \left[\binom{n}{k} - \binom{n+1}{k} (1-\lambda) \right] \lambda^k (1-\lambda)^{n-k} e_n.$$

Using Pascal's identity $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$ leads to

$$\begin{aligned}
(I-S^*)E_\lambda e_k &= -\lambda^k e_{k-1} + \sum_{n=k}^{\infty} \left[\binom{n}{k}\lambda - \binom{n}{k-1}(1-\lambda)\right] \lambda^k (1-\lambda)^{n-k} e_n \\
&= -\lambda^k e_{k-1} + \sum_{n=k}^{\infty} \binom{n}{k} \lambda^{k+1}(1-\lambda)^{n-k} e_n \\
&- \sum_{n=k}^{\infty} \binom{n}{k-1} \lambda^k (1-\lambda)^{n-(k-1)} e_n \\
&= \lambda \sum_{n=k}^{\infty} \binom{n}{k} \lambda^k (1-\lambda)^{n-k} e_n \\
&- \lambda \left[\lambda^{k-1} e_{k-1} + \sum_{n=k}^{\infty} \binom{n}{k-1} \lambda^{k-1}(1-\lambda)^{n-(k-1)} e_n\right] \\
&= \lambda \sum_{n=k}^{\infty} \binom{n}{k} \lambda^k (1-\lambda)^{n-k} e_n \\
&- \lambda \sum_{n=k-1}^{\infty} \binom{n}{k-1} \lambda^{k-1}(1-\lambda)^{n-(k-1)} e_n \\
&= \lambda(E_\lambda e_k - E_\lambda e_{k-1}) \\
&= \lambda E_\lambda (I-S^*) e_k,
\end{aligned}$$

so that $(I-S^*)E_\lambda e_k = \lambda E_\lambda (I-S^*)e_k$ for all $k \in \mathbb{N}$, as we wanted. □

Our next goal is to describe the collection of the extended eigenoperators for $I - S^*$ corresponding to an extended eigenvalue $\lambda \in (0,1]$. It is convenient to have a digression on Toeplitz operators. We shall follow the discussion about Toeplitz operators in the paper of Sheldon Axler [1].

Let $(\alpha_n)_{n\in\mathbb{Z}}$ be a two sided sequence of complex scalars and consider the infinite matrix $A = (a_{nk})$ whose entries are given by the expression $a_{nk} = \alpha_{n-k}$. We say that A is the *Toeplitz matrix* associated with the sequence $(\alpha_n)_{n\in\mathbb{Z}}$. The *Toeplitz operator* associated with a Toeplitz matrix $A = (a_{nk})$ is defined on the complex Hilbert space ℓ^2 by the expression

$$(Af)(n) = \sum_{k=0}^{\infty} \alpha_{n-k} f(k). \tag{2.6}$$

Consider the unit circle $\mathbb{T} = \{z \in \mathbb{C} \colon |z| = 1\}$ and define a function $\varphi \colon \mathbb{T} \to \mathbb{C}$ by the Fourier expansion

$$\varphi(e^{i\theta}) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\theta}. \tag{2.7}$$

It is a standard fact that a Toeplitz matrix A induces a bounded operator if and only if φ is essentially bounded, and moreover,

$$\|A\| = \sup\{|\varphi(z)| \colon z \in \mathbb{T}\}. \tag{2.8}$$

Halmos says [9, Problem 33] that Fourier expansions are formally similar to Laurent expansions, and the analogy motivates calling the functions of $H^2(\mathbb{T})$ the *analytic* elements of $L^2(\mathbb{T})$. Thus, φ is analytic if and only if $\alpha_n = 0$ for all $n < 0$. Also, φ is called *co-analytic* provided that $\alpha_n = 0$ for all $n > 0$.

It turns out that $AS = SA$ if and only if A is an analytic Toeplitz operator, and that $AS^* = S^*A$ if and only A is a co-analytic Toeplitz operator.

Lambert [13] observed that if $X \in \mathcal{B}(H)$ is an extended eigenoperator for an operator $T \in \mathcal{B}(H)$ associated with an extended eigenvalue $\lambda \in \mathbb{C}$, and if $R \in \{T\}'$ then the product XR is also an extended eigenoperator for T associated with λ.

Let A be a co-analytic Toeplitz operator. Since A commutes with S^* and since $(S^*)^{n_0}$ commutes with S^*, it follows that $A(S^*)^{n_0}$ commutes with $I - S^*$. Since E_λ is an extended eigenoperator for $I - S^*$ associated with the extended eigenvalue λ, it follows from Lambert's observation that $E_\lambda A(S^*)^{n_0}$ is also an extended eigenoperator for $I - S^*$ associated with the extended eigenvalue λ. The following result shows that these are all possible extended eigenoperators for the operator $I - S^*$.

Theorem 2.3. *If $0 < \lambda \leq 1$ and X is an extended eigenoperator of $I - S^*$ associated with λ then there is a two sided sequence $(\alpha_n)_{n\in\mathbb{Z}}$ of complex scalars with $\alpha_0 \neq 0$ and $\alpha_n = 0$ for all $n \geq 1$, and there is an $n_0 \in \mathbb{N}$ such that X admits a factorization*

$$X = E_\lambda A(S^*)^{n_0}, \tag{2.9}$$

where E_λ is the Euler operator and where A is the co-analytic Toeplitz matrix associated with $(\alpha_n)_{n\in\mathbb{Z}}$.

Proof. We have $(I - S^*)Xe_0 = \lambda Xe_0$ and $(I - S^*)Xe_n = \lambda(Xe_n - Xe_{n-1})$ for all $n \geq 1$. Since $X \neq 0$, there is some $n \in \mathbb{N}$ such that $Xe_n \neq 0$. Let $n_0 = \min\{n \in \mathbb{N} \colon Xe_n \neq 0\}$.

First step: Let us suppose that $n_0 = 0$ and notice that Xe_0 is an eigenvector of $I - S^*$ corresponding to the eigenvalue λ, so that according to Lemma 2.1, there is a nonzero complex scalar β_0 such that

$$Xe_0 = \beta_0 \sum_{n=0}^{\infty} (1-\lambda)^n e_n. \tag{2.10}$$

We claim that there is a sequence of complex scalars $(\beta_n)_{n\in\mathbb{N}}$ with $\beta_0 \neq 0$ and such that for every $n \in \mathbb{N}$,

$$Xe_n = \sum_{k=0}^{n} \beta_{n-k} E_\lambda e_k. \tag{2.11}$$

We proceed by induction. If $n = 0$, this follows trivially from equation (2.10). Then, suppose that $n \geq 1$ and the complex scalars $\beta_0, \ldots, \beta_{n-1}$ are constructed in such a way that

$$Xe_{n-1} = \sum_{k=0}^{n-1} \beta_{n-1-k} E_\lambda e_k.$$

Notice that

$$\begin{aligned}
[S^* - (1-\lambda)I]Xe_n = \lambda Xe_{n-1} &= \lambda \sum_{k=0}^{n-1} \beta_{n-1-k} E_\lambda e_k \\
&= [S^* - (1-\lambda)I] \left(\sum_{k=0}^{n-1} \beta_{n-1-k} E_\lambda e_{k+1} \right),
\end{aligned}$$

so that

$$Xe_n - \sum_{k=0}^{n-1} \beta_{n-1-k} E_\lambda e_{k+1} \in \ker[S^* - (1-\lambda)I].$$

Finally, according to Lemma 2.1, there is a complex scalar β_n such that

$$Xe_n - \sum_{k=0}^{n-1} \beta_{n-1-k} E_\lambda e_{k+1} = \beta_n E_\lambda e_0,$$

and the claim follows. Now, let $(\alpha_n)_{n\in\mathbb{Z}}$ be the two sided sequence defined by $\alpha_{-n} = \beta_n$ for all $n \geq 1$ and $\alpha_n = 0$ for all $n \in \mathbb{N}$, and let A be the co-analytic Toeplitz matrix associated with the sequence $(\alpha_n)_{n\in\mathbb{Z}}$. We have that $X = E_\lambda A$, so that equation (2.9) holds with $n_0 = 0$.

Second step: Suppose that $n_0 \geq 1$. Notice that $Xe_n = 0$ for all $0 \leq n < n_0$ and $Xe_{n_0} \neq 0$. Thus, $(I - S^*)Xe_{n_0} = \lambda Xe_{n_0}$ and $(I - S^*)Xe_n = \lambda(Xe_n - Xe_{n-1})$ for each $n > n_0$, or in other words, $(I - S^*)XS^{n_0}e_0 = \lambda XS^{n_0}e_0$ and $(I - S^*)XS^{n_0}e_n = \lambda(XS^{n_0}e_n - XS^{n_0}e_{n-1})$ for each $n \geq 1$. This means that XS^{n_0} is a nonzero linear operator as in the first step of the proof. Therefore, there is a sequence (α_n) of complex scalars with $\alpha_0 \neq 0$ and such that $XS^{n_0} = E_\lambda A$, where A is the Toeplitz operator associated with the sequence (α_n). Finally, since $Xe_n = 0$ for $0 \leq n \leq n_0$, it follows that $X = E_\lambda A(S^*)^{n_0}$. ☐

We finish this section with the consideration of the question of boundedness for the Euler operator E_λ. We already mentioned that Rhoades [19] noted that E_λ is bounded for $1/2 < \lambda \leq 1$. He proved that in fact we have $\|E_\lambda\| = \lambda^{-1/2}$. We show in Proposition 2.5 below that E_λ is also bounded for $0 < \lambda \leq 1/2$ and moreover, $\|E_\lambda\| \leq (1-\lambda)^{-1/2}$. Since we could not find a proof of this fact in the literature, we include an argument that is based on a criterion due to Schur. A proof of this criterion, different from the original one, can be found in the paper of Brown, Halmos and Shields [6], where it is applied to show the boundedness of both the continuous and the discrete Cesàro operators.

Lemma 2.4 (Schur test)**.** *If $a_{nk} \geq 0$, if $p_k > 0$, and if $\alpha, \beta > 0$ are such that*

$$\sum_{k=0}^{\infty} a_{nk}p_k \leq \alpha p_n, \tag{2.12}$$

$$\sum_{n=0}^{\infty} a_{nk}p_n \leq \beta p_k, \tag{2.13}$$

then there is a bounded linear operator X with $\|X\|^2 \leq \alpha\beta$ and such that for all $n \in \mathbb{N}$,

$$(Xf)(n) = \sum_{k=0}^{\infty} a_{nk}f(k).$$

Proposition 2.5. *If $0 < \lambda \leq 1/2$ then the Euler operator E_λ is bounded with $\|E_\lambda\| \leq (1-\lambda)^{-1/2}$.*

Proof. We shall apply the Schur test to the infinite matrix

$$a_{nk} = \begin{cases} \binom{n}{k}\lambda^k(1-\lambda)^{n-k}, & \text{if } 0 \leq k \leq n, \\ 0, & \text{if } k > n. \end{cases} \tag{2.14}$$

If we set $p_k = 1$, then it follows from the binomial theorem that

$$\sum_{k=0}^{\infty} a_{nk}p_k = \sum_{k=0}^{n} \binom{n}{k}\lambda^k(1-\lambda)^{n-k} = 1.$$

On the other hand, using the geometric series expansion $(1-\lambda)^{-1} = \sum_{n=0}^{\infty} \lambda^n$, we get

$$\frac{d^k}{d\lambda^k}(1-\lambda)^{-1} = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!}(1-\lambda)^{n-k},$$

so that

$$\begin{aligned} \sum_{n=0}^{\infty} a_{nk}p_n &= \sum_{n=k}^{\infty} \binom{n}{k}\lambda^k(1-\lambda)^{n-k} = \frac{\lambda^k}{k!}\sum_{n=k}^{\infty} \frac{n!}{(n-k)!}(1-\lambda)^{n-k} \\ &= \frac{\lambda^k}{k!}\frac{d^k}{d\lambda^k}(1-\lambda)^{-1} = \lambda^k(1-\lambda)^{-k-1} \leq (1-\lambda)^{-1}, \end{aligned}$$

and we conclude that E_λ is bounded with $\|E_\lambda\| \le (1-\lambda)^{-1/2}$, as we wanted. □

We shall use an elementary fact that can be stated as follows.

Lemma 2.6. *Let λ be a nonzero complex number. Then $|\lambda| + |\lambda - 1| \le 1$ if and only if $\lambda \in (0,1]$.*

Proof. Let us prove the nontrivial implication. If $\lambda \in \mathbb{R}$ and $\lambda > 1$ then we have $|\lambda| + |1-\lambda| = 2\lambda - 1 > 1$, and if $\lambda \in \mathbb{R}$ and $\lambda < 0$ then $|\lambda| + |1-\lambda| = 1 - 2\lambda > 1$. Also, if $\lambda \in \mathbb{C}$ and $\operatorname{Im}\lambda \neq 0$ then $|\lambda| + |1-\lambda| > 1$ because λ and $1-\lambda$ are linearly independent over $\mathbb{R}$. □

Proposition 2.7. *If $\lambda \in \mathbb{C}\backslash(0,1]$ then the Euler operator E_λ is unbounded.*

Proof. We have for every $n \ge 0$

$$E_\lambda^* e_n = \sum_{k=0}^{n} \binom{n}{k} \lambda^k (1-\lambda)^{n-k} e_k.$$

Using the Cauchy-Schwarz inequality gives

$$\begin{aligned}
\|E_\lambda^* e_n\| &= \left\| \sum_{k=0}^{n} \binom{n}{k} \lambda^k (1-\lambda)^{n-k} e_k \right\| \\
&\ge \frac{1}{(n+1)^{1/2}} \sum_{k=0}^{n} \binom{n}{k} |\lambda|^k |1-\lambda|^{n-k} \\
&= \frac{(|\lambda| + |1-\lambda|)^n}{(n+1)^{1/2}}.
\end{aligned}$$

If $\lambda \neq 0$ then it follows from Lemma 2.6 that $\|E_\lambda^* e_n\| \to \infty$ as $n \to \infty$. Finally, if $\lambda = 0$ then according to equation (2.2) we have $(E_0 f)(n) = f(0)$ for all $n \in \mathbb{N}$, so that the constant sequence $E_0 f$ belongs to the complex Hilbert space ℓ^2 only when $f(0) = 0$. □

3. Extended eigenvalues for operators with rich point spectrum

We say that an operator T on a complex Banach space has *rich point spectrum* provided that $\operatorname{int}\sigma_p(T) \neq \emptyset$, and that for every open disc $D \subseteq \sigma_p(T)$, the family of eigenvectors

$$\bigcup_{z \in D} \ker(T - z) \tag{3.1}$$

is a total set. We shall see below that two examples of operators with rich point spectrum are the finite continuous Cesàro operator and the adjoint of the discrete Cesàro operator. There are other natural examples like a bilateral weighted shift whose point spectrum has non empty interior, or the adjoint of an analytic Toeplitz operator with non constant symbol.

Recall that if φ is a bounded analytic function on $\mathbb{D}$ then the *analityc Toeplitz operator* T_φ is defined on the Hardy space $H^2(\mathbb{D})$ by the expression $T_\varphi f = \varphi \cdot f$. Deddens [7] studied intertwining relations between analytic Toeplitz operators. Bourdon and Shapiro [5] generalized his work later on and they applied it to study the extended eigenvalues of an analytic Toeplitz operator.

Deddens showed that if there is a non zero operator X that intertwines two analytic Toeplitz operators T_φ and T_ψ, that is, such that $XT_\varphi = T_\psi X$, then

$$\psi(\mathbb{D}) \subseteq \operatorname{clos}\varphi(\mathbb{D}). \tag{3.2}$$

Bourdon and Shapiro observed that, as a consequence of this, if λ is an extended eigenvalue of an analytic Toeplitz operator T_φ, where φ is not constant, then there is a non zero operator that intertwines $T_{\lambda\varphi}$ and T_φ, so that

$$(1/\lambda) \cdot \varphi(\mathbb{D}) \subseteq \operatorname{clos}\varphi(\mathbb{D}). \tag{3.3}$$

Bourdon and Shapiro say that then the geometry of $\varphi(\mathbb{D})$ quickly determines the extended eigenvalues of T_φ (for instance, if λ is an extended eigenvalue of the shift operator $T_z \in \mathcal{B}(H^2(\mathbb{D}))$ then it follows from Deddens result that $(1/\lambda) \cdot \mathbb{D} \subseteq \operatorname{clos} \mathbb{D}$, and therefore $|\lambda| \geq 1$.)

We prove in Theorem 3.1 that, in general, if an operator has rich point spectrum then the geometry of its point spectrum determines the extended eigenvalues. The precise statement of this result is provided below. Then, we apply Theorem 3.1 to show that if λ is an extended eigenvalue for C_1 on $L^p[0,1]$ then λ is real and $0 < \lambda \leq 1$ (Corollary 4.5) and if λ is an extended eigenvalue for C_0^* on ℓ^p then λ is real and $\lambda \geq 1$ (Corollary 5.3).

As another consequence of our general result, in section 6 we get that if λ is an extended eigenvalue of a bilateral weighted shift W whose point spectrum has non empty interior then $|\lambda| = 1$.

Finally, if λ is an extended eigenvalue of an analytic Toeplitz operator T_φ on the Hardy space $H^2(\mathbb{D})$ with non constant symbol then Deddens result (3.3) can be derived as a consequence of Theorem 3.1.

Theorem 3.1. *Let us suppose that an operator T on a complex Banach space has rich point spectrum. If λ is an extended eigenvalue for T then we have*

$$\lambda \cdot \operatorname{int} \sigma_p(T) \subseteq \operatorname{clos} \sigma_p(T). \tag{3.4}$$

Proof. Let X be an extended eigenoperator of T corresponding to the extended eigenvalue λ, that is, $X \neq 0$ and $TX = \lambda XT$. Let $z \in \operatorname{int} \sigma_p(T)$ and let $n \in \mathbb{N}$ such that $D(z, 1/n) \subseteq \sigma_p(T)$. Since $X \neq 0$ and T has rich point spectrum, there exist $z_n \in D(z, 1/n)$ and $f_n \in \ker(T - z_n) \backslash \{0\}$ such that $Xf_n \neq 0$. Hence,

$$TXf_n = \lambda XTf_n = \lambda z_n Xf_n,$$

and since $Xf_n \neq 0$, this means that $\lambda z_n \in \sigma_p(T)$. Taking limits as $n \to \infty$ yields $\lambda z \in \operatorname{clos} \sigma_p(T)$, as we wanted. □

The following result will be applied at the end of the next section to the finite continuous Cesàro operator and in section 5 to the adjoint of the discrete Cesàro operator.

Theorem 3.2. *Let T be a bounded linear operator with rich point spectrum and such that $\sigma_p(T) = D(r, r)$ for some $r > 0$. If λ is an extended eigenvalue for T then λ is real and $0 < \lambda \leq 1$.*

Proof. Let $\mu = 1/\lambda$. We must show that μ is real and $\mu \geq 1$. First of all, consider the open half plane $\Omega_r = \{w \in \mathbb{C} \colon \operatorname{Re} w > 1/(2r)\}$, and notice that $z \in D(r, r)$ if and only if $1/z \in \Omega_r$. According to Lemma 3.1 we have $\mu w \in \overline{\Omega}_r$ for every $w \in \Omega_r$. This means that the map $\varphi(w) = \mu w$ takes Ω_r into $\overline{\Omega}_r$, and it follows from continuity that φ takes the closed half plane $\overline{\Omega}_r$ into itself. Now start with a point $w \in \Omega_r \cap \mathbb{R}$ and iterate the map φ to get a sequence of points $(\mu^n w)$ in $\overline{\Omega}_r$, so that $\operatorname{Re}(\mu^n w) \geq 1/(2r)$, or in other words,

$$\operatorname{Re}\left[\left(\frac{\mu}{|\mu|}\right)^n\right] \geq \frac{1}{2rw|\mu|^n} > 0.$$

Finally, write $\mu = |\mu|(\cos\theta + i \sin\theta)$ for some $0 \leq \theta < 2\pi$. Observe that $\cos n\theta > 0$ for all $n \in \mathbb{N}$, and this can only happen if $\theta = 0$. This shows that μ is real. It is clear that $\mu \geq 1$ because if $\mu < 1$ then φ maps $\overline{\Omega}_r$ outside $\overline{\Omega}_r$; for instance $\varphi(1/(2r)) = \mu/(2r) < 1/(2r)$, and this is a contradiction. □

The following result will be applied in section 6 to a bilateral weighted shift.

Theorem 3.3. *Let T be a bounded linear operator with rich point spectrum such that for some $0 < r < R$,*

$$\{z \in \mathbb{C} \colon r < |z| < R\} \subseteq \sigma_p(T) \subseteq \{z \in \mathbb{C} \colon r \leq |z| \leq R\}.$$

If λ is an extended eigenvalue of T then $|\lambda| = 1$.

Proof. Consider the region $\Omega = \{z \in \mathbb{C}\colon r < |z| < R\}$. It follows from Lemma 3.1 that the map $\varphi(z) = \lambda z$ takes Ω into $\overline{\Omega}$, and it follows from continuity that φ maps $\overline{\Omega}$ into itself. Start with $z_0 \in \overline{\Omega}$ and iterate the map φ to obtain a sequence of points $(\lambda^n z_0)$ in $\overline{\Omega}$, so that for all $n \in \mathbb{N}$ we have

$$r \leq |\lambda|^n \cdot |z_0| \leq R,$$

and notice that this can only happen if $|\lambda| = 1$. □

The following result provides a sufficient condition for a general operator to have rich point spectrum. We apply that condition in the next section to show that the finite continuous Cesàro operator C_1 on $L^p[0,1]$ has rich point spectrum. We also apply our sufficient condition in section 5 to the adjoint of the discrete Cesàro operator, C_0^* on ℓ^q, and in section 6 to a bilateral weighted shift W whose point spectrum has non empty interior. Finally, in section 7 we apply a suitable modification of that condition to the adjoint of an analytic Toeplitz operator, T_φ^* where φ is non constant.

Lemma 3.4. *Let T be a bounded linear operator on a complex Banach space E and let us suppose that there is an analytic mapping $h\colon \operatorname{int}\sigma_p(T) \to E$ with $h(z) \in \ker(T-z)\backslash\{0\}$ for all $z \in \operatorname{int}\sigma_p(T)$ and such that $\{h(z)\colon z \in \operatorname{int}\sigma_p(T)\}$ is a total subset of E. Then T has rich point spectrum.*

Proof. Let D be an open disc contained in $\sigma_p(T)$ and let $g^* \in E^*$ such that $\langle h(z), g^*\rangle = 0$ for all $z \in D$. We must show that then $g^* = 0$. We consider the analytic function $\varphi\colon \operatorname{int}\sigma_p(T) \to \mathbb{C}$ defined by $\varphi(z) = \langle h(z), g^*\rangle$. We have by assumption that φ vanishes on D. Then, it follows from the principle of analytic continuation that φ vanishes on $\operatorname{int}\sigma_p(T)$. Since the family of eigenvectors $\{h(z)\colon z \in \operatorname{int}\sigma_p(T)\}$ is a total set, it follows that $g^* = 0$, as we wanted. □

We finish this section with a more general formulation of Theorem 3.1 for intertwining operators.

Theorem 3.5. *Let T, S be two bounded linear operators on a complex Banach space, and suppose that there is some X that intertwines T, S, that is, $X \neq 0$ and $XT = SX$. If T has rich point spectrum then*

$$\operatorname{int}\sigma_p(T) \subseteq \operatorname{clos}\sigma_p(S).$$

Proof. Let $z \in \operatorname{int}\sigma_p(T)$ and let $n \in \mathbb{N}$ such that $D(z, 1/n) \subseteq \sigma_p(T)$. Since $X \neq 0$ and since T has rich point spectrum, there exist $z_n \in D(z, 1/n)$ and $f_n \in \ker(T-z)\backslash\{0\}$ such that $Xf_n \neq 0$. Hence,

$$SXf_n = XTf_n = z_n X f_n,$$

and since $Xf_n \neq 0$, this means that $z_n \in \sigma_p(S)$. Taking limits as $n \to \infty$ yields $z \in \operatorname{clos}\sigma_p(S)$. □

Notice that Theorem 3.1 becomes a special case of Theorem 3.5 since λ is an extended eigenvalue for T if and only if there is some non zero operator that intertwines λT and T.

4. The finite continuous Cesàro operator on Lebesgue spaces

Now we focus on the extended eigenvalues and extended eigenoperators for the Cesàro operator C_1 defined on the Lebesgue spaces $L^p[0,1]$ for $1 < p < \infty$ by the integral means (1.2). Leibowitz [15] showed that C_1 is indeed a bounded operator on $L^p[0,1]$ and he computed its spectrum and its point spectrum.

Theorem 4.1. *If $0 < \lambda \leq 1$ then λ is an extended eigenvalue for the Cesàro operator C_1 on $L^p[0,1]$ and a corresponding extended eigenoperator is the weighted composition operator $X_0 \in \mathcal{B}(L^p[0,1])$ defined by*

$$(X_0 f)(x) = x^{(1-\lambda)/\lambda} f(x^{1/\lambda}). \tag{4.1}$$

Proof. First of all, let us show that X_0 is indeed a bounded linear operator. We have for every $f \in L^p[0,1]$

$$\int_0^1 |(X_0 f)(x)|^p \, dx = \int_0^1 x^{p(1-\lambda)/\lambda} |f(x^{1/\lambda})|^p \, dx$$
$$= \lambda \int_0^1 y^{(p-1)(1-\lambda)} |f(y)|^p \, dy \leq \lambda \int_0^1 |f(y)|^p \, dy,$$

and this shows that X_0 is bounded on $L^p[0,1]$ with $\|X_0\| \leq \lambda^{1/p}$.

Now let us show that X_0 is an extended eigenoperator of C_1 associated with the extended eigenvalue λ. Let $n \in \mathbb{N}$ and notice that $X_0 x^n = x^{(n+1-\lambda)/\lambda}$, so that

$$C_1 X_0 x^n = C_1 x^{(n+1-\lambda)/\lambda} = \frac{\lambda}{n+1} x^{(n+1-\lambda)/\lambda} = \frac{\lambda}{n+1} X_0 x^n = \lambda X_0 C_1 x^n,$$

and since the linear subspace $\operatorname{span}\{x^n : n \in \mathbb{N}\}$ is a dense subset of $L^p[0,1]$, it follows that $C_1 X_0 = \lambda X_0 C_1$, that is, X_0 is an extended eigenoperator of C_1 associated with the extended eigenvalue λ. □

Our next goal is to show that if λ is an extended eigenvalue of the finite continuous Cesàro operator $C_1 \in \mathcal{B}(L^p[0,1)])$ then λ is real and $0 < \lambda \leq 1$. First we show that C_1 has rich point spectrum. Let $1 < p, q < \infty$ be a pair of conjugate indices, that is,

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Leibowitz [15] proved the following result about the point spectrum of C_1.

Lemma 4.2. *The point spectrum of the Cesàro operator C_1 on $L^p[0,1]$ is the open disc $D(q/2, q/2)$. Moreover, each $z \in D(q/2, q/2)$ is a simple eigenvalue of C_1 and a corresponding eigenfunction is given by $h_z(x) = x^{(1-z)/z}$.*

The following theorem was conjectured by Borwein and Erdélyi [4] and it was proven by Operstein [16].

Theorem 4.3. (Full Müntz theorem in $L^p[0,1]$.) *Let $1 < p < \infty$ and let (r_n) be a sequence of distinct real numbers greater than $-1/p$. Then the linear subspace* $\operatorname{span}\{x^{r_0}, x^{r_1}, \ldots, x^{r_n}, \ldots\}$ *is dense in $L^p[0,1]$ if and only if*

$$\sum_{n=0}^{\infty} \frac{r_n + 1/p}{(r_n + 1/p)^2 + 1} = \infty. \tag{4.2}$$

Theorem 4.4. *The finite continuous Cesàro operator C_1 on $L^p[0,1]$ has rich point spectrum.*

Proof. Notice that $\sigma_p(C_1) = D(q/2, q/2)$ is open and connected. Also, the mapping $h \colon \sigma_p(C_1) \to L^p[0,1]$ defined by $h(z)(x) = x^{(1-z)/z}$ is analytic, and $h(z) \in \ker(C_1 - z) \setminus \{0\}$. It is a standard consequence of the full Müntz theorem that the family of eigenfunctions $\{h(z) \colon z \in D(q/2, q/2)\}$ is total in $L^p[0,1]$. Indeed, it suffices to consider a sequence of distinct real numbers (z_n) with $q/2 < z_n < q$ and such that $\lim z_n = q$ as $n \to \infty$, since the sequence of exponents $r_n = (1 - z_n)/z_n$ clearly satisfies the condition (4.2). The result now follows from Lemma 3.4. □

Corollary 4.5. *If λ is an extended eigenvalue for C_1 on $L^p[0,1]$ then λ is real and* $0 < \lambda \leq 1$.

Proof. This is a consequence of Theorem 3.2 now that we know that C_1 has rich point spectrum and that its point spectrum is the open disc $D(q/2, q/2)$. □

5. The discrete Cesàro operator on sequence spaces

We shall prove in this section that the set of the extended eigenvalues for the discrete Cesàro operator is the interval $[1,\infty)$ when $p=2$ and that it is contained in the interval $[1,\infty)$ when $1<p<\infty$. Let us recall that the discrete Cesàro operator C_0 is defined on the complex Banach space ℓ^p by the sequence of arithmetic means (1.1). Rhoades [19] showed that C_0 is indeed a bounded linear operator whose point spectrum is empty and he proved the following result about the point spectrum of the adjoint operator C_0^*.

Theorem 5.1. *The point spectrum of C_0^* on the complex Banach space ℓ^q is the open disc $D(q/2,q/2)$. Moreover, every $z\in D(q/2,q/2)$ is a simple eigenvalue for C_0^* and a corresponding eigenvector is the sequence $h(z)=(h_n(z))_{n\in\mathbb{N}}$ defined by the relations*

$$h_0(z)=1,\qquad h_n(z)=\prod_{k=1}^{n}\left(1-\frac{1}{kz}\right)\quad \text{for } n\geq 1. \tag{5.1}$$

Our first goal is to show that if λ is an extended eigenvalue for C_0 on ℓ^p then λ is real and $\lambda\geq 1$. Notice that the method that we applied to C_1 in section 3 does not apply to C_0 because the point spectrum of C_0 is empty. We consider instead its adjoint C_0^*.

Theorem 5.2. *The adjoint of the discrete Cesàro operator $C_0^*\in\mathcal{B}(\ell^q)$ has rich point spectrum.*

Proof. Notice that $\sigma_p(C_0^*)=D(q/2,q/2)$ is open and connected. It is easy to see that the mapping $h\colon\sigma_p(C_0^*)\to\ell^q$ defined by equation (5.1) is analytic, and $h(z)\in\ker(C_0^*-z)\backslash\{0\}$. It is a standard fact that the family of eigenvectors $\{h(z)\colon z\in D(q/2,q/2)\}$ is total in ℓ^q. As a matter of fact, the family of eigenvectors $\{f(1/k)\colon k\in\mathbb{N}\}$ is total in ℓ^q, because $f_n(1/k)\neq 0$ if and only if $n<k$. The result now follows at once from Lemma 3.4. ☐

Corollary 5.3. *If λ is an extended eigenvalue of C_0 on ℓ^p then λ is real and $\lambda\geq 1$.*

Proof. First of all, we have $\lambda\neq 0$ because C_0 is injective. Also, notice that λ is an extended eigenvalue for C_0 if and only if $1/\overline{\lambda}$ is an extended eigenvalue for C_0^*, and therefore it is enough to show that if λ is an extended eigenvalue for C_0^* then λ is real and $0<\lambda\leq 1$. This becomes a consequence of Theorem 3.2 now that we know that C_0^* has rich point spectrum and that its point spectrum is the disc $D(q/2,q/2)$. ☐

Our next goal is to show in the Hilbertian case $p=2$ that if λ is real and $\lambda\geq 1$ then λ is an extended eigenvalue for C_0. Kriete and Trutt [11] showed that C_0 is subnormal using the following construction. Let μ be a positive finite measure defined on the Borel subsets of the complex plane with compact support and let $H^2(\mu)$ be the closure of the polynomials on the Hilbert space $L^2(\mu)$. Consider the shift operator M_z defined on the Hilbert space $H^2(\mu)$ by the expression $(M_zf)(z)=zf(z)$. Kriete and Trutt [11] showed that there is a is a positive finite measure defined on the Borel subsets of the complex plane and supported on $\overline{\mathbb{D}}$, and there is a unitary operator $U\colon\ell^2\to H^2(\mu)$ such that

$$I-C_0=U^*M_zU,$$

or in other words

$$C_0=U^*(I-M_z)U.$$

Then, the extended eigenvalues of C_0 are the extended eigenvalues of $I-M_z$ and the corresponding extended eigenoperators of C_0 are in one to one correspondence with the extended eigenoperators of $I-M_z$ under conjugation with U, that is, if a non-zero operator X satisfies $(I-M_z)X=\lambda X(I-M_z)$ then the operator $Y=U^*XU$ satisfies $C_0Y=\lambda YC_0$.

Theorem 5.4. *If $\lambda\geq 1$ then λ is an extended eigenvalue for $I-M_z$ and a corresponding extended eigenoperator is the composition operator X defined by the expression*

$$(Xf)(z)=f\left(\frac{\lambda-1}{\lambda}+\frac{z}{\lambda}\right). \tag{5.2}$$

Proof. Let $f_n = Xz^n = \left(\frac{\lambda-1}{\lambda} + \frac{z}{\lambda}\right)^n$. We have $f_{n+1} = \left(\frac{\lambda-1}{\lambda} + \frac{z}{\lambda}\right) f_n$ so that

$$\begin{aligned}\lambda f_{n+1} &= [(\lambda-1)+M_z]f_n \\ &= \lambda f_n - (I-M_z)f_n\end{aligned}$$

and it follows that

$$(I-M_z)f_n = \lambda(f_n - f_{n+1})$$

so that

$$\begin{aligned}(I-M_z)Xz^n &= (I-M_z)f_n \\ &= \lambda(f_n - f_{n+1}) \\ &= \lambda(Xz^n - XM_z z^n) \\ &= \lambda X(I-M_z)z^n,\end{aligned}$$

and since the family of monomials $\{z^n : n \in \mathbb{N}\}$ is a total set in $H^2(\mu)$, it follows that $(I-M_z)X = \lambda X(I-M_z)$. □

Corollary 5.5. *If $\lambda \geq 1$ then λ is an extended eigenvalue for the discrete Cesàro operator C_0 on ℓ^2.*

6. Extended eigenvalues for bilateral weighted shifts

The third author [18] showed that the set of extended eigenvalues for an injective unilateral weighted shift is either $\mathbb{C}\backslash\mathbb{D}$ or $\mathbb{C}\backslash\{0\}$. We consider in this section the extended eigenvalues for a bilateral weighted shift W on an infinite dimensional, separable complex Hilbert space H, that is,

$$We_n = w_n e_{n+1}, \qquad n \in \mathbb{Z}, \tag{6.1}$$

where $(e_n)_{n\in\mathbb{Z}}$ is an orthonormal basis of H and the sequence $(w_n)_{n\in\mathbb{Z}}$ of non-zero weights is bounded.

Theorem 6.1. *Let us suppose that an operator T on a complex Banach space is similar to αT for some complex number α. If λ is an extended eigenvalue for T then $\lambda\alpha$ is an extended eigenvalue for T.*

Proof. Let S be an invertible operator such that $\alpha T = S^{-1}TS$. Let X be an extended eigenoperator associated with an extended eigenvalue λ of T. We have

$$TX = \lambda XT = \lambda(XS)(S^{-1}T), \tag{6.2}$$

so that

$$T(XS) = \lambda(XS)(S^{-1}TS) = \lambda\alpha(XS)T. \tag{6.3}$$

Notice that $XS \neq 0$ because $X \neq 0$ and S is onto. This means that $\lambda\alpha$ is an extended eigenvalue for T and XS is a corresponding extended eigenoperator. □

Theorem 6.2. *If W is a bilateral weighted shift then every $\lambda \in \mathbb{T}$ is an extended eigenvalue for W.*

Proof. Notice that if W is a bilateral weighted shift and if $\theta \in \mathbb{R}$ then W is unitarily equivalent to $e^{i\theta}W$. Hence, it follows from Theorem 6.1 with $\alpha = e^{i\theta}$ and $\lambda = 1$ that $e^{i\theta}$ is an extended eigenvalue for W. Thus, the unit circle $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ is contained in the set of extended eigenvalues for W. □

Shkarin [22] constructed an example of a compact, quasinilpotent bilateral weighted shift W so that the set of extended eigenvalues of W is the unit circle.

Now we consider the point spectrum of a bilateral weighted shift. We shall follow the discussion in the classical survey on weighted shift operators by Allen L. Shields [21]. Let us consider the quantities

$$r_3^+(W)\colon = \limsup_{n\to\infty} |w_0 \cdots w_{n-1}|^{1/n}, \tag{6.4}$$

$$r_2^-(W)\colon = \liminf_{n\to\infty} |w_{-1} \cdots w_{-n}|^{1/n}. \tag{6.5}$$

It turns out that when $r_3^+(W) < r_2^-(W)$ we have

$$\{z \in \mathbb{C}\colon r_3^+(W) < |z| < r_2^-(W)\} \subseteq \sigma_p(W) \tag{6.6}$$

$$\sigma_p(W) \subseteq \{z \in \mathbb{C}\colon r_3^+(W) \le |z| \le r_2^-(W)\}. \tag{6.7}$$

Also, every $z \in \mathbb{C}$ with $r_3^+(W) < |z| < r_2^-(W)$ is a simple eigenvalue of W and a corresponding eigenvector is given by the expression

$$h(z) = e_0 + \sum_{n=1}^{\infty} \frac{w_0 \cdots w_{n-1}}{z^n}\, e_n + \sum_{n=1}^{\infty} \frac{z^n}{w_{-1} \cdots w_{-n}}\, e_{-n}. \tag{6.8}$$

Theorem 6.3. *Let W be an injective bilateral weighted shift on an infinite-dimensional, separable complex Hilbert space and suppose that $r_3^+(W) < r_2^-(W)$. If λ is an extended eigenvalue for W then $|\lambda| = 1$.*

Proof. This result becomes a consequence of Theorem 3.3 if we can show that W has rich point spectrum. First of all, the interior of the point spectrum of W is the open annulus

$$G = \{z \in \mathbb{C}\colon r_3^+(W) < |z| < r_2^-(W)\}. \tag{6.9}$$

Notice that this annulus is connected. Consider the analytic function $h\colon G \to H$ defined by equation (8.9). We have $h(z) \in \ker(W - z)\backslash\{0\}$. We must show that the family of eigenvectors $\{h(z)\colon z \in G\}$ is a total subset of H. Take any vector $g = \sum b_n e_n \in H$ and suppose that $\langle f(z), g\rangle = 0$ for all $z \in G$. We ought to show that then $g = 0$. Consider the complex function $\varphi\colon G \to \mathbb{C}$ defined by $\varphi(z) = \langle f(z), g\rangle$, so that

$$\varphi(z) = \overline{b}_0 + \sum_{n=1}^{\infty} \overline{b}_n\, w_0 \cdots w_{n-1}\, \frac{1}{z^n} + \sum_{n=1}^{\infty} \frac{\overline{b}_{-n}}{w_{-1} \cdots w_{-n}}\, z^n, \quad z \in G. \tag{6.10}$$

Thus, φ is analytic and it vanishes identically on G. Hence, $b_n = 0$ for all $n \in \mathbb{Z}$, that is, $g = 0$. ☐

Let $\lambda \in \mathbb{T}$ and let us consider the diagonal operator $X_0 = \operatorname{diag}(\lambda^{-n})_{n\in\mathbb{Z}}$. We have

$$X_0 h(z) = e_0 + \sum_{n=1}^{\infty} \frac{w_0 \cdots w_{n-1}}{\lambda^n z^n}\, e_n + \sum_{n=1}^{\infty} \frac{\lambda^n z^n}{w_{-1} \cdots w_{-n}}\, e_{-n} = h(\lambda z),$$

and it follows that

$$W X_0 h(z) = W h(\lambda z) = \lambda z h(\lambda z) = \lambda z X_0 h(z) = \lambda X_0 W h(z),$$

and since the family of eigenvectors $\{h(z)\colon z \in \Omega\}$ is a total set, it follows that $W X_0 = \lambda X_0 W$, so that X_0 is an extended eigenoperator for W associated with the extended eigenvalue λ. Notice that X_0 is a unitary operator since $|\lambda| = 1$.

7. Extended eigenvalues for analytic Toeplitz operators

Now we focus on Deddens result (3.3) and we show that it can be viewed as a special case of Lemma 3.1. We first show that the adjoint of a non trivial Toeplitz operator has rich point spectrum. The following result is a generalization of Lemma 3.4 that suits the case of the adjoint of an analytic Toeplitz operator.

Lemma 7.1. *Let T be a bounded linear operator on a complex Banach space E and suppose that there is an open connected set $G \subseteq \mathbb{C}$, an analytic mapping $h\colon G \to E$ and a non constant analytic function $\psi\colon G \to \mathbb{C}$ so that*

1. $h(z) \in \ker[T - \psi(z)]\backslash\{0\}$ *for all $z \in G$, and*
2. $\{h(z)\colon z \in G\}$ *is a total set.*

Then T has rich point spectrum.

Proof. Since ψ is a non constant function, it follows from the open mapping theorem that $\psi(G)$ is open. Now it follows from the first condition that $\psi(G)$ is contained in $\sigma_p(T)$, so that $\operatorname{int}\sigma_p(T)$ is non empty. Then let $D \subseteq \sigma_p(T)$ be an open disc, let $G_0 = \psi^{-1}(D)$ and let us show that the family of eigenvectors $\{f(z)\colon z \in G_0\}$ corresponding to eigenvalues $\psi(z) \in D$ is a total subset of E. Let $g^* \in E^*$ be a functional such that $\langle f(z), g^*\rangle = 0$ for all $z \in G_0$. We must show that then $g^* = 0$. Consider the analytic function $\varphi\colon G \to \mathbb{C}$ defined by $\varphi(z) = \langle f(z), g^*\rangle$. We have by assumption that φ vanishes on G_0. Now it follows from the principle of analytic continuation that φ vanishes on G. Since the family of eigenvectors $\{f(z)\colon z \in G\}$ is a total subset of E, it follows that $g^* = 0$, as we wanted. □

Theorem 7.2. *If the symbol φ is not constant then the adjoint operator T_φ^* has rich point spectrum.*

Proof. It suffices to show that T_φ^* satisfies the conditions of Lemma 7.1. Recall that the reproducing kernel K_z is the function defined for every $z \in \mathbb{D}$ by the expression

$$K_z(w) = \frac{1}{1 - \overline{z}w}, \tag{7.1}$$

and it has the property that $\langle f, K_z\rangle = f(z)$ for all $f \in H^2(\mathbb{D})$. It is easy to see that for all $z \in \mathbb{D}$ we have

$$T_\varphi^* K_z = \overline{\varphi(z)} K_z. \tag{7.2}$$

Then, consider the analytic function $f\colon \mathbb{D} \to H^2(\mathbb{D})$ defined by $f(z) = K_{\overline{z}}$. We have $T_\varphi^* f(z) = \overline{\varphi(\overline{z})} f(z)$, so that the first condition in Lemma 7.1 is satisfied by the analytic function $\psi(z) = \overline{\varphi(\overline{z})}$. Moreover, it is clear that the family of eigenvectors $\{f(z)\colon z \in \mathbb{D}\}$ is a total subset of $H^2(\mathbb{D})$. □

Deddens results (3.2) and (3.3) now follow easily.

Corollary 7.3. *If there is an operator X that intertwines two analytic Toeplitz operators T_φ and T_ψ, that is, such that $XT_\varphi = T_\psi X$, then (3.2) holds.*

Proof. Taking adjoints yields $T_\varphi^* X^* = X^* T_\psi^*$ with $X^* \neq 0$. This means that X^* intertwines T_ψ^* and T_φ^*, and from Theorem 3.5 we get $\operatorname{int}\sigma_p(T_\psi^*) \subseteq \operatorname{clos}\sigma_p(T_\varphi^*)$. We have on the one hand $\overline{\psi(\mathbb{D})} \subseteq \operatorname{int}\sigma_p(T_\psi^*)$ and on the other hand $\operatorname{clos}\sigma_p(T_\varphi) \subseteq \sigma(T_\varphi^*) = \operatorname{clos}\overline{\varphi(\mathbb{D})}$, so that $\overline{\psi(\mathbb{D})} \subseteq \operatorname{clos}\overline{\varphi(\mathbb{D})}$, as we wanted. □

Corollary 7.4. *If the symbol φ is not constant and if λ is an extended eigenvalue of T_φ then (3.3) holds.*

8. Factorization of extended eigenoperators in Hilbert space

Now we consider the problem of describing, for an operator on a complex Hilbert space, the family of all the extended eigenoperators corresponding to an extended eigenvalue.

Notice that if X_0 is a particular extended eigenoperator for an operator T corresponding to an extended eigenvalue $\lambda \in \mathbb{C}$ and if $R \in \{T\}'$ then $X_0 R$ is an extended eigenoperator for T corresponding to λ. It is natural to ask whether or not all the extended eigenoperators arise in this fashion. We provide a factorization result in Theorem 8.1 under certain conditions that are fulfilled by any bilateral weighted shift whose point spectrum has non-empty interior.

Our result is based on the construction of an analytic reproducing kernel space $\mathcal{H}$ for an operator T with the nice property that the shift operator M_z is bounded on $\mathcal{H}$ and that T^* is unitarily equivalent to the shift operator M_z on the space $\mathcal{H}$. The construction in the particular case of the operator $T = I - C_0^*$ appears in the paper by Shields and Wallen [20] and also in the papers by Kriete and Trutt [11, 12].

Then we apply this result to show that if W is a bilateral weighted shift whose point spectrum has non-empty interior then W has the property that every extended eigenoperator X corresponding to an extended eigenvalue $\lambda \in \mathbb{T}$ factors as a product $X = X_0 R$, where $X_0 = \operatorname{diag}(\lambda^{-n})_{n\in\mathbb{Z}}$ is a unitary diagonal operator (a particular extended eigenoperator) and where $R \in \{W\}'$.

We also discuss the applicability of this result to the finite continuous Cesàro operator or the adjoint of the discrete Cesàro operator.

Let us recall that an *analytic reproducing kernel space* on an open set $G \subseteq \mathbb{C}$ is a Hilbert space $\mathcal{H}$ of analytic functions $f\colon G \to \mathbb{C}$ such that the point evaluations $f \mapsto f(w)$ are bounded linear functionals. If $\mathcal{H}$ is an analytic reproducing kernel space on G then for each $w \in G$ there exists $K_w \in \mathcal{H}$ such that $f(w) = \langle f, K_w\rangle$ for every $f \in \mathcal{H}$. The function $K\colon G \times G \to \mathbb{C}$ defined by the expression $K(z,w) = K_w(z)$ is called the *reproducing kernel* of $\mathcal{H}$. It follows from the reproducing property that

$$K(z,w) = K_w(z) = \langle K_w, K_z\rangle = \overline{\langle K_z, K_w\rangle} = \overline{K_z(w)} = \overline{K(w,z)}.$$

Since K is analytic in z, it follows that K is co-analytic in w, and K is said to be an *analytic kernel.*

If $\varphi\colon G \to \mathbb{C}$ is an analytic function such that $\varphi \cdot f \in \mathcal{H}$ for every $f \in \mathcal{H}$ then φ is called a *multiplier.* It follows from the closed graph theorem that the operator M_φ defined by $M_\varphi f = \varphi \cdot f$ is bounded.

Theorem 8.1. *Let T be an operator on a complex Hilbert space H, let $G \subseteq \mathbb{C}$ be an open connected set and suppose that there is an analytic mapping $h\colon G \to H$ such that*

(i) $\dim \ker(T - z) = 1$ *for every* $z \in G$,
(ii) $h(z) \in \ker(T - z)\backslash\{0\}$ *for every* $z \in G$,
(iii) $\{h(z)\colon z \in G\}$ *is a total subset of* H.

Then there exists an analytic reproducing kernel space $\mathcal{H}$ on G with the property that M_z is bounded on $\mathcal{H}$, and there exists a unitary operator $U\colon H \to \mathcal{H}$ such that $T^* = U^* M_z U$.

Proof. Let $f \in H$ and let $\hat{f}\colon G \to \mathbb{C}$ be the analytic function defined by the expression $\hat{f}(z) = \langle f, h(\overline{z})\rangle$. Let $\mathcal{H}$ be the Hilbert space of all functions $\hat{f}$ provided with the norm $\|\hat{f}\| = \|f\|$. It is clear that the map $U\colon H \to \mathcal{H}$ defined by $Uf = \hat{f}$ is a unitary operator and that for every $z \in G$ we have

$$\begin{aligned}(UT^*f)(z) &= \langle T^*f, h(\overline{z})\rangle \\ &= \langle f, Th(\overline{z})\rangle \\ &= \langle f, \overline{z}h(\overline{z})\rangle \\ &= \langle zf, h(\overline{z})\rangle \\ &= (M_z Uf)(z).\end{aligned}$$

It follows that $UT^* = M_z U$, so that M_z is bounded on $\mathcal{H}$, and $T^* = U^* M_z U$. ☐

The following result about multipliers is an important tool for the proof of Theorem 8.6. It is stated as Lemma 5 in the paper of Shields and Wallen [20].

Lemma 8.2. *If $\varphi \in H^\infty(G)$ then the multiplication operator M_φ defined by $M_\varphi f = \varphi \cdot f$ is a bounded linear operator on $\mathcal{H}$ with $\|M_\varphi\| = \|\varphi\|_\infty$.*

Another tool for the proof of Theorem 8.6 is a result that has been extracted with slight modifications from the proof of the main theorem in the paper by González and the second author [8].

Lemma 8.3. *Let $T \in \mathcal{B}(H)$ be an operator as in Theorem 8.1 and let $X \in \mathcal{B}(H)$. The following are equivalent:*

(a) $TX = XT$,

(b) there is a bounded analytic function $\varphi \colon G \to \mathbb{C}$ such that for all $z \in G$,

$$Xh(z) = \varphi(z)h(z). \tag{8.1}$$

Proof. First of all, if $TX = XT$ then $TXh(z) = XTh(z) = zXh(z)$, so that $Xh(z) \in \ker(T - z)$ and it follows from (i) that there is a function $\varphi \colon G \to \mathbb{C}$ such that $Xh(z) = \varphi(z)h(z)$. We claim that φ is analytic. Let $z_0 \in G$ and let $g \in H\backslash\{0\}$ such that $\langle h(z_0), g^* \rangle \neq 0$. Then we have

$$\varphi(z) = \frac{\langle Xh(z), g\rangle}{\langle h(z), g\rangle}, \tag{8.2}$$

so that φ is analytic at z_0 because it is the quotient of two analytic functions where the denominator does not vanish in a neighborhood of z_0. Also, it is clear that φ is bounded with $\|\varphi\|_\infty \leq \|X\|$. Conversely, suppose (b) holds. We have

$$\begin{aligned} TXh(z) &= \varphi(z)Th(z) \\ &= z\varphi(z)h(z) \\ &= zXh(z) \\ &= XTh(z). \end{aligned}$$

Finally, it follows from (iii) that $TX = XT$. □

The next result is the key to the factorization of an extended eigenoperator.

Lemma 8.4. *Let T be an operator as in Theorem 8.1 and let λ be an extended eigenvalue of T. Let us suppose that λ satisfies $\lambda \cdot G \subseteq G$ and let X be a corresponding extended eigenoperator. Then there exists an analytic function $\varphi \colon G \to \mathbb{C}$ such that for all $z \in G$ we have*

$$Xh(z) = \varphi(z)h(\lambda z). \tag{8.3}$$

Proof. First of all, since X is an extended eigenoperator corresponding to λ and since $h(z)$ is an eigenvector corresponding to z, we get

$$TXh(z) = \lambda XTh(z) = \lambda zXh(z)$$

for every $z \in G$. This means that $Xh(z) \in \ker(T - \lambda z)$, and it follows from (i) that there is a function $\varphi \colon G \to \mathbb{C}$ such that $Xh(z) = \varphi(z)h(\lambda z)$. We claim that φ is analytic. Indeed, let $z_0 \in G$ and let $g \in H$ such that $\langle f(\lambda z_0), g\rangle \neq 0$. Then

$$\varphi(z) = \frac{\langle Xh(z), g\rangle}{\langle h(\lambda z), g\rangle}, \tag{8.4}$$

so that φ is analytic at z_0 because it is the quotient of two analytic functions where the denominator does not vanish in a neighborhood of z_0. □

We say that an analytic reproducing kernel space $\mathcal{H}$ is *dilation invariant* provided that, for every $\lambda \in \mathbb{C}$ such that $\lambda G \subseteq G$, the composition operator Y_0 defined by the expression

$$(Y_0 \hat{f})(z) = \hat{f}(\lambda z). \tag{8.5}$$

is a bounded linear operator on $\mathcal{H}$.

Lemma 8.5. *Let us suppose that the model space $\mathcal{H}$ of Theorem 8.1 is dilation invariant, let λ be a complex scalar such that $\lambda G \subseteq G$, let Y_0 be the composition operator defined on $\mathcal{H}$ by equation (8.5), and set $X_0 = U^* Y_0 U$. Then λ is an extended eigenvalue for T and X_0 is a corresponding extended eigenoperator.*

Proof. We claim that $X_0 h(z) = h(\lambda z)$ for every $z \in G$. The result then follows easily because

$$\begin{aligned} TX_0 h(z) &= Th(\lambda z) \\ &= \lambda z h(\lambda z) \\ &= \lambda z X_0 h(z) \\ &= \lambda X_0 T h(z), \end{aligned}$$

and from (iii) we get $TX_0 = \lambda X_0 T$. Now, for the proof of our claim, observe that $UX_0 = Y_0 U$, so that $UX_0 h(z) = Y_0 U h(z) = Uh(\lambda z)$, and the claim follows. □

Theorem 8.6. *Suppose that the model space $\mathcal{H}$ of Theorem 8.1 is dilation invariant and that the extended eigenoperator X_0 of Theorem 8.5 is bounded below, i.e., there is a constant $c > 0$ such that $\|X_0 f\| \geq c\|f\|$. If X is an extended eigenoperator for T corresponding to λ then there exists $R \in \{T\}'$ such that $X = X_0 R$.*

Proof. First of all, apply Lemma 8.4 to find an analytic function $\varphi\colon G \to \mathbb{C}$ such that for all $z \in G$,

$$Xh(z) = \varphi(z) h(\lambda z). \tag{8.6}$$

Notice that $Xh(z) = \varphi(z) X_0 h(z)$, and since X_0 is bounded below, we get

$$|\varphi(z)| = \frac{\|Xh(z)\|}{\|X_0 h(z)\|} \leq \frac{1}{c} \cdot \frac{\|Xh(z)\|}{\|h(z)\|} \leq \frac{1}{c} \cdot \|X\|,$$

so that φ is bounded. Then, consider the analytic function $\psi(z) = \overline{\varphi(\overline{z})}$. Thus, $\psi \in H^\infty(G)$, and according to Lemma 8.1, the multiplication operator M_ψ defined by $M_\psi f = \psi \cdot f$ is a bounded linear operator on $\mathcal{H}$. Next, consider the operator $R = U^* M_\psi^* U$. We claim that for all $z \in G$ we have

$$Rh(z) = \varphi(z) h(z). \tag{8.7}$$

Indeed, from the definition of R we have

$$URh(z) = M_\psi^* U h(z),$$

so that for all $z, \xi \in G$ we get

$$\begin{aligned} [URh(z)](\xi) &= [M_\psi^* U h(z)](\xi) \\ &= \langle M_\psi^* U h(z), Uh(\overline{\xi}) \rangle \\ &= \langle Uh(z), M_\psi U h(\overline{\xi}) \rangle \\ &= \overline{\langle M_\psi U h(\overline{\xi}), Uh(z) \rangle} \\ &= \overline{[M_\psi U h(\overline{\xi})](\overline{z})} \\ &= \overline{\psi(\overline{z})} \cdot \overline{[Uh(\overline{\xi})](\overline{z})} \\ &= \varphi(z) \cdot \overline{\langle Uh(\overline{\xi}), Uh(z) \rangle} \\ &= \varphi(z) \cdot \langle Uh(z), Uh(\overline{\xi}) \rangle \\ &= \varphi(z) \cdot [Uh(z)](\xi), \end{aligned}$$

so that $URh(z) = \varphi(z)Uh(z)$ for all $z \in G$ and the claim follows. Finally, it follows from equation (8.7) and Lemma 8.3 that $R \in \{T\}'$. Moreover, $Xh(z) = \varphi(z)X_0h(z) = X_0Rh(z)$ for all $z \in G$, and it follows from (iii) that $X = X_0R$, as we wanted. □

Let W be an injective bilateral weighted shift on an infinite-dimensional, separable complex Hilbert space H, so that for every $n \in \mathbb{Z}$ we have

$$We_n = w_n e_{n+1}, \tag{8.8}$$

where $(e_n)_{n\in\mathbb{Z}}$ is an orthonormal basis of H and the sequence $(w_n)_{n\in\mathbb{Z}}$ of non-zero weights is bounded. Recall that the point spectrum of W is the open annulus $G = \{z \in \mathbb{C} \colon r_3^+(W) < |z| < r_2^-(W)\}$. Also, recall that every $z \in G$ is a simple eigenvalue of W and a corresponding eigenvector is given by

$$h(z) = e_0 + \sum_{n=1}^{\infty} \frac{w_0 \cdots w_{n-1}}{z^n}\, e_n + \sum_{n=1}^{\infty} \frac{z^n}{w_{-1} \cdots w_{-n}}\, e_{-n}. \tag{8.9}$$

It is easy to see that conditions (i), (ii) and (iii) of Theorem 8.1 are satisfied. Then, let $\lambda \in \mathbb{T}$ and consider the unitary diagonal operator $X_0 = \operatorname{diag}(\lambda^{-n})_{n\in\mathbb{Z}}$. A direct computation shows that that X_0 is an extended eigenoperator for W corresponding to the extended eigenvalue λ, and moreover, $X_0h(z) = h(\lambda z)$. Therefore, the model space $\mathcal{H}$ of Theorem 2.3 is dilation invariant, and the operator X_0 is bounded below. Thus, we get the following

Corollary 8.7. *Let W be an injective bilateral weighted shift on an infinite dimensional, separable complex Hilbert space and suppose that $r_3^+(W) < r_2^-(W)$. Let X be an extended eigenoperator for W correponding to some extended eigenvalue $\lambda \in \mathbb{T}$. Then X admits a factorization*

$$X = X_0R,$$

where $X_0 = \operatorname{diag}(\lambda^{-n})_{n\in\mathbb{Z}}$ is a unitary diagonal operator (a particular extended eigenoperator for T) and where $R \in \{W\}'$.

Let us see if Theorem 8.1 can be applied to C_1. Let $G = \{z \in \mathbb{C} \colon |z-1| < 1\}$ and let $h \colon G \to L^2[0,1]$ be the analytic mapping defined by the expression

$$h(z)(x) = x^{(1-z)/z}. \tag{8.10}$$

We have already seen that the conditions (i), (ii) and (iii) of Theorem 8.1 are fulfilled. Then, let $0 < \lambda \le 1$ and consider the weighted composition operator X_0 defined on $L^2[0,1]$ by the expression

$$(X_0f)(x) = x^{(1-\lambda)/\lambda} f(x^{1/\lambda}).$$

We know that X_0 is bounded with $\|X_0\| \le \lambda^{1/2}$ and that $X_0h(z) = h(\lambda z)$. It follows that the model space $\mathcal{H}$ is dilation invariant. However, we cannot apply Theorem 8.6 because X_0 is not bounded below. Indeed, if X_0 is bounded below then there is a constant $c > 0$ such that $\|X_0f\| \ge c\|f\|$ for all $f \in L^2[0,1]$, so that

$$\begin{aligned}
\frac{1}{c^2} &\ge \frac{\|h(z)\|_2^2}{\|X_0h(z)\|_2^2} \\
&= \frac{\|f(z)\|_2^2}{\|f(\lambda z)\|_2^2} \\
&= \frac{2\operatorname{Re}\dfrac{1-\lambda z}{\lambda z} + 1}{2\operatorname{Re}\dfrac{1-z}{z} + 1} \\
&= \frac{2\left(\dfrac{1-\lambda}{\lambda} + \dfrac{1}{\lambda}\operatorname{Re}\dfrac{1-z}{z}\right) + 1}{2\operatorname{Re}\dfrac{1-z}{z} + 1} \to \infty \quad \text{as } z \to 2,
\end{aligned}$$

and this is a contradiction.

Let us see if Theorem 8.1 can be applied to the adjoint of the discrete Cesàro operator. We consider the operator $T = VC_0^*V^* \in \mathcal{B}(H^2(\mathbb{D}))$ and the analytic mapping $h\colon G \to H^2(\mathbb{D})$ defined by the expression $h(z) = Vg(z)$, so that $h(z)(\xi) = (1-\xi)^{(1-z)/z}$. It is easy to see that h is analytic on G and that the conditions (i), (ii) and (iii) of Theorem 8.1 are satisfied. However, we cannot apply Theorem 8.5 because the model space $\mathcal{H}$ fails to be dilation invariant. Indeed, if $\mathcal{H}$ is dilation invariant then for every $0 < \lambda < 1$ there is a constant $c > 0$ such that $\|h(\lambda z)\| \le c\|h(z)\|$. When $\lambda = 1/2$, we set $z = 1/(n+1)$ and we get

$$h(z)(\xi) = (1-\xi)^n,$$
$$h(z/2)(\xi) = (1-\xi)^{2n+1},$$

so that for every $n \in \mathbb{N}$ we have

$$\|(1-\xi)^{2n+1}\|^2_{H^2(\mathbb{D})} \le c^2 \|(1-\xi)^n\|^2_{H^2(\mathbb{D})}.$$

Use the binomial theorem to get

$$(1-\xi)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \xi^k.$$

It follows from Parseval's identity that

$$\|(1-\xi)^n\|^2_{H^2(\mathbb{D})} = \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Then we have

$$\begin{aligned} c^2 &\ge \frac{\|(1-\xi)^{2n+1}\|^2_{H^2(\mathbb{D})}}{\|(1-\xi)^n\|^2_{H^2(\mathbb{D})}} \\ &= \frac{\binom{4n+2}{2n+1}}{\binom{2n}{n}} \\ &= \frac{(4n+2)!\, n!\, n!}{(2n+1)!\,(2n+1)!\,(2n)!}, \end{aligned}$$

but using Stirling's formula, the last expression is approximately 2^{2n+2}, and this is a contradiction.

9. The infinite continuous Cesàro operator on Hilbert space

As we mentioned in the introduction, in this section we show that, in contrast with the operator C_1, the set of extended eigenvalues for the operator C_∞ is as small as it can be, that is, it reduces to $\{1\}$.

There are several examples of Hilbert space operators with this property in the literature. It is worth mentioning some of them. Biswas and the third author [3] showed that if $Q \in \mathcal{B}(H)$ is a quasinilpotent operator then the set of extended eigenvalues for $\alpha + Q$ for every complex number $\alpha \ne 0$ reduces to $\{1\}$. They also showed when $\dim H < \infty$ that the set of extended eigenvalues for $T \in \mathcal{B}(H)$ reduces to $\{1\}$ if and only if $\sigma(T) = \{\alpha\}$ for some complex number $\alpha \ne 0$. Finally, an example was given by Shkarin [22] of a compact quasinilpotent operator on a Hilbert space whose set of extended eigenvalues reduces to $\{1\}$, answering at once two questions raised by Biswas, Lambert and the third author [2].

Brown, Halmos and Shields [6] proved that C_∞ is indeed a bounded linear operator, and they also proved that $I - C_\infty^*$ is unitarily equivalent to a bilateral shift of multiplicity one.

Recall that a bounded linear operator U on a complex Hilbert space H is a *bilateral shift of multiplicity one* provided that there is an orthonormal basis (e_n) of H such that $Ue_n = e_{n+1}$ for all $n \in \mathbb{Z}$.

Consider a bilateral shift of multiplicity one $U \in \mathcal{B}(L^2[0,1])$ and a unitary operator $V \in \mathcal{B}(L^2[0,1])$ such that $I - C_\infty^* = V^*UV$. We have

$$C_\infty = V^*(I - U^*)V,$$

and it follows that the extended eigenvalues of C_∞ are precisely the extended eigenvalues of $I - U^*$, and that the extended eigenoperators of C_∞ are in one to one correspondence with the extended eigenoperators of $I - U^*$ under conjugation with V.

Lemma 9.1. *Let X be an operator satisfying $(I - U^*)X = \lambda X(I - U^*)$, and let $\dots X_{-1}, X_0, X_1, X_2, \dots$ be the rows of the matrix of X. Then*

$$X_{n+1} = (\lambda U + 1 - \lambda)\, X_n,$$

for all $n \in \mathbb{Z}$. Consequently, for any $m, n \in \mathbb{N}$,

$$X_{m+n} = (\lambda U + 1 - \lambda)^n\, X_m.$$

In particular, if $m = 0$, $X_n = (\lambda U + 1 - \lambda)^n X_0$, for all $n \in \mathbb{N}$.

Proof. Taking adjoints we obtain $X^*(I - U) = \overline{\lambda}(I - U)X^*$ so that $X^*e_n - X^*e_{n+1} = \overline{\lambda}(I - U)X^*e_n$ and therefore $X^*e_{n+1} = (\overline{\lambda}U + 1 - \overline{\lambda})X^*e_n$. Hence, $X_{n+1} = \overline{X^*e_{n+1}} = (\lambda U + 1 - \lambda)\overline{X^*e_n} = (\lambda U + 1 - \lambda)X_n$. □

Theorem 9.2. *Let U be a bilateral shift of multiplicity one, and let λ be a complex number with $\lambda \neq 1$. Then the equation $(I - U^*)X = \lambda X(I - U^*)$ has only the trivial solution $X = 0$.*

Proof. Let A be a subset of the interval $[0, 2\pi)$ such that $|\lambda e^{it} + 1 - \lambda| > 1$ for all $t \in A$. Each row X_n of the matrix for X is a doubly infinite, square summable sequence of complex numbers, so it can be identified with a function in $L^2(\mathbb{T})$, with these complex numbers as its Fourier coefficients. Since every point on the unit circle is of the form e^{it} for a unique $t \in [0, 2\pi)$, the set A corresponds to a subset A' of $\mathbb{T}$. We will show that X_0 is equal to 0 almost everywhere on A'. Indeed, if that was not the case, there would exist a set $A_0 \subset A$ of positive measure and a constant $c > 0$ such that $|X_0(t)| \geq c$ and $|\lambda e^{it} + 1 - \lambda| \geq 1 + c$ for all $t \in A_0$. It would then follow that for every $n \in \mathbb{N}$,

$$\begin{aligned}
\|X_n\|^2 &= \int_0^{2\pi} |X_n(t)|^2\, dt \\
&= \int_0^{2\pi} |\,(\lambda U + 1 - \lambda)^n\, X_0(t)|^2\, dt \\
&= \int_0^{2\pi} |\,\left(\lambda e^{it} + 1 - \lambda\right)^n|^2\, |X_0(t)|^2\, dt \\
&\geq \int_{A_0} |\,\left(\lambda e^{it} + 1 - \lambda\right)^n|^2 |X_0(t)|^2\, dt \\
&\geq \int_{A_0} (1 + c)^{2n} c^2\, dt \to \infty, \text{ as } n \to \infty.
\end{aligned}$$

Now we turn our attention to the set $B \subset [0, 2\pi)$ such that $|\lambda e^{it} + 1 - \lambda| < 1$ for all $t \in B$. Once again, X_0 is equal to 0 for almost every $t \in B$. Otherwise, there would be a set $B_0 \subset B$ of positive measure and a constant $d \in (0, 1)$ such that $|X_0(t)| \geq d$ and $d \leq |\lambda e^{it} + 1 - \lambda| \leq 1 - d$ for all $t \in B_0$. It would then follow that for every negative integer n,

$$\begin{aligned}
\|X_n\|^2 &= \textstyle\int_0^{2\pi} |X_n(t)|^2\, dt \geq \int_{B_0} |X_n(t)|^2\, dt = \int_{B_0} |X_0(t)|^2 |\lambda e^{it} + 1 - \lambda|^{2n}\, dt \\
&\geq \textstyle\int_{B_0} d^2 (1 - d)^{2n}\, dt \to \infty, \text{ as } n \to -\infty.
\end{aligned}$$

Thus, the function X_0 is zero almost everywhere on $A \cup B$. The complement of this set in $[0, 2\pi)$ consists of two points. These are the points of intersection of the unit circle and the circle with center $(1-\lambda)/\lambda$ and radius $1/|\lambda|$. The only exceptions occurs when $\lambda = 1$ and $\lambda = 0$. In the former case, the two circles coincide, and in the latter $|\lambda e^{it} + 1 - \lambda| = 1$ for all $t \in [0, 2\pi)$. However, the case $\lambda = 0$ has been ruled out since the kernel of $I - U^*$ is trivial.

We conclude that, unless $\lambda = 1$, X_0 is the zero function in $L^2([0, 2\pi))$ and, by Lemma 9.1, the same is true of X_n for any $n \in \mathbb{Z}$. Consequently, $X = 0$ and the theorem is proved. □

10. The infinite continuous Cesàro operator on Lebesgue spaces

Let $1 < p, q < \infty$ be conjugate indices, that is,

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Our aim in this section is to show that the set of extended eigenvalues for the infinite continuous Cesàro operator C_∞ on the complex Banach space $L^p[0, \infty)$ reduces to the singleton $\{1\}$.

Before we present our result we define a sequence of functions $\{e_n\}_{n\in\mathbb{Z}}$ in $L^q(0, \infty)$. This construction is modeled after the one in [6] for the case $q = 2$. Let $e_0 = \chi_{(0,1)}$, and let

$$e_n = (1 - 2/q\, C_\infty^*)^n e_0, \quad \text{for } n \in \mathbb{N}.$$

Next, we define an operator R on the linear span of $\{e_n\}_{n\in\mathbb{N}}$ by

$$Rf(x) = -x^{-2/q} f\left(\frac{1}{x}\right),$$

and define $e_{-n} = Re_{n-1}(x)$, for $n \in \mathbb{N}$.

Proposition 10.1. *Let the sequence of functions $\{e_n\}_{n\in\mathbb{Z}}$ be defined as above. Then $\{e_n\}_{n\in\mathbb{Z}}$ is a linearly independent set of functions in $L^q(0, \infty)$ and its closed linear span is $L^q(0, \infty)$. Further, the operator $1 - 2/q\, C_\infty^*$ shifts this sequence, i.e., $(1 - 2/q\, C_\infty^*)e_n = e_{n+1}$ for all $n \in \mathbb{Z}$. Finally, for any $\gamma \in (0, 1)$ there exists $K = K(\gamma)$ such that $\|e_n\| \le K\gamma^{-n}$ if $n \ge 0$, and $\|e_n\| \le K\gamma^n$ if $n < 0$.*

Proof. We start with the observation that the Cesaro operator C_∞ is a bounded operator on $L^p(0, \infty)$, so its adjoint C_∞^* is bounded on $L^q(0, \infty)$. Therefore, $e_n \in L^q(0, \infty)$ for $n \ge 0$. Furthermore, it is straightforward to verify that $\|Re_n\|_q = \|e_n\|_q$, so $e_n \in L^q(0, \infty)$ for $n < 0$ as well.

Next we will show that $\{e_n\}_{n\in\mathbb{Z}}$ is a total set in $L^q(0, \infty)$. First we notice that for $n \ge 0$, each function e_n vanishes outside $[0, 1]$, and for $n < 0$ outside of $(1, +\infty)$. In both cases it suffices to demonstrate that if a bounded linear functional vanishes on all $\{e_n\}$ then it must be the zero functional. Further, each functional on $L^q(0, 1)$ can be represented by a function $g \in L^p(0, 1)$. So, suppose that g is such a function and that $\int_0^1 e_n \overline{g} = 0$ for all $n \ge 0$. Let $g_n = (I - C_\infty^*)^n e_0$, for $n \ge 0$. It was proved in [6] that $\{g_n\}$ is an orthonormal system in $L^2(0, 1)$. Further,

$$g_n = \left[1 - \frac{q}{2} + \frac{q}{2}\left(1 - \frac{2}{q} C_\infty^*\right)\right]^n e_0 = \sum_{i=0}^{n} \binom{n}{i} \left(1 - \frac{q}{2}\right)^{n-i} \left(\frac{q}{2}\right)^i e_i,$$

so $\int_0^1 g_n \overline{g} = 0$ for all $n \ge 0$. Thus, any bounded linear functional that vanishes on $\{e_n\}$ must vanish on $\{g_n\}$, hence on $L^2(0, 1)$, and it must be zero. When $n < 0$, we will assume that $g \in L^p(1, \infty)$ and that $\int_1^\infty e_{-n} \overline{g} = 0$ for all $n \ge 1$. However, using the substitution $t = 1/x$,

$$\begin{aligned} \int_1^\infty e_{-n}(x)\overline{g}(x)\, dx &= -\int_1^\infty x^{-2/q} e_{n-1}(1/x)\overline{g}(x)\, dx \\ &= -\int_0^1 t^{-2/p} e_{n-1}(t)\overline{g}(1/t)\, dt. \end{aligned}$$

So, the previous case implies that $t^{-2/p}\overline{g}(1/t)$ is the zero function, whence $g = 0$.

Next we consider the set $\mathcal{F}$ defined as follows. A function $f \in L^q(0,\infty)$ belongs to $\mathcal{F}$ if there exists a sequence of complex numbers $\{c_n\}_{n\in\mathbb{Z}}$ such that $f = \sum_{n\in\mathbb{Z}} c_n e_n$. Since $\{e_n\}_{n\in\mathbb{Z}}$ is a total set, $\mathcal{F}$ is dense in $L^q(0,\infty)$. Now we will show that if $f \in \mathcal{F}$, there is exactly one sequence $\{c_n\}_{n\in\mathbb{Z}}$. In order to do that it suffices to demonstrate that, if $\sum_{k\in\mathbb{Z}} c_k e_k = 0$ then $c_k = 0$ for all $k \in \mathbb{Z}$. Notice that

$$\left\|\sum_{k\in\mathbb{Z}} c_k e_k\right\|^q = \int_0^\infty \left|\sum_{k\in\mathbb{Z}} c_k e_k\right|^q = \int_0^1 \left|\sum_{k=0}^{\infty} c_k e_k\right|^q + \int_1^\infty \left|\sum_{k=-\infty}^{-1} c_k e_k\right|^q$$
$$= \left\|\sum_{k=-\infty}^{-1} c_k e_k\right\|^q + \left\|\sum_{k=0}^{\infty} c_k e_k\right\|^q,$$

so we can consider separately $n \geq 0$ and $n < 0$. We start with $n \geq 0$. Let $\alpha \in D(q/2, q/2)$ and $f_\alpha(x) = x^{(1-\alpha)/\alpha}$. Since $\|f\|_q \geq |\int_0^1 f\overline{f_\alpha}|/\|f_\alpha\|_p$ for any $f \in L^q(0,1)$ and $f_\alpha \in L^p(0,1)$ it follows that

$$\int_0^1 \left(\sum_{k=0}^{\infty} c_k e_k\right)\overline{f_\alpha} = 0.$$

Notice that, if $k \geq 0$

$$\int_0^1 e_k\overline{f_\alpha} = \int_0^1 \left(1 - \frac{2}{q}C_\infty^*\right)^k e_0\overline{f_\alpha} = \int_0^1 e_0\left(1 - \frac{2}{q}C_\infty^k\right)\overline{f_\alpha}.$$

Further, $(1 - 2/q\, C_\infty)^k\overline{f_\alpha} = (1 - 2/q\,\overline{\alpha})^k\overline{f_\alpha} + v_k$, where v_k is a function that vanishes on $(0,1)$. Thus,

$$\int_0^1 \sum_{k=0}^{\infty} c_k\left(1 - \frac{2}{q}\alpha\right)^k\overline{f_\alpha} = 0.$$

It is easy to see that $\int_0^1 \overline{f_\alpha} \neq 0$, so we obtain that

$$\sum_{k=0}^{\infty} c_k\left(1 - \frac{2}{q}\alpha\right)^k = 0.$$

This implies that the analytic function $\sum_{k=0}^{\infty}(1 - 2z/q)^k$ vanishes in the disc $D(q/2, q/2)$, whence $c_k = 0$ for all k. This settles the case $n \geq 0$ and we turn our attention to $n < 0$. We will use the identity

$$e_{-n}(x) = -x^{-2/q}e_{n-1}(1/x) \tag{10.1}$$

which holds for all $n \in \mathbb{N}$, and follows directly from the definition of e_{-n}. Suppose that there exist complex numbers $\{c_k\}$ such that

$$\left\|\sum_{k=1}^{\infty} c_k e_{-k}\right\| = 0.$$

Using (10.1), it follows that

$$\int_1^\infty \left|\sum_{k=1}^{\infty} c_k x^{-2/q} e_{k-1}(1/x)\right|^q dx = 0.$$

With the substitution $t = 1/x$ we obtain

$$\int_0^1 \left|\sum_{k=1}^{\infty} c_k e_{k-1}(t)\right|^q dt = 0,$$

so the result follows from the previous case.

Our next step is to establish the desired estimate on the norm of e_n. To that end, we notice that the spectrum of $1-(2/q)C^*$ is the unit circle. Thus, if $\gamma \in (0,1)$, the spectral radius of $\gamma(1-(2/q)C^*)$ is less than one. It follows that this operator is similar to a strict contraction, hence power bounded. That is, there exists $K>0$ such that for $n \geq 0$, $\|(\gamma-(2\gamma/q)C^*)^n\| \leq K$. Therefore,

$$\|e_n\| = \left\| \left(1-\frac{2}{q}C^*_\infty\right)^n e_0 \right\| \leq \left(\frac{1}{\gamma}\right)^n K\|e_0\| = K\left(\frac{1}{\gamma}\right)^n.$$

As we had already noticed, $\|e_{-n}\| = \|e_{n-1}\|$ so the analogous estimate for e_n indexed by negative integers follows.

Finally, we will prove that $(1-2/q\,C^*_\infty)e_n = e_{n+1}$ for all $n \in \mathbb{Z}$. For $n \geq 0$ this is just the definition of e_n, so we focus on the case $n<0$. We will show that, for $n \geq 0$,

$$\left(1-\frac{2}{q}C^*_\infty\right) R \left(1-\frac{2}{q}C^*_\infty\right) e_n = Re_n. \tag{10.2}$$

Once this is established the result will easily follow. Indeed, if $n>1$ then

$$\begin{aligned}\left(1-\frac{2}{q}C^*_\infty\right) e_{-n} &= \left(1-\frac{2}{q}C^*_\infty\right) Re_{n-1}\\ &= R\left(1-\frac{2}{q}C^*_\infty\right)^{-1} e_{n-1}\\ &= Re_{n-2} = e_{-n+1}.\end{aligned}$$

When $n=1$

$$\begin{aligned}\left(1-\frac{2}{q}C^*_\infty\right) e_{-1}(x) &= \left(1-\frac{2}{q}C^*_\infty\right) Re_0(x)\\ &= -\left(1-\frac{2}{q}C^*_\infty\right) x^{-2/q} e_0\left(\frac{1}{x}\right)\\ &= -x^{-2/q}e_0\left(\frac{1}{x}\right) + \left(\frac{2}{q}\right)\int_x^\infty \frac{t^{-2/q}e_0(1/t)}{t}\,dt.\end{aligned}$$

Since $e_0 = \chi_{(0,1)}$, if $0<x<1$ then $e_0(1/x)=0$ and the domain of integration is reduced to $(1,+\infty)$. Thus, we obtain

$$\left(\frac{2}{q}\right)\int_1^\infty \frac{t^{-2/q}}{t}\,dt = 1.$$

If $x \geq 1$ then $e_0(1/x)=1$ so we obtain

$$-x^{-2/q} + \left(\frac{2}{q}\right)\int_x^\infty \frac{t^{-2/q}}{t}\,dt = 0.$$

We conclude that $(1-2/qC^*_\infty)e_{-1} = e_0$.

Thus it remains to establish the identity (10.2). Let f be any function in $L^q(0,\infty)$ that vanishes outside the interval $(0,1)$. Then

$$\begin{aligned}&\left(1-\frac{2}{q}C^*_\infty\right) R \left(1-\frac{2}{q}C^*_\infty\right) f =\\ &= -x^{-2/q} f\left(\frac{1}{x}\right) + \frac{2}{q}x^{-2/q}\int_{1/x}^\infty \frac{f(t)}{t}\,dt \\ &+\frac{2}{q}\int_x^\infty \frac{t^{-2/q}f(1/t)}{t}\,dt - \frac{4}{q^2}\int_x^\infty \frac{t^{-2/q}}{t}\,dt\int_{1/t}^\infty \frac{f(s)}{s}\,ds.\end{aligned} \tag{10.3}$$

If $0 < x < 1$ the first two terms are equal to 0, and in the remaining two, the domains of integration are changed. We obtain

$$\frac{2}{q}\int_1^\infty \frac{t^{-2/q}f(1/t)}{t}\,dt - \frac{4}{q^2}\int_1^\infty \frac{t^{-2/q}}{t}\,dt \int_{1/t}^1 \frac{f(s)}{s}\,ds.$$

Now the substitution $u = 1/t$ followed by the change in the order of integration in the second term yields

$$\begin{aligned} &\frac{2}{q}\int_0^1 \frac{u^{2/q}f(u)}{u}\,du - \frac{4}{q^2}\int_0^1 \frac{u^{2/q}}{u}\,du \int_u^1 \frac{f(s)}{s}\,ds \\ &= \frac{2}{q}\int_0^1 \frac{u^{2/q}f(u)}{u}\,du - \frac{4}{q^2}\int_0^1 \frac{f(s)}{s}\,ds \int_0^s \frac{u^{2/q}}{u}\,du \\ &= \frac{2}{q}\int_0^1 \frac{u^{2/q}f(u)}{u}\,du - \frac{4}{q^2}\int_0^1 \frac{f(s)}{s}\,\frac{q}{2}\,s^{2/q}\,ds = 0. \end{aligned}$$

If $x \geq 1$, we will obtain that all the terms in (10.3) except for the first cancel. Once again, we use the substitution $u = 1/t$ in the last two terms and obtain

$$-x^{-2/q}f\left(\frac{1}{x}\right) + \frac{2}{q}x^{-2/q}\int_{1/x}^\infty \frac{f(t)}{t}\,dt \tag{10.4}$$

$$+\frac{2}{q}\int_0^{1/x} \frac{u^{2/q}f(u)}{u}\,du - \frac{4}{q^2}\int_0^{1/x} \frac{u^{2/q}}{u}\,du \int_u^\infty \frac{f(s)}{s}\,ds. \tag{10.5}$$

Further, after interchanging the order of integration in the iterated integral, it becomes

$$\begin{aligned} &\int_0^{1/x} \frac{f(s)}{s}\,ds \int_0^s \frac{u^{2/q}}{u}\,du + \int_{1/x}^\infty \frac{f(s)}{s}\,ds \int_0^{1/x} \frac{u^{2/q}}{u}\,du \\ &= \int_0^{1/x} \frac{f(s)}{s}\,\frac{q}{2}\,s^{2/q}\,ds + \int_{1/x}^\infty \frac{f(s)}{s}\,\frac{q}{2}\,x^{-2/q}\,ds, \end{aligned}$$

so it is easy to see that we have the announced cancelation. Combining these two cases we conclude that

$$\left(1 - \frac{2}{q}C_\infty^*\right) R \left(1 - \frac{2}{q}C_\infty^*\right) f = Rf.$$

whenever f vanishes outside $(0, 1)$. In particular, if $f = e_n$ for $n \geq 0$, we obtain (10.2). □

Proposition 10.2. *Let $\{e_n\}_{n\in\mathbb{Z}}$ and $\mathcal{F}$ be as in Proposition 10.1 and let $0 < \theta < 1$. Let $W_\theta : \mathcal{F} \to L^q(0, 2\pi)$ be a linear transformation defined by*

$$W_\theta e_n = \frac{\theta^{|n|}}{(1-\theta)^{\max\{1/p,1/q\}}}e^{int}, \text{ for } n \in \mathbb{Z},$$

and extended linearly. Then there is a constant $K = K(p, q)$ such that, for any $\theta \in (0, 1)$ and any $f \in \mathcal{F}$, $\|W_\theta f\| \leq K\|f\|$. Consequently, W_θ extends to a bounded linear operator $W_\theta : L^q(0,\infty) \to L^q(0, 2\pi)$.

Proof. We will show that there exists such a constant K that does not depend on θ and such that, for any $f = \sum_{k=-\infty}^\infty c_k e_k \in L^q(0,\infty)$ and any $n \in \mathbb{N}$,

$$\left\|\sum_{k=-n}^n c_k\theta^{|k|}e^{ikt}\right\| \leq \frac{K}{(1-\theta)^{\max\{1/p,1/q\}}}\left\|\sum_{k=-n}^n c_k e_k\right\|. \tag{10.6}$$

We start with the fact that $\sum_{k=-n}^{n} c_k \theta^{|k|} e^{ikt}$ is continuous, so its modulus attains its maximum at some $t_0 \in [0, 2\pi]$. Consequently,

$$\begin{aligned}\left\|\sum_{k=-n}^{n} c_k \theta^{|k|} e^{ikt}\right\|^q &= \int_0^{2\pi} \left|\sum_{k=-n}^{n} c_k \theta^{|k|} e^{ikt}\right|^q dt \\ &\leq 2\pi \left|\sum_{k=-n}^{n} c_k \theta^{|k|} e^{ikt_0}\right|^q \\ &= 2\pi \left|\sum_{k=0}^{n} c_k \left(1 - \frac{2}{q}\overline{\alpha}\right)^k + \sum_{k=-n}^{-1} c_k \left(1 - \frac{2}{q}\overline{\beta}\right)^{-k}\right|^q,\end{aligned}$$

where $\alpha = q/2(1-\theta e^{-it_0})$ and $\beta = q/2(1-\theta e^{it_0})$. Let

$$\begin{aligned} g_1(x) &= \left(1 - \frac{2}{q}\beta\right)^{-1} \beta \chi_{(0,1)}(x)\, x^{(1-\alpha)/\alpha}, \\ g_2(x) &= -\alpha \chi_{(1,\infty)}(x)\, x^{-2/p-(1-\beta)/\beta}, \end{aligned}$$

and $g = g_1 + g_2$. Notice that g belongs to $L^p(0,\infty)$. Indeed, it suffices to establish that

$$\mathrm{Re}\left(\frac{p(1-\alpha)}{\alpha}\right) > -1 \text{ and } \mathrm{Re}\left(-2 - \frac{p(1-\beta)}{\beta}\right) < -1.$$

These inequalities can be reduced to $\mathrm{Re}(1/\alpha) > 1/q$ and $\mathrm{Re}(1/\beta) > 1/q$, which in turn is equivalent to $\alpha, \beta \in D(q/2, q/2)$. Since these are obvious, $g \in L^p(0,\infty)$. Moreover,

$$\begin{aligned}\|g\|^p &= \int_0^\infty |g_1 + g_2|^p \\ &= \int_0^1 \left|1 - \frac{2}{q}\beta\right|^{-p} |\beta x^{(1-\alpha)/\alpha}|^p + \int_1^\infty |\alpha x^{-2/p-(1-\beta)/\beta}|^p \\ &= \frac{1}{\theta^p}|\beta|^p \frac{1}{1 + \mathrm{Re}\, p\frac{1-\alpha}{\alpha}} + |\alpha|^p \frac{1}{1 + \mathrm{Re}\, p\frac{1-\beta}{\beta}}.\end{aligned}$$

Further,

$$\begin{aligned}1 + \mathrm{Re}\frac{p(1-\alpha)}{\alpha} &= 1 - p + p\,\mathrm{Re}\frac{1}{\alpha} \\ &= 1 - p + \frac{p}{|\alpha|^2}\mathrm{Re}(\overline{\alpha}) \\ &= 1 - p + \frac{2p}{q|1-\theta e^{-it_0}|^2}\mathrm{Re}(1 - \theta e^{it_0}) \\ &= \frac{p}{q}\left(-1 + 2\,\frac{1 - \theta\cos t_0}{1 - 2\theta\cos t_0 + \theta^2}\right) \\ &= \frac{p}{q}\frac{1-\theta^2}{1 - 2\theta\cos t_0 + \theta^2},\end{aligned}$$

and the same equality holds with β in place of α. Using the relation $\alpha = \overline{\beta}$, we obtain that

$$\|g\| = \left(\frac{1}{\theta^p}\frac{|\beta|^p}{1+\mathrm{Re}\frac{p(1-\alpha)}{\alpha}} + \frac{|\alpha|^p}{1+\mathrm{Re}\frac{p(1-\beta)}{\beta}}\right)^{1/p} \tag{10.7}$$

$$= \left(\tfrac{1}{\theta^p} + 1\right)^{1/p} |\alpha| \left(\frac{q}{p}\frac{1-2\theta\cos t_0+\theta^2}{1-\theta^2}\right)^{1/p} \tag{10.8}$$

$$= \left(\frac{(\theta^p+1)q}{(\theta+1)p}\right)^{1/p} \frac{|\alpha|^{1+2/p}}{\theta(1-\theta)^{1/p}}. \tag{10.9}$$

Next,

$$\begin{aligned}
&\left\|\sum_{k=-n}^{n} c_k e_k\right\| \\
&\geq \left|\int_0^\infty \sum_{k=-n}^{n} c_k e_k\,\overline{g}\right| \frac{1}{\|g\|} \\
&= \left|\int_0^\infty \sum_{k=-n}^{n} c_k e_k\,\overline{g_1+g_2}\right| \frac{1}{\|g\|} \\
&= \left|\int_0^\infty \sum_{k=0}^{n} c_k e_k\,\overline{g_1} + \int_0^\infty \sum_{k=-n}^{-1} c_k e_k\,\overline{g_2}\right| \frac{1}{\|g\|} \\
&= \left|\int_0^\infty \sum_{k=0}^{n} c_k \left(1-\frac{2}{q}C_\infty^*\right)^k e_0\,\overline{g_1} + \int_0^\infty \sum_{k=-n}^{-1} c_k R e_{-k-1}\,\overline{g_2}\right| \frac{1}{\|g\|} \\
&= \left|\int_0^\infty \sum_{k=0}^{n} c_k e_0 \left(1-\frac{2}{q}C_\infty\right)^k \overline{g_1} + \int_0^\infty \sum_{k=-n}^{-1} c_k R \left(1-\frac{2}{q}C_\infty^*\right)^{-k-1} e_0\,\overline{g_2}\right| \frac{1}{\|g\|} \\
&= \left|\int_0^1 \sum_{k=0}^{n} c_k e_0 \left(1-\frac{2}{q}\overline{\alpha}\right)^k \overline{g_1} + \int_0^\infty \sum_{k=-n}^{-1} c_k e_0 \left(1-\frac{2}{q}C_\infty\right)^{-k-1} R^*\overline{g_2}\right| \frac{1}{\|g\|}.
\end{aligned}$$

It is not hard to see that the operator R^* is given by the formula $R^*f(x) = -x^{-2/p}f(1/x)$, so

$$R^*\overline{g_2}(x) = \overline{\alpha}x^{-2/p}\chi_{(1,\infty)}(1/x)\,x^{2/p+\overline{(1-\beta)/\beta}} = \overline{\alpha}\chi_{(0,1)}(x)x^{\overline{(1-\beta)/\beta}} = \overline{\alpha}f_{\overline{\beta}}(x).$$

Therefore, the second integral can be written as

$$\begin{aligned}
&\overline{\alpha}\int_0^1 \sum_{k=-n}^{-1} c_k e_0(x) \left(1-\frac{2}{q}C\right)^{-k-1} f_{\overline{\beta}}(x)\,dx \\
&= \overline{\alpha}\int_0^1 \sum_{k=-n}^{-1} c_k e_0(x) \left(1-\frac{2}{q}\,\overline{\beta}\right)^{-k-1} f_{\overline{\beta}}(x)\,dx \\
&= \overline{\alpha}\overline{\beta} \sum_{k=-n}^{-1} c_k \left(1-\frac{2}{q}\overline{\beta}\right)^{-k-1} \\
&= \overline{\alpha}\overline{\beta}\left(1-\frac{2}{q}\overline{\beta}\right)^{-1} \sum_{k=-n}^{-1} c_k \left(1-\frac{2}{q}\overline{\beta}\right)^{-k}.
\end{aligned}$$

Since the first integral equals

$$\overline{\alpha}\overline{\beta}\left(1-\frac{2}{q}\overline{\beta}\right)^{-1} \sum_{k=0}^{n} c_k \left(1-\frac{2}{q}\overline{\alpha}\right)^{k},$$

we obtain that

$$\begin{aligned}
\left\|\sum_{k=-n}^{n} c_k e_k\right\| &\geq \left|\overline{\alpha}\overline{\beta}\left(1-\frac{2}{q}\overline{\beta}\right)^{-1}\right| \left|\sum_{k=0}^{n} c_k \left(1-\frac{2}{q}\overline{\alpha}\right)^{k} + \sum_{k=-n}^{-1} c_k \left(1-\frac{2}{q}\overline{\beta}\right)^{-k}\right| \frac{1}{\|g\|} \\
&= \frac{|\alpha|^2}{\theta}\,\frac{1}{(2\pi)^{1/q}}\|\sum_{k=-n}^{n} c_k\theta^{|k|}e^{ikt}\| \left(\frac{(\theta+1)p}{(\theta^p+1)q}\right)^{1/p} \frac{\theta(1-\theta)^{1/p}}{|\alpha|^{1+2/p}} \\
&\geq \left(\frac{p}{q}\right)^{1/p} \frac{1}{(2\pi)^{1/q}}\,|\alpha|^{1-2/p}(1-\theta)^{1/p}\left\|\sum_{k=-n}^{n} c_k\theta^{|k|}e^{ikt}\right\|
\end{aligned}$$

If $1 < p \leq 2$ then $1 - 2/p \leq 0$, so

$$|\alpha|^{1-2/p} \geq \left(\frac{q}{2}\right)^{1-2/p} (1+\theta)^{1-2/p} > \left(\frac{q}{2}\right)^{1-2/p} 2^{1-2/p}.$$

If $p > 2$ then $1 - 2/p > 0$, so

$$|\alpha|^{1-2/p} \geq \left(\frac{q}{2}\right)^{1-2/p} (1-\theta)^{1-2/p}$$

and it follows that, in this case,

$$|\alpha|^{1-2/p}(1-\theta)^{1/p} \geq \left(\frac{q}{2}\right)^{1-2/p} (1-\theta)^{1-2/p+1/p} = \left(\frac{q}{2}\right)^{1-2/p} (1-\theta)^{1/q}.$$

Therefore, there exists $K = K(p,q)$ such that (10.6) holds. We conclude that W is a bounded linear transformation and that $\|W\| \leq K$. □

Theorem 10.3. *Let C_∞ be the Cesaro operator on $L^p(0,\infty)$ for $1 < p \leq \infty$, and let $\lambda \neq 1$ be a complex number. If X is a bounded linear operator on $L^p(0,\infty)$ such that $C_\infty X = \lambda X C_\infty$, then $X = 0$.*

Proof. Let q be the exponent conjugate to p, i.e., $1/p + 1/q = 1$. Since C_∞ acts on $L^p(0,\infty)$, its conjugate operator C_∞^* is a bounded operator acting on $L^q(0,\infty)$. Let $\{e_n\}_{n\in\mathbb{Z}}$ be set of functions inf $L^q(0,\infty)$ as defined above, let $\theta \in (0,1)$, and let $W = W_\theta$ be as in Proposition 10.2.

Next, let M_z be the operator of multiplication by e^{it} on $L^q(0,2\pi)$, and let Γ be a weighted shift on $L^q(0,2\pi)$ with weight sequence $\{\mu_n\}$, i.e.,

$$\Gamma e^{int} = \mu_n e^{i(n+1)t}, \text{ with } \mu_n = \begin{cases} \theta, & \text{if } n \geq 0, \\ 1/\theta & \text{if } n < 0, \end{cases} = \frac{\theta^{|n+1|}}{\theta^{|n|}}.$$

Then

$$\begin{aligned} \left(1 - \frac{2}{q} C_\infty^*\right) e_n = W e_{n+1} &= \frac{\theta^{|n+1|}}{(1-\theta)^{\max\{1/p,1/q\}}} e^{i(n+1)t} \\ &= \frac{\mu_n \theta^{|n|}}{(1-\theta)^{\max\{1/p,1/q\}}} M_z e^{int} \\ &= \Gamma W e_n \end{aligned}$$

so $W(1 - 2/q\, C_\infty^*) = \Gamma W$. Further if $C_\infty X = \lambda X C_\infty$ then $X^* C_\infty^* = \overline{\lambda} C_\infty^* X^*$, so we have

$$X^* \left(1 - \frac{2}{q} C_\infty^*\right) = \left(1 - \frac{2}{q} \overline{\lambda} C_\infty^*\right) X^*.$$

This implies that $(1 - 2/q\overline{\lambda} C_\infty^*) X^* e_n = X^*(1 - 2/q C_\infty^*) e_n = X^* e_{n+1}$ and, inductively, that

$$X^* e_n = (1 - \frac{2}{q}\overline{\lambda} C_\infty^*)^n X^* e_0, \tag{10.10}$$

for all $n \in \mathbb{Z}$. Notice that

$$\begin{aligned} W\left(1 - \frac{2}{q}\overline{\lambda} C_\infty^*\right) &= W\left(1 - \overline{\lambda} + \overline{\lambda} - \frac{2}{q}\overline{\lambda} C_\infty^*\right) \\ &= (1 - \overline{\lambda}) W + \overline{\lambda} W \left(1 - \frac{2}{q} C_\infty^*\right) = UW, \end{aligned}$$

where $U = 1 - \overline{\lambda} + \overline{\lambda}\Gamma$. By the definition of Γ, we have

$$Ue^{int} = \left[(1-\overline{\lambda}) + \overline{\lambda}\theta e^{it}\right] e^{int}, \text{ if } n \geq 0, \text{ and}$$
$$Ue^{int} = \left[(1-\overline{\lambda}) + \overline{\lambda}\frac{1}{\theta} e^{it}\right] e^{int}, \text{ if } n < 0.$$

The estimates established in Proposition 10.1 allow us to obtain an estimate on the operator norm $\|X^*\|$. We have

$$\|X^*\| \geq \frac{\|X^* e_n\|}{\|e_n\|} \geq \frac{1}{K}\gamma^n \|X^* e_n\|, \text{ if } n \geq 0, \text{ and}$$
$$\|X^*\| \geq \frac{\|X^* e_n\|}{\|e_n\|} \geq \frac{1}{K}\gamma^{-n} \|X^* e_n\|, \text{ if } n < 0.$$

As for $\|X^* e_n\|$ we have

$$\begin{aligned}
\|X^* e_n\| &= \| \left(1 - \frac{2}{q}\overline{\lambda} C_\infty^*\right)^n X^* e_0 \| \\
&\geq \frac{1}{\|W\|} \|W \left(1 - \frac{2}{q}\overline{\lambda} C_\infty^*\right)^n X^* e_0 \| \\
&= \frac{1}{\|W\|} \|U^n W X^* e_0\| \\
&= \frac{1}{\|W\|} \|U^n f\|
\end{aligned}$$

where $f = WX^* e_0$. Combining with the previous estimates, we obtain that

$$\|X^*\| \geq \frac{1}{K}\gamma^n \frac{1}{\|W\|} \|U^n f\|, \text{ if } n \geq 0,$$

and

$$\|X^*\| \geq \frac{1}{K\gamma^n} \frac{1}{\|W\|} \|U^n f\|, \text{ if } n < 0.$$

Let

$$A_\gamma = \{t \in [0, 2\pi] : |\gamma(1-\overline{\lambda}) + \overline{\lambda}\gamma\theta e^{it}| > 1\},$$
$$B_\gamma = \{t \in [0, 2\pi] : |\gamma^{-1}(1-\overline{\lambda}) + \overline{\lambda}\gamma^{-1}/\theta e^{it}| < 1\}.$$

Using the same argument as in the proof of Theorem 9.2, we see that f must be 0 on $A_\gamma \cup B_\gamma$. Since this must be true for any $\gamma \in (0,1)$, we see that f must vanish on $A = \cup_{\gamma\in(0,1)} A_\gamma$ and $B = \cup_{\gamma\in(0,1)} B_\gamma$. Thus, f can be different from 0 only on the complement of $A \cup B$. But,

$$(A \cup B)^c = \{t \in [0, 2\pi] : |(1-\overline{\lambda}) + \overline{\lambda}\theta e^{it}| \leq 1 \text{ and } |(1-\overline{\lambda}) + \frac{\overline{\lambda}}{\theta} e^{it}| \geq 1\}.$$

Let $re^{i\varphi}$ be the polar form of $(1-\overline{\lambda})/\overline{\lambda}$. Since we are assuming that $\lambda \neq 1$, this complex number is not zero, so φ is well defined. Then

$$\begin{aligned}
&(A \cup B)^c \\
&= \{t \in [0, 2\pi] : |r + \theta e^{i(t-\varphi)}| \leq \frac{1}{|\lambda|} \text{ and } |r + \frac{1}{\theta} e^{i(t-\varphi)}| \geq \frac{1}{|\lambda|}\} \\
&= \{t \in [0, 2\pi] : r^2 + \theta^2 + 2r\theta\cos(t-\varphi) \leq \frac{1}{|\lambda|^2}\} \cap \\
&\cap \{t \in [0, 2\pi] \colon r^2 + \frac{1}{\theta^2} + 2\frac{r}{\theta}\cos(t-\varphi) \geq \frac{1}{|\lambda|^2}\} \\
&= \{t \in [0, 2\pi] : \frac{\theta}{2r}\left(\frac{1}{|\lambda|^2} - r^2 - \frac{1}{\theta^2}\right) \leq \cos(t-\varphi) \leq \frac{1}{2r\theta}\left(\frac{1}{|\lambda|^2} - r^2 - \theta^2\right)\}.
\end{aligned}$$

Notice that, as $\theta \uparrow 1$, both bounds for $\cos(t-\varphi)$ converge to the same number. It follows that, for a fixed $t \in [0, 2\pi]$ there exists $\Theta \in (0,1)$ such that, if $\theta \geq \Theta$ then $t \notin (A \cup B)^c$. In other words, if $\theta \geq \Theta$ then $f(t) = 0$.

Let us write $X^* e_0 = \sum_{n \in \mathbb{Z}} c_n e_n$. Then

$$f(t) = (WX^* e_0)(t) = \sum_{n=-\infty}^{\infty} c_k \frac{\theta^{|n|}}{(1-\theta)^{\max\{1/p, 1/q\}}} e^{int}.$$

For a fixed $t \in [0, 2\pi]$ the power series above is an analytic function of θ, for $|\theta| < 1$, and this function vanishes on the line segment $(\Theta, 1)$, so it must be zero. Consequently, $c_{-n} e^{-int} + c_n e^{int} = 0$ for every $n \in \mathbb{N}$. Since this is true for all $t \in (A \cup B)^c$, it is easy to see that $c_n = 0$ for all $n \in \mathbb{Z}$. Thus $X^* e_0 = 0$ and (10.10) implies that $X^* e_n = 0$ for all $n \in \mathbb{Z}$, whence $X = 0$. □

11. Some open problems

Here is a list of problems that we find interesting and that we have not been able to solve.

1. Show that the co-analytic Toeplitz matrix A of Theorem 2.3 induces a bounded linear operator on ℓ^2, or in other words, show that the supremum in equation (2.8) is finite.
2. Show that if X is an extended eigenoperator for C_1 on $L^p[0,1]$ then there exists $R \in \{C_1\}'$ such that $X = X_0 R$, where X_0 is the weighted composition operator of Lemma 4.1.
3. Show that if $1 < p < \infty$ and if λ is real and $\lambda \geq 1$ then λ is an extended eigenvalue for C_0 on ℓ^p.
4. Let $T \in \mathcal{B}(E)$ and consider the *Deddens algebra* $\mathcal{D}_T$ associated with T, that is, the family of all $X \in \mathcal{B}(E)$ for which there is a constant $M > 0$ such that for every $n \in \mathbb{N}$ and for every $f \in E$,

$$\|T^n X f\| \leq M \|T^n f\|. \tag{11.1}$$

When T is invertible this is equivalent to saying that

$$\sup_{n \in \mathbb{N}} \|T^n X T^{-n}\| < \infty. \tag{11.2}$$

The Deddens algebra $\mathcal{D}_T$ is a not necesarily closed subalgebra of $\mathcal{B}(E)$ that contains all extended eigenoperators corresponding to extended eigenvalues λ with $|\lambda| \leq 1$. Show that $\mathcal{D}_{C_\infty} = \{C_\infty\}'$. A consequence of this result would be that the set of extended eigenvalues for C_∞ reduces to $\{1\}$.

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