Function algebras with a strongly precompact unit ball

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Abstract

Let μ be a finite positive Borel measure with compact support $K \subseteq \mathbb{C}$, and regard $L^\infty(\mu)$ as an algebra of multiplication operators on the Hilbert space $L^2(\mu)$. Then consider the subalgebra $A(K)$ of all continuous functions on K that are analytic on the interior of K, and the subalgebra $R(K)$ defined as the uniform closure of the rational functions with poles outside K. Froelich and Marsalli showed that if the restriction of the measure μ to the boundary of K is discrete then the unit ball of $A(K)$ is strongly precompact, and that if the unit ball of $R(K)$ is strongly precompact then the restriction of the measure μ to the boundary of each component of $\mathbb{C}\backslash K$ is discrete. The aim of this paper is to provide three examples that go to clarify the results of Froelich and Marsalli; in particular, it is shown that the converses to both statements are false.

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1. Introduction

Let $\mathcal{B}(H)$ denote the algebra of all bounded linear operators on a separable, infinite dimensional complex Hilbert space H. We are interested in operator algebras $\mathcal{R} \subseteq \mathcal{B}(H)$ such that the unit ball $\{R \in \mathcal{R} \colon \|R\| \leq 1\}$ is precompact in the strong operator topology.

Marsalli [7] showed that a von Neumann algebra has a strongly precompact unit ball if and only if it can be decomposed as a direct sum of finite dimensional von Neumann algebras.

Let μ be a finite measure of compact support defined on the Borel subsets of the complex plane and let $K \subseteq \mathbb{C}$ denote the support of μ. Any function algebra $\mathcal{R} \subseteq L^\infty(\mu)$ can be regarded as an algebra of multiplication operators on $L^2(\mu)$. The following is a characterization of the subalgebras $\mathcal{R} \subseteq L^\infty(\mu)$ with the property that the unit ball of $\mathcal{R}$ is precompact in the strong operator topology.

Theorem 1.1 (Froelich and Marsalli [4])**.** *Let $\mathcal{R}$ be a subalgebra of $L^\infty(\mu)$. The following are equivalent:*

1. *The unit ball of $\mathcal{R}$ is precompact in the strong operator topology.*
2. *The natural embedding $\mathcal{R} \hookrightarrow L^2(\mu)$ is a compact operator.*
3. *Any bounded sequence in $\mathcal{R}$ has a subsequence that converges μ-almost everywhere.*

We consider the following subalgebras of $L^\infty(\mu)$. First of all, $P(K)$ is the uniform closure of the analytic polynomials. Next, $R(K)$ is the uniform closure of the rational functions with poles outside K, and $A(K)$ is the algebra of all continuous functions on K that are analytic on $\operatorname{int}(K)$. As customary, $C(K)$ is the algebra of all continuous functions on K.

Let G_0 denote the unbounded component of $\mathbb{C}\backslash K$. Recall that the polynomially convex hull of K is the compact set $\widehat{K} = \mathbb{C}\backslash G_0$. It is plain that $K \subseteq \widehat{K}$ and that $\mathbb{C}\backslash\widehat{K}$ is connected.

Froelich and Marsalli [4] considered the problem of characterizing among the above function algebras the ones that have a unit ball that is precompact in the strong operator topology. They proved the following

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Theorem 1.2. 1. *The algebra $L^\infty(\mu)$ has a strongly precompact unit ball if and only if the measure μ is discrete.*
2. *The algebra $C(K)$ has a strongly precompact unit ball if and only if the measure μ is discrete.*
3. *The algebra $P(K)$ has a strongly precompact unit ball if and only if the restriction of the measure μ to the boundary of $\widehat{K}$ is discrete.*
4. *The algebra $R(K)$ has a strongly precompact unit ball provided that the restriction of the measure μ to the boundary of K is discrete.*
5. *The algebra $A(K)$ has a strongly precompact unit ball provided that the restriction of the measure μ to the boundary of K is discrete.*

The authors [6] obtained independently a characterization of those normal operators that generate an algebra whose unit ball is precompact in the strong operator topology. This characterization is similar to part (3) of Theorem 1.2.

Froelich and Marsalli [4] claimed that the sufficient conditions for the unit ball to be strongly precompact stated in parts (4) and (5) of Theorem 1.2 were also necessary.

We were told in a private communication that they discovered after their paper was published that there was an inaccuracy in their argument and that they did not really know if those conditions were necessary. A negative answer to this question is given by Theorem 2.2.

Let us explain why their argument is inaccurate. Let $\{G_n\}_{n=1}^\infty$ denote the sequence of the bounded components of $\mathbb{C}\backslash K$. Froelich and Marsalli [4] showed that if the function algebra $R(K)$ has a strongly precompact unit ball then the measure μ restricted to each ∂G_n is discrete. Then, they concluded that the measure μ restricted to ∂K is also discrete by using the identity

$$\partial K = \bigcup_{n=0}^{\infty} \partial G_n.$$

Although this is the case, for instance, when $\mathbb{C}\backslash K$ has finitely many components, in general this identity does not hold and the best that can be said about the boundary of K is that

$$\partial K \supseteq \bigcup_{n=0}^{\infty} \partial G_n.$$

Nevertheless, there is something that remains valid: if the measure μ restricted to ∂K is discrete then the algebra $A(K)$ has a strongly precompact unit ball, and if the algebra $R(K)$ has a strongly precompact unit ball then the measure μ restricted to each ∂G_n is discrete.

The aim of this paper is to provide several examples that go to illustrate the following phenomena. First, in Theorem 2.2, it is shown that there is a measure μ that restricted to ∂K is not discrete although the unit ball of $A(K)$ is strongly precompact. Second, in Theorem 2.7, it is shown that there is a measure μ that restricted to each ∂G_n is discrete, although the unit ball of $R(K)$ fails to be strongly precompact. Third, in Theorem 2.11, it is shown that there is a measure μ such that the unit ball of $R(K)$ is strongly precompact but the unit ball of $A(K)$ fails to be strongly precompact.

A key for our approach is the fact that, for a compact subset $L \subseteq K$, a measure μ supported on L, and for $\mathcal{R} = A(K)$ or $R(K)$, the natural mapping $\mathcal{R} \hookrightarrow L^2(\mu)$ is compact provided that the restriction mapping $R\colon \mathcal{R} \to C(L)$ is compact. In Theorem 2.12, it is shown that, given a compact subset $L \subseteq K$ and a subalgebra $\mathcal{R} \subseteq C(K)$, the natural mapping $\mathcal{R} \hookrightarrow L^2(\mu)$ is compact for every measure μ supported on L if and only if the restriction mapping $R\colon \mathcal{R} \to C(L)$ does not fix a copy of ℓ_1.

2. The main results

Example 2.1. Let $\{q_n\}$ be a countable dense subset of $[0,1]$ with the property that $1/2^n \le q_n \le 1 - 1/2^n$, let $z_n = q_n + i/2^n$, and let $0 < r_n < 1/2^{n+3}$ to be chosen later on. Then, consider the open set

$$G = \bigcup_{n=1}^{\infty} D(z_n, r_n).$$

Notice that the discs $D(z_n, r_n)$ are all contained in the square $[0,1] \times [0,1]$ and that their projections onto the imaginary axis are pairwise disjoint. Now, consider the compact set $K = [-2,2] \times [-2,2] \setminus G$, and notice that its boundary is the set

$$\partial K = \left(\bigcup_{n=0}^{\infty} \gamma_n \right) \cup [0,1],$$

where $\gamma_0 = \partial(\,[-2,2] \times [-2,2]\,)$ and $\gamma_n = \partial D(z_n, r_n)$ for each $n \geq 1$.

Our first result concerns measures supported on this compact set K, and it can be stated as follows.

Theorem 2.2. *There is a suitable choice for the sequence of radii (r_n) such that for any finite measure μ defined on the Borel subsets of the complex plane whose support is the compact set K, if the restriction of the measure μ to each γ_n is discrete then the algebra $A(K)$ has a strongly precompact unit ball.*

Notice that the restriction of the measure μ to the interval $[0,1]$ is not necessarily discrete, and in fact, it can have any prescribed values. Such a measure μ can be constructed in the following way. Take a sequence of complex numbers (z_k) dense in $\bigcup_{n=0}^{\infty} \gamma_n$ and let δ_{z_k} denote the Dirac measure at z_k. Next, let λ_1 denote the one dimensional Lebesgue measure on the real line, and let λ_2 denote the two dimensional Lebesgue measure on the complex plane. Then, let μ be the measure defined on the Borel subsets of the complex plane by the expression

$$\mu(B) = \lambda_1(B \cap [0,1]) + \lambda_2(B \cap K) + \sum_{k=1}^{\infty} \frac{1}{2^k} \delta_{z_k}(B).$$

It is easy to see that $\operatorname{supp}(\mu) = K$, each restriction $\mu_{|\gamma_n}$ is discrete, and the restriction $\mu_{|[0,1]}$ is not discrete. Now we state and prove three lemmas before we proceed with the proof of Theorem 2.2.

Lemma 2.3. *If $f \in A(K)$ and $z \in \operatorname{int}(K)$ then*

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \oint_{\gamma_n} \frac{f(\xi)}{\xi - z} d\xi,$$

where γ_n is oriented counterclockwise when $n = 0$ and clockwise when $n \geq 1$.

Proof of Lemma 2.3. Fix $z \in \operatorname{int} K$, let $N \geq 1$, and consider the rectangle

$$R_N = [0,1] \times [0, (2^{-N} + 2^{-(N+1)})/2].$$

Then, choose N large enough, so that $z \notin R_N$. It follows from Cauchy's integral formula that

$$f(z) = \left(\sum_{n=0}^{N} \frac{1}{2\pi i} \oint_{\gamma_n} \frac{f(\xi)}{\xi - z}\, d\xi \right) + \frac{1}{2\pi i} \oint_{\partial R_N} \frac{f(\xi)}{\xi - z}\, d\xi,$$

where ∂R_N is oriented clockwise. Although the contour of integration is not contained in the interior of K, we may use a standard approximation argument by slightly enlarging the radii of the circles γ_n and lowering the bottom of R_N a little bit, just enough to fit the contour inside the interior of K and to obtain the desired identity after a limiting process. Finally,

$$\lim_{N \to \infty} \frac{1}{2\pi i} \oint_{\partial R_N} \frac{f(\xi)}{\xi - z} d\xi = 0,$$

because, f being continuous, the integrals along the vertical segments of ∂R_N can be made arbitrarily small, while the integrals along the horizontal segments of ∂R_N tend to cancel each other. □

Lemma 2.4. *If the sequence of radii* (r_n) *satisfies* $\sum_{n=1}^{\infty} 2^n r_n < \infty$, *then for every* $f \in A(K)$ *and for every* $x \in [0,1]$ *we have*

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \oint_{\gamma_n} \frac{f(\xi)}{\xi - x} d\xi.$$

Proof of Lemma 2.4. Consider for each $z \in [0,1] \times [-1,0]$ the series of functions

$$\sum_{n=0}^{\infty} \frac{1}{2\pi i} \oint_{\gamma_n} \frac{f(\xi)}{\xi - z}\, d\xi.$$

and notice that

$$\left| \frac{1}{2\pi i} \oint_{\gamma_n} \frac{f(\xi)}{\xi - z} d\xi \right| \leq \frac{\|f\|_\infty\, r_n}{d(z, \gamma_n)} \leq \frac{\|f\|_\infty\, r_n}{2^{-n} - r_n} = \frac{\|f\|_\infty\, 2^n r_n}{1 - 2^n r_n}.$$

It follows from the Weierstrass M-test that the series of functions converges uniformly on $[0,1] \times [-1,0]$ to a continuous function, say g. According with Lemma 2.3, we have $f(z) = g(z)$ for every $z \in [0,1] \times [-1,0)$, and it follows from continuity that $f(x) = g(x)$ for each $x \in [0,1]$. □

Lemma 2.5. *If* $\sum_{n=1}^{\infty} 2^n r_n < \infty$ *then the restriction mapping* $T : A(K) \to C[0,1]$ *is a compact operator.*

Proof of Lemma 2.5. According with Lemma 2.4, we have the series expansion for the restriction operator

$$T = \sum_{n=0}^{\infty} T_n,$$

where each operator $T_n\colon A(K) \to C[0,1]$ is given by the expression

$$(T_n f)(x) = \frac{1}{2\pi i} \oint_{\gamma_n} \frac{f(\xi)}{\xi - x} d\xi.$$

Also, it follows from the computations in Lemma 2.4 that we have the estimate

$$\|T_n\| \leq \frac{r_n}{2^{-n} - r_n},$$

so that the series converges in the uniform topology. Moreover, each T_n is a compact operator. Indeed, if $f \in A(K)$ satisfies $\|f\|_\infty \leq 1$ then a quick computation with $x, y \in [0,1]$ leads to the inequality

$$|(T_n f)(x) - (T_n f)(y)| \leq \frac{r_n}{(2^{-n} - r_n)^2} \cdot |x - y|.$$

Hence, $T_n f$ satisfies a Lipschitz condition with a Lipschitz constant that only depends on n. Therefore, $\{T_n f : f \in A(K), \|f\|_\infty \leq 1\}$ is an equicontinuous family and it follows from Ascoli's Theorem that T_n is a compact operator. Hence, T is the limit in the uniform topology of a sequence of compact operators, so that T is a compact operator itself, as we wanted. □

Proof of Theorem 2.2. Let (f_n) be a bounded sequence in $A(K)$. It follows from Montel's theorem that there is a subsequence (f_{n_j}) that converges uniformly on compact subsets of $\text{int}(K)$. Since the restriction of μ to each γ_n is discrete, we may apply a standard diagonal procedure to extract a subsequence $(f_{n_{j_k}})$ that converges μ-almost everywhere on the union $\bigcup_n \gamma_n$. Finally, we may apply Lemma 2.5 to extract a further subsequence $(f_{n_{j_{k_l}}})$ that converges uniformly on the interval $[0,1]$. Since we have the decomposition

$$K = \text{int}(K) \cup \left(\bigcup_{n=0}^{\infty} \gamma_n \right) \cup [0,1],$$

we conclude that the subsequence $(f_{n_{j_{k_l}}})$ converges μ-almost everywhere on K. It follows from Theorem 1.1 that the algebra $A(K)$ has a strongly precompact unit ball. $\square$

Example 2.6. Consider the compact set $K \subseteq \mathbb{C}$ defined by the expression

$$K = (\{0,1\} \times [-1,1]) \cup ([0,1] \times \{\frac{1}{n} : n \in \mathbb{Z}, n \neq 0\}) \cup ([0,1] \times \{0\}).$$

Since K has zero planar measure, we get from the Hartogs-Rosenthal Theorem that $R(K) = C(K)$. (See the book of Gamelin [5, p.47] for a reference on the Hartogs-Rosenthal Theorem.) The complement of K in the complex plane is the union of the sequence of disjoint components,

$$\mathbb{C} \backslash K = \bigcup_{n=0}^{\infty} G_n,$$

and the following relationship with the boundaries is fulfilled

$$\partial K \backslash \bigcup_{n=0}^{\infty} \partial G_n = (0,1).$$

Theorem 2.7. *There is a measure μ whose support is K and such that the unit ball of the algebra $R(K)$ fails to be strongly precompact although the restriction of μ to each ∂G_n is discrete.*

Proof of Theorem 2.7. Let μ be a finite positive Borel measure whose support is K, such that the restriction of μ to each ∂G_n is discrete and the restriction of μ to $(0,1)$ is the Lebesgue measure. Then, the sequence of functions (f_n) defined by $f_n(z) = e^{2\pi i n \mathrm{Re} z}$ is bounded in $C(K)$ and its restriction to $(0,1)$ has no convergent subsequence. Hence, the unit ball of $R(K)$ fails to be strongly precompact. $\square$

Example 2.8 (The Swiss cheese). Consider an infinite sequence of open discs $D_n = D(z_n, r_n)$ such that $\overline{D}_n \subseteq \mathbb{D}$ and $\overline{D}_n \cap \overline{D}_m = \emptyset$ when $n \neq m$, and such that the compact set

$$K = \overline{\mathbb{D}} \backslash \bigcup_{n=1}^{\infty} D_n$$

has empty interior. Such a compact set is called a Swiss cheese. See the book of Gamelin [5, p.25] for more information on this issue. There is some variety in the literature in the definition of Swiss cheeses. Frequently, the requirement that the sum of the radii is finite is included. Some authors allow the discs to touch or overlap, but strengthen other conditions to compensate. The reader can also find a survey of the use of Swiss cheese constructions in the theory of uniform algebras in the paper of Feinstein and Heath [3].

Since K has empty interior, we have $A(K) = C(K)$. If μ is a non discrete measure supported on K then it follows from Theorem 1.2 that the unit ball of the algebra $A(K)$ fails to be strongly precompact.

Our next goal is to show that there are a Swiss cheese $K \subseteq \mathbb{C}$ and a non discrete measure μ supported on K such that the unit ball of the algebra $R(K)$ is strongly precompact.

Lemma 2.9. *If (R_n) is a sequence of positive numbers such that $\sum_{n=1}^{\infty} R_n^2 < 1$ then there is a sequence of positive numbers (r_n) such that $0 < r_n < R_n$ and there is a sequence of complex numbers (z_n) in the unit disc such that the set $K = \overline{\mathbb{D}} \backslash \bigcup_{n=1}^{\infty} D(z_n, r_n)$ is a Swiss cheese and such that $\sum_{n=1}^{\infty} r_n/R_n < \infty$.*

Proof of Lemma 2.9. Let (s_n) be a sequence of positive numbers with $s_n < R_n$ and with $\sum_{n=1}^{\infty} s_n/R_n < \infty$. Then, let $W \subseteq \mathbb{D}$ be a countable, dense subset, say $W = \{w_n : n \in \mathbb{N}\}$. Now, define the sequence of discs $\{D(r_n, z_n)\}$ inductively. Start with $z_1 = w_1$ and choose $0 < r_1 \leq s_1$ such that $\overline{D}(z_1, r_1) \subseteq \mathbb{D}$. Since the open set $\mathbb{D} \backslash \overline{D}(z_1, r_1)$ is non empty, we may take

$$m_2 = \min\{m \in \mathbb{N} : w_m \in \mathbb{D} \backslash \overline{D}(z_1, r_1)\}.$$

It is clear that $m_2 \geq 2$. Define $z_2 = w_{m_2}$ and let $0 < r_2 \leq s_2$ be such that $\overline{D}(z_2, r_2) \subseteq \mathbb{D} \setminus \overline{D}(z_1, r_1)$. Now, suppose that we have defined the indices m_j for $1 \leq j \leq k$, so that $m_k \geq j$ and $w_m \in \bigcup_{j=1}^{k} \overline{D}(z_j, r_j)$ for $1 \leq m \leq m_k$. Also, suppose that we have constructed the discs $\overline{D}(z_j, r_j)$ for $1 \leq j \leq k$, so that

$$\overline{D}(z_j, r_j) \subseteq \mathbb{D} \setminus \bigcup_{h=1}^{j-1} \overline{D}(z_h, r_h).$$

Set $m_{k+1} = \min\{m \in \mathbb{N} \colon w_m \in \mathbb{D} \setminus \bigcup_{j=1}^{k} \overline{D}(z_j, r_j)\}$ and notice that $m_{k+1} \geq k+1$. Set $z_{k+1} = w_{m_{k+1}}$. Then choose $0 < r_{k+1} \leq s_{k+1}$ such that

$$\overline{D}(z_{k+1}, r_{k+1}) \subseteq \mathbb{D} \setminus \bigcup_{j=1}^{k} \overline{D}(z_j, r_j).$$

It is plain that $0 < r_k < R_k$ and $\sum_{n=1}^{\infty} r_k / R_k < \infty$. We claim that the compact set $K = \overline{\mathbb{D}} \setminus \bigcup_{k=1}^{\infty} D(z_k, r_k)$ has empty interior. Otherwise we have $\operatorname{int}(K) \subseteq \mathbb{D}$ and $\operatorname{int}(K) \cap D(z_n, r_n) = \emptyset$, so that $\operatorname{int}(K) \cap \overline{D}(z_n, r_n) = \emptyset$. Therefore we have

$$\emptyset \neq \operatorname{int}(K) \subseteq \mathbb{D} \setminus \bigcup_{k=1}^{\infty} \overline{D}(z_k, r_k).$$

Since W is dense in $\mathbb{D}$, we have $W \cap \operatorname{int} K \neq \emptyset$, so that there is an $l \in \mathbb{N}$ such that $w_l \in \mathbb{D} \setminus \bigcup_{k=1}^{\infty} \overline{D}(z_k, r_k)$. Finally, choose some $k \geq 1$ such that $l \leq m_k$, and notice that $w_l \in \bigcup_{j=1}^{k} \overline{D}(z_j, r_j)$, which is a contradiction. □

Lemma 2.10. *There is a suitable choice of the sequence (r_n) for which there is a compact subset $L \subseteq K$ of positive planar measure such that the restriction mapping $T \colon R(K) \to C(L)$ is a compact operator.*

Proof of Lemma 2.10. Choose two sequences $(r_n), (R_n)$ such that $0 < r_n < R_n$ and such that the following inequalities hold

$$\sum_{n=1}^{\infty} R_n^2 < 1, \qquad \sum_{n=1}^{\infty} \frac{r_n}{R_n} < \infty.$$

Consider a rational function f with poles outside K, and let $z \in (K \setminus \bigcup_{n=1}^{\infty} \partial D_n) \setminus \partial\mathbb{D}$. It follows from Cauchy's integral formula that

$$f(z) = \frac{1}{2\pi i} \oint_{\partial\mathbb{D}} \frac{f(\xi)}{\xi - z} \, d\xi - \frac{1}{2\pi i} \sum_{n=1}^{\infty} \oint_{\partial D_n} \frac{f(\xi)}{\xi - z} \, d\xi.$$

Now, consider the compact set

$$L_0 = \overline{\mathbb{D}} \setminus \bigcup_{n=1}^{\infty} D(z_n, R_n).$$

Notice that $L_0 \subseteq K$ and L_0 has positive planar measure. Next, there is an $r < 1$ such that the compact set $L = L_0 \cap \overline{D}(0, r)$ still has positive planar measure. Thus, for every $z \in L$ we have

$$f(z) = (T_0 f)(z) - \sum_{n=1}^{\infty} (T_n f)(z),$$

where the operators T_n are given by the expression

$$(T_0 f)(z) = \frac{1}{2\pi i} \oint_{\partial\mathbb{D}} \frac{f(\xi)}{\xi - z} \, d\xi, \qquad (T_n f)(z) = \frac{1}{2\pi i} \oint_{\partial D_n} \frac{f(\xi)}{\xi - z} \, d\xi.$$

Notice that

$$|(T_0 f)(z)| \leq \frac{1}{1-r}\|f\|_\infty, \qquad |(T_n f)(z)| \leq \frac{r_n}{R_n - r_n}\|f\|_\infty,$$

so that we can extend the operators $T_n \colon R(K) \to C(L)$ in such a way that

$$\|T_0\| \leq \frac{1}{1-r}, \qquad \|T_n\| \leq \frac{r_n}{R_n - r_n}.$$

Moreover, each T_n is a compact operator. Indeed, if f is a rational function with poles outside K that satisfies $\|f\|_\infty \leq 1$ then a quick computation with $z, w \in L$ leads to the inequalities

$$\begin{aligned} |(T_0 f)(z) - (T_0 f)(w)| &\leq \frac{1}{(1-r)^2} \cdot |z - w|, \\ |(T_n f)(z) - (T_n f)(w)| &\leq \frac{r_n}{(R_n - r_n)^2} \cdot |z - w|. \end{aligned}$$

Hence, $T_n f$ satisfies a Lipschitz condition with a Lipschitz constant that only depends on n. Therefore, $\{T_n f : f \in R(K), \|f\|_\infty \leq 1\}$ is an equicontinuous family and it follows from Ascoli's Theorem that T_n is a compact operator. Finally, the restriction operator $T \colon R(K) \to C(L)$ is represented as a uniformly convergent series expansion of compact operators

$$T = T_0 - \sum_{n=1}^{\infty} T_n,$$

and we conclude that T itself is a compact operator. □

Theorem 2.11. *There is a suitable choice of the sequence (r_n) and there is a measure μ such that the unit ball of $R(K)$ is strongly precompact but the unit ball of $A(K)$ fails to be strongly precompact.*

Proof of Theorem 2.11. Let $L \subseteq K$ be a compact subset as in Lemma 2.10. Let μ be a measure whose support is K and whose restriction to L equals the planar Lebesgue measure and such that the restriction of μ to $K \backslash L$ is discrete. Since $A(K) = C(K)$, it follows from Theorem 1.2 that the unit ball of $A(K)$ fails to be strongly precompact. Now, let (f_n) be a bounded sequence in $R(K)$. Since μ restricted to $K \backslash L$ is discrete, we may extract a subsequence (f_{n_j}) that converges μ-almost everywhere on $K \backslash L$. Since the restriction mapping $T \colon R(K) \to C(L)$ is a compact operator, we may extract a further subsequence $(f_{n_{j_k}})$ that converges uniformly on L. It follows from Theorem 1.1 that the algebra $R(K)$ has a strongly precompact unit ball. □

Notice that in the proofs of Theorem 2.2 and Theorem 2.11 we have used the following fact. Let $L \subseteq K$ be a compact subset and let $\mathcal{R} \subseteq C(K)$ be a subalgebra. If the restriction mapping $R \colon \mathcal{R} \to C(L)$ is compact then, for any measure μ supported on L, the natural mapping $\mathcal{R} \hookrightarrow L^2(\mu)$ is compact. The following result shows how the assumption on the restriction mapping being compact can be relaxed to obtain a necessary and sufficient condition for the natural mapping being compact. Recall that, according to Theorem 1.1, the latter condition is equivalent for the algebra $\mathcal{R}$ to have a strongly precompact unit ball.

Theorem 2.12. *Let $L \subseteq K$ be two compact sets and let $\mathcal{R} \subseteq C(K)$ be a closed subspace. Then the following conditions are equivalent:*

1. *The natural mapping $\mathcal{R} \hookrightarrow L^2(\mu)$ is compact for every measure μ supported on L.*
2. *The restriction mapping $R \colon \mathcal{R} \to C(L)$ does not fix a copy of ℓ_1.*

Proof of Theorem 2.12. First, suppose that R does not fix a copy of ℓ_1 and take any sequence (f_n) in $\mathrm{ball}(\mathcal{R})$. Then (Rf_n) does not admit a subsequence equivalent to the standard unit vector basis of ℓ_1. It follows from Rosenthal's Theorem that there is a weakly Cauchy subsequence (Rf_{n_k}). (See the book of Diestel [1, p.201] for a reference on Rosenthal's Theorem.) Hence, for every $x^* \in C(L)^*$, the scalar sequence $x^*(Rf_{n_k})$ is convergent. In particular, taking $x^* = \delta_z$ for $z \in L$, it follows that there is a function $g\colon L \to \mathbb{C}$ such that $f_{n_k}(z) \to g(z)$ for every $z \in L$. Since the sequence (f_{n_k}) is uniformly bounded, it follows from the bounded convergence theorem that $f_{n_k} \to g$ in $L^2(\mu)$. Hence, the natural mapping $\mathcal{R} \hookrightarrow L^2(\mu)$ is compact.

Next, suppose that R fixes a copy of ℓ_1, so that there is a bounded sequence (f_n) in $\mathcal{R}$ such that (Rf_n) is equivalent to the standard unit vector basis of ℓ_1. Now, let $X \subseteq C(L)$ be the closed linear span of (Rf_n), let (e_n) be the standard unit vector basis of ℓ_2, and consider the operator $T\colon X \to \ell_2$ defined by $TRf_n = e_n$. Since X is isomorphic to ℓ_1, it follows from Grothendieck's Theorem that T is 2-summing. Then, it follows from Pietsch's Factorization Theorem that there is a measure μ supported on L and there is some $C > 0$ such that $\|Tg\| \leq C\|g\|_{L^2(\mu)}$ for all $g \in X$. (See the book of Diestel, Jarchow and Tonge [2, p.15 and p.48] for a reference on Grothendieck's Theorem and Pietsch's Factorization Theorem.) Finally, we have

$$\sqrt{2} = \|e_n - e_m\| = \|TRf_n - TRf_m\| \leq C\|Rf_n - Rf_m\|_{L^2(\mu)} = C\|f_n - f_m\|_{L^2(\mu)},$$

so that the sequence (f_n) has no convergent subsequence in $L^2(\mu)$. Hence, the natural mapping $\mathcal{R} \hookrightarrow L^2(\mu)$ fails to be compact. $\square$

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References

[1] J. Diestel, *Sequences and series on Banach spaces,* Springer-Verlag, New York (1984).
[2] J. Diestel, H. Jarchow and A. Tonge, *Absolutely Summing Operators,* Cambridge University Press (1995).
[3] J. F. Feinstein and M. J. Heath, *Swiss cheeses, rational approximation and universal plane curves,* Studia Math. 196 (2010), 289–306.
[4] J. Froelich and M. Marsalli, *Operator semigroups, flows, and their invariant sets,* J. Funct. Anal. 115 (1993), 454–479.
[5] T.W. Gamelin, *Uniform algebras,* AMS Chelsea Publishing, Providence, Rhode Island (2005).
[6] M. Lacruz and L. Rodríguez-Piazza, *Strongly compact normal operators,* Proc. Amer. Math. Soc. 137 (2009), 2623–2630.
[7] M. Marsalli, *A classification of operator algebras,* J. Operator Theory 24 (1990), 155–163.