APPROXIMATION OF LIPSCHITZ FUNCTIONS BY Δ-CONVEX FUNCTIONS IN BANACH SPACES

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Abstract. In this paper we give some results about the approximation of a Lipschitz function on a Banach space by means of Δ-convex functions. In particular, we prove that the density of Δ-convex functions in the set of Lipschitz functions for the topology of uniform convergence on bounded sets characterizes the superreflexivity of the Banach space. We also show that Lipschitz functions on superreflexive Banach spaces are uniform limits on the whole space of Δ-convex functions.

0. Introduction and Notations

A function defined on a Banach space X is called Δ-*convex* if it can be expressed as a difference of continuous convex functions or, equivalently, if it belongs to the linear span of the continuous convex functions on X. The purpose of this paper is to give some necessary and sufficient conditions for the approximation of Lipschitz functions by Δ-convex functions. The initial motivation for our work comes from two recent articles of R. Deville, V. Fonf and P. Hájek ([DFH_1] and [DFH_2]). A consequence of their results is that, under certain conditions on the Banach space X, any convex function on X which is bounded on bounded sets can be approximated by smooth convex functions. It is therefore natural to consider the class of Banach spaces for which the Δ-convex functions are dense in the class of Lipschitz functions in order to extend this property of smooth approximations.

Our main result is the following characterization of superreflexivity.

Theorem 0. *Let X be a Banach space. Then X is superreflexive if and only if every Lipschitz function on X can be approximated uniformly on bounded sets by differences of convex functions on X which are bounded on bounded sets.*

A consequence of **Theorem 0** is that the above mentioned approach does not provide any new result, because it works only for superreflexive spaces (for which the smooth approximation property is known using the existence of partitions of unity, see Ch. VIII of [DGZ]; for the analytic approximation case, see [K]).

For the superreflexive case, we give explicit formulas for the Δ-convex approximation of a Lipschitz function. These formulas, which are simpler than those from

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[St], also provide uniform convergence on the whole space X. Specifically, the degree of convergence given by our formulas relies directly on the rotundity of the equivalent norm that it is used. Similar ideas in this direction can be found in [A] and [PVZ].

We thank P. Hájek for bringing to our attention the link between our work and the *distortion theorem* (see [OS] for definitions and details). Our results provide simple formulas for deducing, on minimal superreflexive Banach spaces (such as ℓ_p, $1 < p < \infty$), the existence of a convex function which is not oscillation stable from the existence of a Lipschitz function which is not oscillation stable.

Let us fix some notation used in this paper. For a real Banach space X, we denote an equivalent norm on X by $\|\cdot\|$ and by B_X its closed unit ball under this norm. By convex function we will always mean continuous convex function. We will consider two fundamental topologies on the set of continuous functions defined on X: τ_κ (respectively τ_b) is the topology of uniform convergence on compact sets of X (resp. uniform convergence on bounded sets of X).

The *modulus of convexity* of the norm $\|\cdot\|$ defined by

$$\delta_{\|\cdot\|}(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in B_X; \|x-y\| \geq \varepsilon\right\} \quad (0 < \varepsilon < 2)$$

is called *of power type* p ($p \geq 2$) if $\delta_{\|\cdot\|}(\varepsilon) \geq K\varepsilon^p$, for some $K > 0$.

The concept of *dyadic tree* will play an important role in the second part of this work. Our trees are geometric trees contained in X and defined as follows. The symbol α denote a multi-index $\alpha = (\alpha_1 \frown \alpha_2 \frown \ldots \frown \alpha_n) \in \{-1,1\}^{<\mathbb{N}}$ and $|\alpha| := n$. For $n \in \mathbb{N}$, a dyadic (n,θ)-tree T in X is a set of the form $\{x_\alpha \in X : \alpha \in \{-1,1\}^{<n}\}$ satisfying the following two conditions:

(1) $x_\alpha = \frac{1}{2}x_{\alpha\frown 1} + \frac{1}{2}x_{\alpha\frown -1}$, for all $|\alpha| < n$.
(2) $\|x_\alpha - x_{\alpha'}\| \geq \theta > 0$, for $\alpha \neq \alpha'$.

The point $x_\varnothing$ will be called the *root* of the tree T.

1. The positive results

Theorem 1.
Let $(X, \|\cdot\|)$ be a Banach space. The norm $\|\cdot\|$ is locally uniformly convex (respectively uniformly convex) if and only if the following property holds: for every Lipschitz function f on X, the sequence of functions $(f_n)_{n\in\mathbb{N}}$ defined by the formula

$$f_n(x) := \inf_{y\in X}\left\{f(y) + n\big(2\|x\|^2 + 2\|y\|^2 - \|x+y\|^2\big)\right\} \quad (n \in \mathbb{N},\ x \in X)$$

is τ_κ-converging (resp. τ_b-converging) to f.

Remark. For any function f on X and $n \in \mathbb{N}$, the function f_n defined as above is a Δ-convex function. This follows immediately from the decomposition $f_n = c_n - d_n$, with $c_n(x) := 2n\|x\|^2$ and

$$d_n(x) := \sup_{y\in X}\left\{n\|x+y\|^2 - 2n\|y\|^2 - f(y)\right\} \quad (x \in X).$$

The functions c_n and d_n are clearly convex.

Proof of the Theorem 1.

Let us see first that the property is necessary. So, let f be a Lipschitz function on X. We have to show that the previously defined sequence $(f_n)_{n\in\mathbb{N}}$ (τ_K or τ_b)-converges to f if the norm $\|\cdot\|$ satisfies the corresponding rotundity condition.

We begin with the following general result:

Fact. For any point $x \in X$, $(f_n(x))_{n\in\mathbb{N}}$ is an increasing sequence bounded above by $f(x)$.

This *Fact* follows immediately from taking $y = x$ in the f_n's infimum formula and from the inequality

$$2\|x\|^2+2\|y\|^2-\|x+y\|^2 \geq 2\|x\|^2+2\|y\|^2-\big(\|x\|+\|y\|\big)^2 = \big(\|x\|-\|y\|\big)^2 \geq 0. \quad (1)$$

Since for $K \in \mathbb{N}$ we have that $f_{(K\cdot n)} = K\big(\frac{f}{K}\big)_n$, the previous *Fact* allows us to suppose without loss of generality that the Lipschitz constant of f is less than 1 (*i.e.* $|f(x)-f(y)| \leq \|x-y\|$ for all $x, y \in X$).

We need to study of the infimum formula defining f_n at a point $x \in X$. Thus, consider any point y for which we have

$$f(y) + n\big(2\|x\|^2 + 2\|y\|^2 - \|x+y\|^2\big) \leq f(x). \quad (2)$$

As f is 1-Lipschitz, we deduce from **(1)** and **(2)** that

$$n\big(\|x\|-\|y\|\big)^2 \leq n\big(2\|x\|^2+2\|y\|^2-\|x+y\|^2\big) \leq f(x)-f(y) \leq \|x-y\|. \quad (3)$$

This last condition gives a relation between the norms of x and y. Actually, $\|y\|$ can be controlled by $\|x\|$ in the following way. Suppose that $\|y\| \geq 1+\|x\|$, then by **(3)** we get that

$$1 \leq \big|\|x\|-\|y\|\big| \leq \big(\|x\|-\|y\|\big)^2 \leq \frac{1}{n}\|x-y\| \leq \frac{1}{n}\|x\| + \frac{1}{n}\|y\|. \quad (4)$$

And we conclude for $n \geq 3$ that $\|y\| \leq \frac{n+1}{n-1}\|x\| \leq 2\|x\|$. Therefore, we see that if y satisfies **(2)** and $n \geq 3$ then

$$\|y\| \leq 2(1+\|x\|). \quad (5)$$

Then, for $n \geq 3$ we deduce that

$$f_n(x) = \inf_{\|y\|\leq 2(1+\|x\|)} \big\{f(y) + 2n\|x\|^2 + 2n\|y\|^2 - n\|x+y\|^2\big\}. \quad (6)$$

In particular, the boundedness on bounded sets of f_n (and, consequently, of $d_n = c_n - f_n$) follows immediately.

Moreover, the upper bound on $\|y\|$ given by **(5)** together with the condition **(3)** gives

$$0 \leq 2\|x\|^2+2\|y\|^2-\|x+y\|^2 \leq \frac{1}{n}\|x-y\| \leq \frac{1}{n}\|x\|+\frac{1}{n}\|y\| \leq \frac{3}{n}(1+\|x\|). \quad (7)$$

The condition **(7)** gives the crucial step of the proof. In fact, if the norm $\|\cdot\|$ verifies one of the rotundity properties stated in theorem then by **(7)** n can be chosen big enough to necessarily enforce y to be close to x. Therefore, $f_n(x)$ must be close to $f(x)$. Let us justify this assertion.

Suppose that the sequence of functions f_n is not compactly converging to f. Since the sequence of Δ-convex functions $(f_n)_n$ is increasing and f is continuous, Dini's theorem tells us that τ_κ-convergence of f_n to f is equivalent to the pointwise convergence. Then, there exists a point $x_0 \in X$ such that $(f_n(x_0))_n$ does not converge to $f(x_0)$.

As $(f_n(x_0))_n$ is increasing, there exists some $\varepsilon_0 > 0$ such that for any $n \in \mathbb{N}$ we have $f_n(x_0) + \varepsilon_0 < f(x_0)$. By definition of f_n, we can find a sequence of $(y_n)_n$ in X so that

$$f_n(x_0) + \varepsilon_0 \leq f(y_n) + 2n\|x_0\|^2 + 2n\|y_n\|^2 - n\|x_0 + y_n\|^2 + \varepsilon_0 \leq f(x_0). \tag{8}$$

Since f is 1-Lipschitz, we get from **(8)** that

$$\|x_0 - y_n\| \geq f(x_0) - f(y_n) \geq \varepsilon_0 > 0. \tag{9}$$

Since y_n verifies the condition **(8)**, it follows from **(7)** that we have

$$0 \leq 2\|x_0\|^2 + 2\|y_n\|^2 - \|x_0 + y_n\|^2 \leq \frac{3}{n}(1 + \|x_0\|) \xrightarrow[n\to\infty]{} 0. \tag{10}$$

But **(9)** and **(10)** show that the norm $\|\cdot\|$ can not be locally uniformly convex at x_0 (cf .Ch. II. Prop. 1.2 of [DGZ]). That proves the compact convergence of f_n to f if the norm $\|\cdot\|$ is locally uniformly convex.

A simple proof of the uniform convergence of the sequence f_n in the uniformly convex case follows the same lines. We will give later a direct quantitative approach (see the proof of **Theorem 3**). If the sequence f_n does not converge uniformly on a bounded set of X, there is an $\varepsilon_0 > 0$ and a bounded sequence $(x_n)_n$ so that $f_n(x_n) + \varepsilon_0 < f(x_n)$, $n \in \mathbb{N}$. Then, for each x_n ($n \in \mathbb{N}$) we can choose y_n verifying

$$f(y_n) + 2n\|x_n\|^2 + 2n\|y_n\|^2 - n\|x_n + y_n\|^2 + \varepsilon_0 \leq f(x_n). \tag{11}$$

Similar reasonings as before imply from **(11)** that the next two statements are fulfiled:

$$\|x_n - y_n\| \geq f(x_n) - f(y_n) \geq \varepsilon_0 > 0 \tag{12}$$

$$0 \leq 2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2 \leq \frac{3}{n}(1 + \|x_n\|) \xrightarrow[n\to\infty]{} 0 \tag{13}$$

We have $\lim_{n\to\infty} \frac{3}{n}(1+\|x_n\|) = 0$ since the sequence $(x_n)_n$ is bounded. Moreover, **(5)** implies that the sequence (y_n) is also bounded. Therefore, **(12)** and **(13)** show that the norm $\|\cdot\|$ cannot be uniformly convex (cf. Ch. IV Lemma 1.5 [DGZ]). The necessity of the property is proved.

Conversely, the property stated in the theorem is sufficient. We first show that the property of compact convergence implies the following claim.

Claim 1.1. *For any sequence* $(x_n)_{n\in\mathbb{N}} \subset X$ *and any point* $x_0 \in X$*, the condition* $\lim_{n\to\infty}\big(2\|x_0\|^2 + 2\|x_n\|^2 - \|x_0 + x_n\|^2\big) = 0$ *implies* $d\big(x_0, (x_n)_n\big) = 0$*.*

Since the **Claim 1.1** is also valid for any subsequence of the given sequence $(x_n)_n$, we can strengthen the conclusion of the claim to $\lim_{n\to\infty}\|x_n - x_0\| = 0$ and the locally convexity of the norm $\|\cdot\|$ is verified (cf .Ch. II. Prop. 1.2 of [DGZ]).

Proof of the Claim 1.1. Consider the Lipschitz function $g(x) = \mathrm{d}\big(x, (x_n)_n\big)$, $x \in X$. For $x_0 \in X$ and $\varepsilon > 0$, the pointwise convergence of the sequence $(g_n)_n$ to g at x_0 tells us that there exists $N_\varepsilon \in \mathbb{N}$ so that

$$\mathrm{d}\big(x_0, (x_n)_n\big) = g(x_0) \le \inf_{y\in X}\Big\{g(y) + N_\varepsilon\big(2\|x_0\|^2 + 2\|y\|^2 - \|x_0 + y\|^2\big)\Big\} + \varepsilon. \quad (14)$$

Using $g(x_n) = 0$, we can evaluate the inequality **(14)** at $y = x_n$ $(n \ge N_\varepsilon)$ and deduce that

$$\mathrm{d}\big(x_0, (x_n)_n\big) \le N_\varepsilon\big(\lim_{n\to\infty} 2\|x_0\|^2 + 2\|x_n\|^2 - \|x_0 + x_n\|^2\big) + \varepsilon = \varepsilon.$$

Therefore, $\mathrm{d}\big(x_0, (x_n)_n\big) = 0$ and the claim is proved.

If the property of uniform convergence on bounded sets of X holds, the next claim, analogous to the previous one, also does.

Claim 1.2. *Let* $(x_n)_{n\in\mathbb{N}}$ *and* $(y_n)_{n\in\mathbb{N}}$ *be two bounded sequences in* X *such that* $\lim_{n\to\infty}\big(2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2\big) = 0$*. Then* $\lim_{m\to\infty} d\big(y_m, (x_n)_n\big) = 0$*.*

Proof of the Claim 1.2. As before, consider the Lipschitz function $\mathrm{d}\big(\cdot, (x_n)_n\big)$ and $\varepsilon > 0$. This time, the property of uniform convergence gives a positive integer N_ε so that for all $m \in \mathbb{N}$ the next inequality is satisfied.

$$\mathrm{d}\big(y_m, (x_n)_n\big) \le \inf_{z\in X}\Big\{\mathrm{d}\big(z, (x_n)_n\big) + N_\varepsilon\big(2\|z\|^2 + 2\|y_m\|^2 - \|z + y_m\|^2\big)\Big\} + \varepsilon. \quad (15)$$

Taking $z = x_m$ in the infimum of **(15)** we obtain

$$\mathrm{d}\big(y_m, (x_n)_n\big) \le N_\varepsilon\big(2\|x_m\|^2 + 2\|y_m\|^2 - \|x_m + y_m\|^2\big) + \varepsilon \le 2\varepsilon,$$

for m large enough, and the claim is proved.

In order to finish with the proof, we shall show that the validity of **Claim 1.2** implies that the norm $\|\cdot\|$ is uniformly convex. If not, (cf. Ch. IV Lemma 1.5 [DGZ]) there exists two bounded sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in X satisfying that

$$\lim_{n\to\infty}\big(2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2\big) = 0 \text{ and } \|x_n - y_n\| \ge 1 \text{ (for all } n \in \mathbb{N}). \quad (16)$$

First, the norm $\|\cdot\|$ is locally uniformly convex (since **Claim 1.1** clearly holds). It follows that the sequence $(x_n)_n$ has no norm cluster point. Indeed, if for some $x_0 \in X$ there exists $(x_{n_k})_k \underset{k}{\longrightarrow} x_0$, then

$$\lim_{k\to\infty} 2\|x_0\|^2 + 2\|y_{n_k}\|^2 - \|x_0 + y_{n_k}\|^2 = \lim_{k\to\infty} 2\|x_0\|^2 + 2\|x_{n_k}\|^2 - \|x_0 + x_{n_k}\|^2 = 0$$

and therefore $(y_{n_k})_k \underset{k}{\longrightarrow} x_0$. A contradiction with the fact $\|x_n - y_n\| \ge 1$, for all $n \in \mathbb{N}$, of **(16)**.

Hence, passing to a subsequence, we can suppose that for some $1 > \alpha > 0$ we have that $\|x_n - x_m\| \ge \alpha$, for all $n \ne m$. Now, we need the following technical lemma, whose proof will be given later.

Lemma 1.3. *Let be $(\lambda_n)_{n\in\mathbb{N}} \subset [0,1]$ and $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ two bounded sequences in X so that $2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n+y_n\|^2 \xrightarrow[n\to\infty]{} 0$. Then the sequence of convex combinations $z_n = \lambda_n x_n + (1-\lambda_n)y_n$ ($n \in \mathbb{N}$) satisfies that $2\|x_n\|^2 + 2\|z_n\|^2 - \|x_n+z_n\|^2 \xrightarrow[n\to\infty]{} 0$.*

For any $n \in \mathbb{N}$, take $0 \le \lambda_n \le 1$ such that for $z_n := \lambda_n x_n + (1-\lambda_n)y_n$ we have $\|x_n - z_n\| = \frac{\alpha}{2}$. Then, the **Lemma 1.3** and the **Claim 1.2** used jointly imply that $\lim_{m\to\infty} \mathrm{d}(z_m, (x_n)) = 0$. But, $\|x_n - z_n\| = \frac{\alpha}{2} > 0$ and also for $n \neq m$ $\|x_n - z_m\| \ge \|x_n - x_m\| - \|x_m - z_m\| \ge \alpha - \frac{\alpha}{2} = \frac{\alpha}{2} > 0$, a contradiction. $\square$

Proof of the Lemma 1.3. By the inequality

$$0 \le \big(\|x_n\| - \|y_n\|\big)^2 \le 2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n+y_n\|^2 \xrightarrow[n\to\infty]{} 0, \tag{17}$$

we have that

$$\lim_{n\to\infty} \big(\|x_n\| - \|y_n\|\big) = 0.$$

Since the sequences $(x_n)_n$ and $(y_n)_n$ are bounded we also have that

$$\lim_{n\to\infty} \big(\|x_n\|^2 - \|y_n\|^2\big) = 0. \tag{18}$$

But then we deduce from **(17)** and **(18)** that

$$\lim_{n\to\infty} \left(\|x_n\|^2 - \left\|\frac{x_n+y_n}{2}\right\|^2\right) = \lim_{n\to\infty} \left(\|y_n\|^2 - \left\|\frac{x_n+y_n}{2}\right\|^2\right) = 0. \tag{19}$$

On the other hand, using the convexity of the function $\|\cdot\|^2$ we get the following general lower estimate for $0 \le \lambda \le 1$

$$\|\lambda x + (1-\lambda)y\|^2 \ge \left\|\frac{x+y}{2}\right\|^2 - |1-2\lambda| \max\left\{\left|\left\|\frac{x+y}{2}\right\|^2 - \|x\|^2\right|, \left|\left\|\frac{x+y}{2}\right\|^2 - \|y\|^2\right|\right\}.$$

Putting **(17)**, **(18)**, **(19)** and the last inequality together, we obtain that

$$\begin{aligned}
0 &\le 2\|x_n\|^2 + 2\|z_n\|^2 - \|x_n+z_n\|^2 \\
&= 2\|x_n\|^2 + 2\|\lambda_n x_n + (1-\lambda_n)y_n\|^2 - \|(1+\lambda_n)x_n + (1-\lambda_n)y_n\|^2 \\
&\le 2\|x_n\|^2 + 2\lambda_n\|x_n\|^2 + 2(1-\lambda_n)\|y_n\|^2 - 4\left\|\frac{1+\lambda_n}{2}x_n + \frac{1-\lambda_n}{2}y_n\right\|^2 \\
&\le 2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n+y_n\|^2 + \lambda_n\big(\|x_n\|^2 - \|y_n\|^2\big) \\
&\quad + 4\lambda_n \max\left\{\left|\left\|\frac{x_n+y_n}{2}\right\|^2 - \|x_n\|^2\right|, \left|\left\|\frac{x_n+y_n}{2}\right\|^2 - \|y_n\|^2\right|\right\} \xrightarrow[n\to\infty]{} 0.
\end{aligned}$$

The lemma is proved and this concludes the proof of **Theorem 1**.

Remark. The infimum formula used for the definition of (f_n) in the **Theorem 1** is closely related to the well-known *inf-convolution formula* of f by $n\|\cdot\|^2$:

$$\big(f \,\square\, (n\|\cdot\|^2)\big)(x) := \inf_{y\in X} \big\{f(y) + n\|x-y\|^2\big\} \qquad (n \in \mathbb{N},\, x \in X). \tag{20}$$

In fact, these two infimum formulas are identical if the norm $\|\cdot\|$ is a Hilbertian norm, because of the parallelogram identity. However, for a non-Hilbertian norm $\|\cdot\|$ the functions given by the inf-convolution formula can not be expressed in general as Δ-convex functions.

As a corollary of the above remark, we have that the formula of **Theorem 1** converges uniformly on X for case of a Hilbertian norm $\|\cdot\|$ (since it is well-known that the inf-convolution formula of **(20)** converges uniformly on the whole space X, see [LL]). The question that naturally arises is whether this remains true or not for a general uniformly convex norm. The answer is given in the following proposition.

Proposition 2. *Let $(X,\|\cdot\|)$ be a Banach space. If for every Lipschitz function f on X the sequence of functions*

$$f_n(x) := \inf_{y\in X}\left\{f(y) + n\big(2\|x\|^2 + 2\|y\|^2 - \|x+y\|^2\big)\right\}$$

converges to f uniformly on X, then the modulus of convexity of the norm $\|\cdot\|$ is of power type 2.

Proof of the Proposition 2. This uniform convergence property has the following consequence analogous to **Claim 1.1** and **Claim 1.2** and whose proof is identical to that of **Claim 1.2**.

Claim 2.1. *If $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are two sequences (not necessarily bounded) in X satisfying that $\lim_{n\to\infty}\big(2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n+y_n\|^2\big) = 0$, then they also verify that the $\lim_{m\to\infty} d\big(y_m,(x_n)_n\big) = 0$.*

By **Claim 1.2**, any norm which satisfies the conclusion of **Claim 2.1** is uniformly convex. In fact, **Claim 2.1** insures that the modulus of convexity of the norm $\|\cdot\|$ is of power type 2. If not (see [H]), there exists two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ such that

$$2\|x_n\|^2 + 2\|y_n\|^2 < \|x_n+y_n\|^2 + \frac{1}{n}\|x_n-y_n\|^2 \quad x_n \neq y_n,\ n\in\mathbb{N}. \tag{21}$$

If we take $u_n := \frac{x_n}{\|x_n-y_n\|}$ and $v_n := \frac{y_n}{\|x_n-y_n\|}$ in **(21)** we obtain that

$$0 \le 2\|u_n\|^2 + 2\|v_n\|^2 - \|u_n+v_n\|^2 < \frac{1}{n} \xrightarrow[n\to\infty]{} 0, \tag{22}$$

$$\|u_n - v_n\| = 1. \tag{23}$$

Since the norm $\|\cdot\|$ is uniformly convex and conditions **(22)** and **(23)** holds, the sequences $(u_n)_n$ and $(v_n)_n$ can not be bounded. Notice that we have also from **(22)** that $\lim_{n\to\infty}\big(\|u_n\| - \|v_n\|\big) = 0$. Thus, passing to a subsequence we can suppose that

$$\begin{aligned} &\sup\{\|u_n\|,\|v_n\|\} + 1 < \min\{\|u_{n+1}\|,\|v_{n+1}\|\} \text{ for all } n\in\mathbb{N}\\ &\quad\Longrightarrow \|u_n - v_m\| \ge \big|\|u_n\| - \|v_m\|\big| \ge 1 \quad (n\neq m). \end{aligned} \tag{24}$$

Assembling **(23)** and **(24)** we deduce that $\mathrm{d}\big(v_m,(u_n)_n\big) = 1$ $(m\in\mathbb{N})$. But **Claim 2.1** and **(22)** imply that $\lim_m \mathrm{d}\big(v_m,(u_n)_n\big) = 0$ which is a contradiction. □

Theorem 3 below provides a converse to **Proposition 2** and an explicit formula for uniform approximation of Lipschitz functions by Δ-convex functions on superreflexive Banach spaces. The validity of this formula depends upon the existence in the superreflexive space of an equivalent norm which is enough rotund.

Theorem 3. *Let $(X, \|\cdot\|)$ be a Banach space whose norm $\|\cdot\|$ has its modulus of convexity of power type p $(p \geq 2)$. Then for every Lipschitz function f on X the following sequence of Δ-convex functions*

$$f_n^p(x) = \inf_{y \in X} \left\{ f(y) + n\big(2^{p-1}\|x\|^p + 2^{p-1}\|y\|^p - \|x+y\|^p\big) \right\} \quad (n \in \mathbb{N},\ x \in X)$$

converges to f uniformly on X.

Proof of the Theorem 3. We essentially need the following lemma to prove the result.

Lemma 3.1. *If the modulus of convexity of the norm $\|\cdot\|$ is of power type p $(p \geq 2)$ then there exists a positive constant $C_{\|\cdot\|} \leq 1$ such that for every pair $x, y \in X$ the following inequality holds:*

$$C_{\|\cdot\|}\|x-y\|^p \leq 2^{p-1}\|x\|^p + 2^{p-1}\|y\|^p - \|x+y\|^p.$$

Proof of the Lemma 3.1. Under the assumption of the lemma (see [H]), there exists a positive constant $C'_{\|\cdot\|} \leq 2$ such that for every pair $u, v \in X$ we have that

$$\|u+v\|^p + \|u-v\|^p \geq 2\|u\|^p + C'_{\|\cdot\|}\|v\|^p. \tag{25}$$

The lemma follows from the change of variables $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$ in **(25)**. $\square$

Given a Lipschitz function f on X, we use the previous **Lemma 3.1** and the fact that $f_n^p(x) \leq f(x)$ (for all $x \in X$ and $n \in \mathbb{N}$) to obtain the following chain of inequalities:

$$\big(f \,\square\, (nC_{\|\cdot\|}\|\cdot\|^p)\big) \leq f_n^p \leq f \quad (n \in \mathbb{N}). \tag{26}$$

Since $p \geq 2$, the previous inf-convolution formula in **(26)** converges to f uniformly on X (see [LL]). Therefore, the same is true for $(f_n^p)_n$. For instance, if f is a 1-Lipschitz function we get the following estimate

$$\|f - f_n^p\|_\infty \leq \ \|f - \big(f \,\square\, (nC_{\|\cdot\|}\|\cdot\|^p)\big)\|_\infty \leq \left(\frac{1}{nC_{\|\cdot\|}}\right)^{\frac{1}{p-1}}. \quad \square$$

The first consequence that stems from the **Theorem 1** is another proof of a result due to G.A. Edgar ([E]).

Corollary 4. *Let X Banach space with an equivalent locally uniformly convex norm. Then the σ-fields of Borel sets for the norm and weak topologies are the same.*

Proof of the Corollary 4. The first part of the **Theorem 1** affirms that the existence of a locally uniformly convex norm on X implies the following:

$$\forall f : X \to \mathbb{R} \text{ Lipschitz } \exists (c_n)_n, (d_n)_n \subset \mathrm{Conv}(X) \text{ such that } f = \tau_\kappa\text{-}\lim_{n\to\infty}(c_n - d_n).$$

Let us check that this property implies the equivalence of the two Borel families, $\mathrm{Bor}(X, \|\cdot\|)$ and $\mathrm{Bor}(X, w)$. Obviously, $\mathrm{Bor}(X, w) \subseteq \mathrm{Bor}(X, \|\cdot\|)$. To see the other inclusion, take F a $\|\cdot\|$-closed set of X. Then consider the Lipschitz function $f(\cdot) = \mathrm{dist}(\cdot, F)$. Note that the previous property implies that f is the pointwise limit of a sequence of w-Borel functions (because every continuous convex function is w-lower-semicontinuous). Therefore, f is w-Borel and so F is w-Borel. $\square$

Now, we proceed to state two applications of **Theorem 1** and **Theorem 3** to the study of superreflexive Banach spaces. Actually, we will show in the next section that both of them characterize superreflexivity.

Corollary 5. *Let X be a superreflexive Banach space. Denote by* $\mathrm{Conv}(X)$ *the set of continuous convex functions on X,* $\mathrm{Conv}_b(X)$ *the subset of* $\mathrm{Conv}(X)$ *consisting of the functions which are bounded on bounded sets and* $\mathcal{UC}_b(X)$ *the class of functions on X which are uniformly continuous on bounded sets of X. Then*

$$\overline{\mathrm{span}}^{\tau_b}\big\{\mathrm{Conv}_b(X)\big\} = \mathcal{UC}_b(X).$$

Proof of the Corollary 5. From the Prop. 1.6 of [Ph] follows immediately that $\mathrm{Conv}_b(X) = \mathrm{Conv}(X) \cap \mathcal{UC}_b(X) \subseteq \mathcal{UC}_b(X)$. Let us show the τ_b-density of the Δ-convex functions in the set $\mathcal{UC}_b(X)$.

Notice that the set of Lipschitz functions on X is dense in $\mathcal{UC}_b(X)$ under the topology τ_b. This fact can be proved using again the inf-convolution formula. More precisely, given $f \in \mathcal{UC}_b(X)$ and bounded on X, $\big(f \,\Box\, (n\|\cdot\|)\big)(x)$ is a sequence of Lipschitz functions τ_b-converging to f. Since the bounded functions of $\mathcal{UC}_b(X)$ are clearly τ_b-dense in $\mathcal{UC}_b(X)$, the density of the Lipschitz functions is therefore deduced.

Thus, the corollary holds if we show that the convex functions $\{c_n, d_n\}_{n\in\mathbb{N}}$ obtained in the proof of **Theorem 1** are in $\mathrm{Conv}_b(X)$. Clearly, $c_n(\cdot) = 2n\|\cdot\|^2 \in \mathrm{Conv}_b(X)$ and, as we remarked during the proof of this theorem, $d_n = c_n - f_n$ is also bounded on bounded sets of X. $\Box$

Corollary 6. *Let X be a superreflexive Banach space. With the same notations of the previous corollary, one has*

$$\overline{\mathrm{span}}^{\tau_u}\big\{\mathrm{Conv}_b(X)\big\} \supset \mathcal{UC}(X)$$

(where τ_u is the topology of uniform convergence on X).

Proof of the Corollary 6. As we remarked during the proof of the previous **Corollary 5**, the result is proved if we show the τ_u-density of the subset of Δ-convex functions $\mathrm{Conv}_b(X)$ in the set of uniformly continuous functions $\mathcal{UC}(X)$.

But this follows from **Theorem 3** and Pisier's renorming theorem ([P]) that gives an equivalent norm with modulus of convexity of power type p (for some $p \geq 2$) on every superreflexive Banach space.

Remark. For a simpler and more geometrical proof of Pisier's theorem we refer to [L].

2. The negative results

In this part, we will show that the rotundity conditions on the norm needed in **Theorem 1** can not be dropped. For instance, even the pointwise convergence fails for some Banach spaces, as the following counter-example shows.

Example 7. There exists a Lipschitz function on ℓ_∞ which can not be a pointwise limit of a sequence of Δ-convex functions.

In the article [T], M. Talagrand proved that $\mathrm{Bor}(\ell_\infty, w) \subsetneqq \mathrm{Bor}(\ell_\infty, \|\cdot\|_\infty)$. Taking a $\|\cdot\|_\infty$-closed, non w-Borel set B the function $\mathrm{d}(\cdot, B)$ is a Lipschitz function which can not be the pointwise limit of any sequence of Δ-convex functions (since Δ-convex functions are w-Borel).

On the other hand, the τ_b-density property of the span of $\mathrm{Conv}_b(X)$ in $\mathcal{UC}_b$ is a characterization of superreflexivity. This conclusion comes from the following theorem.

Theorem 8. *For any non-superreflexive Banach space X there exists a 1–Lipschitz function defined on X such that for every pair $\{c, d\}$ of continuous, bounded on B_X, convex functions we have* $\sup_{x\in B_X} \left|f(x) - (c-d)(x)\right| \geq \frac{1}{4}$.

Proof of the Theorem 8. Our main tool is James' Finite Tree Property (defined in [J]). Specifically, we need the following well-known lemma.

Lemma 8.1. *Let X be a non-superreflexive Banach space. For any $0 < \theta < \frac{1}{2}$ and any $n \in \mathbb{N}$ there exists a dyadic (n, θ)-tree in the unit ball of X.*

Proof of the lemma 8.1. This lemma follows from the equivalence between the superreflexivity and super–Radon-Nikodym properties (see [Sn]). That means for any non-superreflexive Banach space X that there exists a dual Banach space failing the Radon-Nikodym Property which is finitely representable on X (namely, the bidual of one of its ultrapowers $(X^{\mathcal{U}})^{**}$). The lemma is then deduced from the existence of an infinite bounded dyadic tree in any dual Banach space failing the Radon-Nikodym Property ([Sg]).

This lemma allow us to construct a convenient sequence of trees in the unit ball of X. By "convenient" we understand the following:

Claim 8.2. *Let $\{\rho_n\}_{n\in\mathbb{N}}$ be a increasing sequence of positive numbers tending to* 1*. Then there exists a sequence of dyadic trees T_n ($n \in \mathbb{N}$) in the unit ball of X such that*

(1) $T_n = \left\{x^n_\alpha : \alpha \in \{-1,1\}^{<n}\right\}$ *is a* $(n, \frac{\rho_n}{2})$*-tree.*
(2) $\operatorname{dist}(T_p, T_q) \geq \frac{1}{2}\rho_{\max\{p,q\}}$.

Once the sequence $\{T_n\}_n$ of the previous claim is constructed, we define the 1-Lipschitz function $f(\cdot) = \operatorname{dist}(\cdot, \cup_n S_n)$ where $S_n = \{x_\alpha \in T_n : |\alpha| \text{ is even}\}$.

Obviously, if $x \in S_n$ then $f(x) = 0$. On the other hand, the special construction of the sequence $\{T_n\}$ gives that $f(x) \geq \frac{\rho_n}{2}$, for $x \in T_n \setminus S_n$. Let us check that f is the function we are looking for.

If not, there is a pair of continuous convex functions c, d bounded on the unit ball of X such that $|f(x) - (c-d)(x)| < \delta < \frac{1}{4}$, $x \in B_X$. Take n big enough so that $\delta < \frac{\rho_n}{4}$. Now, we proceed to show that f is strictly increasing along one of the branches of $T_n = \{x^n_\alpha\}$ by a two-step algorithmic method.

First, as $f(x^n_\varnothing) = 0$ we have $d(x^n_\varnothing) \geq c(x^n_\varnothing) - \delta$. Then, using the convexity of d, we can find $\alpha_1 \in \{-1,1\}$ so that

$$d(x^n_{\alpha_1}) \geq d(x^n_\varnothing) \geq c(x^n_\varnothing) - \delta. \tag{27}$$

But, $\left(c(x^n_{\alpha_1}) - d(x^n_{\alpha_1})\right)$ is δ-close to $f(x^n_{\alpha_1}) \geq \frac{\rho_n}{2}$, so we conclude from **(27)** that

$$c(x^n_{\alpha_1}) \geq f(x^n_{\alpha_1}) + d(x^n_{\alpha_1}) - \delta \geq c(x^n_\varnothing) + \frac{\rho_n}{2} - 2\delta > 0. \tag{28}$$

Secondly, appealing this time to the convexity of c, we can choose $\alpha_2 \in \{-1,1\}$ such that $c(x^n_{\alpha_1\frown\alpha_2}) \geq c(x^n_{\alpha_1})$. Using **(28)**, we deduce that

$$c(x^n_{\alpha_1\frown\alpha_2}) - c(x^n_\varnothing) \geq \frac{\rho_n}{2} - 2\delta > 0. \tag{28}$$

And at this level we can repeat the same process as above.

Iterating this process n times up to the end of a branch of the tree T_n, we find a point $x^n_{\alpha_1\frown\cdots\frown\alpha_n} \in T_n$ that satisfies

$$c(x^n_{\alpha_1\frown\cdots\frown\alpha_n}) - c(x^n_{\varnothing}) \geq c(x^n_{\alpha_1\frown\cdots\frown\alpha_n}) - c(x^n_{\alpha_1\frown\cdots\frown\alpha_{n-2}}) + \cdots +$$
$$+ \cdots + c(x^n_{\alpha_1\frown\alpha_2}) - c(x^n_{\varnothing}) \geq \Big(\frac{\rho_n}{2} - 2\delta\Big)\frac{n}{2}.$$

As n can be chosen arbitrarily large, this last inequality contradicts the boundedness of c in B_X. $\square$

Proof of the claim 8.2. First, let us show that we can define a sequence $\{T_n\}_{n\in\mathbb{N}}$ of trees such that for $n \in \mathbb{N}$ and the quotient map $\Pi_n : X \to X/\mathrm{span}\{T_1,\ldots,T_n\}$ one gets that $\Pi_n(T_{n+1})$ is a dyadic $(n+1, \frac{\rho_{n+1}}{2})$-tree rooted at $\overline{0}$. By induction, consider that $T_1,\ldots,T_n$ are already defined. Since X is non-superreflexive, the quotient space $X/\mathrm{span}\{T_1,\ldots,T_n\}$ is non-superreflexive. So, there exists a dyadic $\overline{0}$-rooted $(n+1, \frac{\rho_{n+1}}{2})$-tree $\overline{T_{n+1}}$ in $X/\mathrm{span}\{T_1,\ldots,T_n\}$. Then the tree T_{n+1} which we are looking for is simply a *lifting* of the tree $\overline{T_{n+1}}$, obtained as follows: take $\big\{x^{n+1}_\alpha : \alpha \in \{-1,1\}^{n+1}\big\} \subset B_X$ in such a way that $\big\{\Pi_n(x^{n+1}_\alpha) : \alpha \in \{-1,1\}^{n+1}\big\}$ is the set of end points of $\overline{T_{n+1}}$; then reconstruct from the $\{x^{n+1}_\alpha\}_{|\alpha|=n+1}$ the tree T_{n+1} as

$$T_{n+1} := \Big\{x^{n+1}_\beta = \sum_{\beta\preceq\alpha} \frac{x^{n+1}_\alpha}{2^{n+1-|\beta|}} : \beta \in \{-1,1\}^{<n+1}\Big\}.$$

This sequence $\{T_n\}_{n\in\mathbb{N}}$ does not satisfy the conditions of the claim yet; for example, the set of root points $\{x^n_\varnothing\}_n$ of the $\{T_n\}_{n\in\mathbb{N}}$ might have cluster points. However, we can avoid this problem by selecting sub-trees of the trees T_n which do not contain these root points. For each $n \in \mathbb{N}$, consider

$$T'_n = \{y^n_\alpha : y^n_\alpha = x^{n+1}_{1\frown\alpha} \in T_{n+1}\}$$

Let us see that the special quotient properties of T_n's imply that if $p \neq q$ then $\mathrm{dist}(T'_p, T'_q) \geq \frac{1}{2}\rho_{\max\{p,q\}}$. For $q > p$ positive integers take $y^q_\alpha \in T'_q$ and $y^p_\beta \in T'_p$. Then we have that

$$\Pi_q(y^p_\beta) = \Pi_q(x^{p+1}_{1\frown\beta}) = \overline{0} = \Pi_q(x^{q+1}_\varnothing) \text{ and } \Pi_q(y^q_\alpha) = \Pi_q(x^{q+1}_{1\frown\alpha}). \tag{30}$$

Since $\Pi_q(T_{q+1}) = \overline{T_{q+1}}$ is a dyadic $(q+1, \frac{\rho_{q+1}}{2})$-tree rooted at $\overline{0}$ in the quotient space $X/\mathrm{span}\{T_1,\ldots,T_{q-1}\}$, we deduce from **(30)** that

$$\|y^q_\alpha - y^p_\beta\| \geq \|\Pi_q(y^q_\alpha - y^p_\beta)\| \geq \|\Pi_q(x^{q+1}_\varnothing) - \Pi_q(x^{q+1}_{1\frown\alpha})\| \geq \frac{\rho_{q+1}}{2} \geq \frac{1}{2}\rho_{\max\{p,q\}}. \quad \square$$

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