PACIFIC JOURNAL OF MATHEMATICS
Vol. 120, No. 2, 1985

# GROTHENDIECK LOCALLY CONVEX SPACES OF CONTINUOUS VECTOR VALUED FUNCTIONS

FRANCISCO J. FRENICHE

**Let $\mathscr{C}(X, E)$ be the space of continuous functions from the completely regular Hausdorff space $X$ into the Hausdorff locally convex space $E$, endowed with the compact-open topology. Our aim is to characterize the $\mathscr{C}(X, E)$ spaces which have the following property: weak-star and weak sequential convergences coincide in the equicontinuous subsets of $\mathscr{C}(X, E)'$. These spaces are here called Grothendieck spaces. It is shown that in the equicontinuous subsets of $E'$ the $\sigma(E', E)$- and $\beta(E', E)$-sequential convergences coincide, if $\mathscr{C}(X, E)$ is a Grothendieck space and $X$ contains an infinite compact subset. Conversely, if $X$ is a $G$-space and $E$ is a strict inductive limit of Fréchet-Montel spaces $\mathscr{C}(X, E)$ is a Grothendieck space. Therefore, it is proved that if $E$ is a separable Fréchet space, then $E$ is a Montel space if and only if there is an infinite compact Hausdorff $X$ such that $\mathscr{C}(X, E)$ is a Grothendieck space.**

**1. Introduction.** In this paper $X$ will always denote a completely regular Hausdorff topological space, $E$ a Hausdorff locally convex space, and $\mathscr{C}(X, E)$ the space of continuous functions from $X$ into $E$, endowed with the compact-open topology. When $E$ is the scalar field of reals or complex numbers, we write $\mathscr{C}(X)$ instead $\mathscr{C}(X, E)$.

It is well known that $\mathscr{C}(X, E)$ is a Montel space whenever $\mathscr{C}(X)$ and $E$ so are, hence, if and only if $X$ is discrete and $E$ is a Montel space (see **[5]**, **[16]**).

We study what happens when $X$ has the following weaker property: the compact subsets of $X$ are $G$-spaces (see below for definitions).

We obtain in Theorem 4.4 that if $E$ is a Fréchet-Montel space and $X$ has that property, then $\mathscr{C}(X, E)$ is a Grothendieck locally convex space. The key in the proof is the following fact: every countable equicontinuous subset of $\mathscr{C}(X, E)'$ lies, via a Radon-Nikodým theorem, in a suitable $L^1(\tau, E'_\beta)$. As a consequence of a theorem of Mújica **[10]**, the same result is true when $E$ is a strict inductive limit of Fréchet-Montel spaces.

In §3 we study the converse of 4.4. In Corollary 3.3 it is proved that if $X$ contains an infinite compact subset, $E$ is a Fréchet separable space and $\mathscr{C}(X, E)$ is a Grothendieck space, then $E$ is a Montel space. This property characterizes the Montel spaces among the Fréchet separable spaces.

Finally, in §5 we study the Grothendieck property in $\mathscr{B}(\Sigma, E)$, the space of $\Sigma$-totally measurable functions, by using the results for $\mathscr{C}(X, E)$.

**2. Generalities.** A compact Hausdorff topological space $K$ is called a $G$-space whenever $\mathscr{C}(K)$ is a Grothendieck Banach space, i.e. the weak-star and weak sequential convergences coincide in $\mathscr{C}(K)'$ [**6**].

We extend here this concept to completely regular spaces.

2.1. DEFINITION. $X$ is a $G$-space if every compact subset $K$ of $X$ is a $G$-space.

If $X$ is compact, both definitions coincide [**6**]. Let us remark that there exist non-compact non-discrete $G$-spaces. Indeed, the topological subspace of the Stone-Čech compactification of a countable discrete set obtained removing a cluster point, is such a space.

We introduce a new definition of Grothendieck locally convex space, so that $\mathscr{C}(X)$ is a Grothendieck space if and only if $X$ is a $G$-space.

2.2. DEFINITION. $E$ is a Grothendieck space whenever the $\sigma(E', E)$- and $\sigma(E', E'')$-sequential convergences coincide in the equicontinuous subsets of $E'$.

In [**17**] the $TG$-spaces are defined as those spaces $E$ in which the $\sigma(E', E)$- and $\sigma(E', E'')$-sequential convergences coincide. When one deals with $\mathscr{C}(X)$ spaces, our definition seems to be more reasonable than that of [**17**] (see 2.4 and 2.5).

The following permanence properties of the class of Grothendieck locally convex spaces are easy to see, thus we state them without proof.

2.3. PROPOSITION. (a) *$E$ is a Grothendieck space if and only if every, or some, dense subspace of $E$ so is.*

(b) *Let $T$: $E \to F$ be a linear continuous operator such that for every bounded subset $B$ of $F$ there is a bounded subset $C$ of $E$ so that $B$ is contained in the closure of $T(C)$. Then $F$ is a Grothendieck space if $E$ so is.*

(c) *If $E$ is the inductive limit of the sequence $(E_n)$ of Grothendieck spaces, and if every bounded subset of $E$ is contained in some $E_n$, then $E$ is a Grothendieck space.*

2.4. THEOREM. *$\mathscr{C}(X, E)$ is a Grothendieck space if and only if $\mathscr{C}(K, E)$ so is for every compact subset $K$ of $X$. In particular, $X$ is a $G$-space if and only if $\mathscr{C}(X)$ is a Grothendieck space.*

*Proof*. Let us recall that, if $K$ is a compact subset of $X$, the restriction map $T$ is a continuous linear operator from $\mathscr{C}(X, E)$ into $\mathscr{C}(K, E)$.

If $B \subset \mathscr{C}(K, E)$ is bounded, then the bounded subset $C$ of $\mathscr{C}(X, E)$, whose elements $g$ can be written $g = \sum_{n \leq m} f_n(\cdot)e_n$ with $f_n \in \mathscr{C}(X)$, $0 \leq f_n \leq 1$, $\sum_{n \leq m} f_n \leq 1$, and $e_n \in \bigcup\{h(K)\colon h \in B\}$, satisfies $\overline{T(C)} \supset B$ (see [**14**, I.5.3]).

If $\mathscr{C}(X, E)$ is a Grothendieck space, $\mathscr{C}(K, E)$ so is by 2.3(b).

Conversely, let $(g_n')$ be an equicontinuous and $\sigma(\mathscr{C}(X, E)', \mathscr{C}(X, E))$-null sequence. By [**14**, III.3 and III.4], there exist a compact subset $K$ of $X$ and an equicontinuous sequence $(h_n')$ in $\mathscr{C}(K, E)'$ such that $g_n' = h_n' \circ T$ for all $n \in \mathbf{N}$. Since $(h_n')$ is $\sigma(\mathscr{C}(K, E)', T(\mathscr{C}(K, E)))$-null and equicontinuous, it is also $\sigma(\mathscr{C}(K, E)', \mathscr{C}(K, E)'')$-null if $\mathscr{C}(K, E)$ is a Grothendieck space. It follows that $(g_n')$ is $\sigma(\mathscr{C}(X, E)', \mathscr{C}(X, E)'')$-null.

2.5. Remark. We use an example of Haydon [**4**] to show that, while in the class of barrelled spaces the *TG*-spaces and the Grothendieck spaces do coincide, this is not true in general.

Choose, for each infinite sequence in $\mathbf{N}$, a cluster point in the Stone-Čech compactification of $\mathbf{N}$, and let $X$ be the topological subspace of that compactification, formed by $\mathbf{N}$ and these cluster points. Then every compact subset of $X$ is finite, $\mathscr{C}(X)$ is infrabarrelled and every $f \in \mathscr{C}(X)$ is bounded. By Theorem 2.4, $X$ is a $G$-space. Let $f_n'(f) = n^{-1}f(n)$ for all $f \in \mathscr{C}(X)$ and $n \in \mathbf{N}$. Then $(f_n')$ is a $\sigma(\mathscr{C}(X)', \mathscr{C}(X))$-null sequence in $\mathscr{C}(X)'$, that is not $\sigma(\mathscr{C}(X)', \mathscr{C}(X)'')$-null because it is not equicontinuous.

**3. Necessary conditions for $\mathscr{C}(X, E)$ to be a Grothendieck space.** It is well known, and easy to see, that $\mathscr{C}(X)$ and $E$ are topologically isomorphic to complemented subspaces of $\mathscr{C}(X, E)$. By 2.3(b), $\mathscr{C}(X)$ and $E$ must be Grothendieck spaces if $\mathscr{C}(X, E)$ is such a space.

However, unless $X$ is pseudofinite, i.e. their compact subsets are finite (hence $\mathscr{C}(X, E)$ is a Grothendieck space if and only if $E$ so is, by Theorem 2.4), $E$ has a stronger property if $\mathscr{C}(X, E)$ is a Grothendieck space, as we prove in the next theorem. To prove it we recall the following result of [**2**]:

Theorem A. *Let $E$ and $F$ be Hausdorff locally convex spaces, and suppose that $F$ contains a subspace topologically isomorphic to the subspace of $c_0$ whose elements have only finitely many non-zero coordinates.*

*If the injective tensor product $F \otimes_\varepsilon E$ is a Grothendieck space, then the $\sigma(E', E)$- and $\beta(E', E)$-sequential convergences coincide in the equicontinuous subsets of $E'$.*

As was noted in [2], if $X$ is not pseudofinite, then $\mathscr{C}(X)$ contains a subspace topologically isomorphic to the above mentioned subspace of $c_0$. Moreover, the injective tensor product $\mathscr{C}(X) \otimes_\varepsilon E$ can be linear and topologically identified with a dense subspace of $\mathscr{C}(X, E)$, namely, the subspace of all finite dimensional valued elements of $\mathscr{C}(X, E)$. Thus we obtain from Theorem A and Proposition 2.3 (a):

3.1. THEOREM. *If $\mathscr{C}(X, E)$ is a Grothendieck space and $X$ contains an infinite compact subset, then the $\sigma(E', E)$- and $\beta(E', E)$-sequential convergences coincide in the equicontinuous subsets of $E'$.*

3.2. REMARK. By Theorem 2.4, if $X$ is pseudofinite and $E$ is a Grothendieck Banach space, $\mathscr{C}(X, E)$ is a Grothendieck space. However, if $E$ is infinite dimensional, the conclusion of Theorem 3.1 does not hold [11].

Using Theorem 3.1 and [7, 11.6.2], we obtain the following corollary, converse of Theorem 4.4:

3.3. COROLLARY. *If $E$ is a Fréchet separable space, $X$ is not pseudofinite and $\mathscr{C}(X, E)$ is a Grothendieck space, then $E$ is a Montel space.*

3.4. REMARK. It is unknown for us if Corollary 3.3 is true without the separability assumption on $E$. This is related with the following question raised in [7, pg. 247]: is a Fréchet space $E$ already a Montel space if every $\sigma(E', E)$-convergent sequence in $E'$ converges for $\beta(E', E)$?

**4. Sufficient conditions for $\mathscr{C}(X, E)$ to be a Grothendieck space.** We shall need some facts about vector integration, many of those can be found in [1] and [15].

Let $(X, \Sigma, \tau)$ be a complete measure space with $\tau(X) \leq 1$. We denote by $\mathscr{S}(\Sigma, E)$ (resp. $\mathscr{B}(\Sigma, E)$, $L^1(\tau, E)$, $L^\infty(\tau, E)$) the vector space of $\Sigma$-simple (resp. $\Sigma$-totally measurable, $\tau$-integrable, $\tau$-essentially bounded) $E$-valued (classes of) functions. Recall that $\mathscr{S}(\Sigma, E)$ and $\mathscr{B}(\Sigma, E)$ are endowed with the uniform convergence topology, and that the topology of $L^1(\tau, E)$ is defined by the seminorms $u \to \int p(u(x))\, d\tau(x)$, where $p$ runs over the set of all continuous seminorms in $E$ (unless contrary specification, all integrals will be extended to $X$).

The following Radon-Nikodým theorem is proved in [1]:

THEOREM B. *If $E$ is a quasi-complete* (CM)*-space, $\mu\colon \Sigma \to E$ is a countably additive vector measure, of bounded variation and $\tau$-absolutely continuous, then there exists $u \in L^1(\tau, E)$ such that $\mu(A) = \int_A u(x)\, d\tau(x)$ for every $A \in \Sigma$.*

Let us recall that $E$ is a quasi-complete (CM)-space, if, for instance, it is either a Fréchet-Montel space or a (DF)-Montel space [**1**].

Firstly we extend the classical duality theorem $L^1 - L^\infty$ to $L^1(\tau, E'_\beta)$, where $E$ is a Fréchet-Montel space.

The following lemma can be easily proved. As usual, $p_L$ will denote the gauge of the absolutely convex set $L$ in its linear span.

4.1. LEMMA. *If $u \in \mathscr{S}(\Sigma, E')$, namely, $u = \sum_{i \leq m} \chi_{A_i} e'_i$ with $(A_i)_{i \leq m}$ disjoint in $\Sigma$, then*

$$\int p_{B^0}(u(x))\, d\tau(x) \leq \tau\Big(\bigcup_{i \leq m} A_i\Big) \sup_{i \leq m} p_{B^0}(e'_i)$$

*for every bounded subset $B$ of $E$.*

4.2. THEOREM. *Let $E$ be a Fréchet-Montel space. The relation*

$$u'(u) = \int u(x)(v(x))\, d\tau(x) \quad \text{for all } u \in L^1(\tau, E'_\beta) \tag{1}$$

*defined for $u' \in L^1(\tau, E'_\beta)'$ and $v \in L^\infty(\tau, E)$, is an algebraic isomorphism between $L^1(\tau, E'_\beta)'$ and $L^\infty(\tau, E)$.*

*Proof.* Let $v \in L^\infty(\tau, E)$. The map $x \to u(x)(v(x))$ is measurable for every $u \in L^1(\tau, E'_\beta)$, because $v$ is strongly measurable and the assertion is clearly true when $v \in \mathscr{S}(\Sigma, E)$.

Furthermore, if $Z \in \Sigma$ is a $\tau$-null set such that $B = v(S \setminus Z)$ is bounded, then we have

$$|u(x)(v(x))| \leq p_{B^0}(u(x)) \tag{2}$$

for every $x \in X \setminus Z$.

Hence $x \to u(x)(v(x))$ is $\tau$-integrable, and we can define a linear form $u'$ on $L^1(\tau, E'_\beta)$ by (1). Moreover, it follows from (2) that $u'$ is continuous.

Conversely, fix $u' \in L^1(\tau, E'_\beta)'$. There exists a bounded subset $B$ of $E$ such that

$$\int p_{B^0}(u(x))\, d\tau(x) \leq 1 \quad \text{implies } |u'(u)| \leq 1 \tag{3}$$

for every $u \in L^1(\tau, E'_\beta)$.

We define a map $\mu\colon \Sigma \to E''$ by

$$\mu(A)(e') = u'(\chi_A e') \tag{4}$$

for every $A \in \Sigma$ and $e' \in E'$ (it follows easily from Lemma 4.1 and (3) that $\mu(A) \in E''$). Since $E$ is reflexive we can suppose that $\mu(A) \in E$.

Clearly, $\mu: \Sigma \to E$ is a finitely additive vector measure. We shall show that $\mu$ is countably additive: let $A$ be the union of the disjoint sequence $(A_n)$ in $\Sigma$. Given an absolutely convex zero-neighborhood $U$ in $E$ and $\varepsilon > 0$, we choose $\lambda$ with $0 < \lambda < \infty$ such that $B \subset \lambda U$, and $m_0 \in \mathbf{N}$ such that $\lambda\tau(\bigcup_{n>m} A_n) \leq \varepsilon$ for every $m \geq m_0$. Since

$$e'(\mu(A)) - \sum_{n \leq m} e'(\mu(A_n)) = u'\left(\chi_{\bigcup_{n>m} A_n} e'\right)$$

it follows from Lemma 4.1 and (3) that

$$\left|e'(\mu(A)) - \sum_{n \leq m} e'(\mu(A_n))\right| \leq \varepsilon$$

for every $m \geq m_0$ and $e' \in U^0$, as desired.

Furthermore, if $A = \bigcup_{n \leq m} A_n$ where $(A_n)_{n \leq m}$ is disjoint in $\Sigma$, and if $\varepsilon > 0$, there exists $(e'_n)_{n \leq m}$ in $U^0$ such that

$$\sum_{n \leq m} p_U(\mu(A_n)) \leq \sum_{n \leq m} e'_n(\mu(A_n)) + \varepsilon = u'\left(\sum_{n \leq m} \chi_{A_n} e'_n\right) + \varepsilon.$$

Hence the $p_U$-variation of $\mu$ satisfies the inequality $V_{p_U}\mu(A) \leq \lambda\tau(A)$, from Lemma 4.1 and (3) again.

Thus $\mu$ is $\tau$-absolutely continuous and has bounded variation. By Theorem B, there exists $v \in L^1(\tau, E)$ such that

$$\mu(A) = \int_A v(x)\, d\tau(x) \quad \text{for every } A \in \Sigma. \tag{5}$$

We claim that $v$ is $\tau$-essentially bounded and satisfies (1). Indeed, let $(U_j)_j$ be a countable basis in $E$ of absolutely convex zero-neighborhoods. Choose, for each $j \in \mathbf{N}$, $\lambda_j$ such that $0 < \lambda_j < \infty$ and $B \subset \lambda_j U_j$.

By Lemma 4.1, (3), (4) and (5), we have

$$\left|\int_A e'(v(x))\, d\tau(x)\right| \leq \lambda_j \tau(A) \tag{6}$$

for all $e' \in U_j^0$, $A \in \Sigma$ and $j \in \mathbf{N}$.

Let $(e'_{j,k})_k$ be a sequence in $U_j^0$ such that $p_{U_j}(e) = \sup_k |e'_{j,k}(e)|$ for every $e \in E$.

By (6), there exists $Z \in \Sigma$ with $\tau(Z) = 0$ such that $|e'_{j,k}(v(x))| \leq \lambda_j$ for all $x \in X \setminus Z$ and all $j, k \in \mathbf{N}$. Hence $v(X \setminus Z)$ is bounded in $E$.

Finally, it follows from (4) that (1) is true for all $u \in \mathcal{S}(\Sigma, E')$, and, by density, for every $u \in L^1(\tau, E'_\beta)$. This concludes the proof.

Assume that $X$ is compact Hausdorff and $\Sigma$ contains the Borel subsets of $X$. For each $u \in L^1(\tau, E'_\beta)$, denote by $\nu_u$ the vector measure of density $u$ with respect to $\tau$. If $p$ is a continuous seminorm in $E$, the subset

$F$ of $L^1(\tau, E'_\beta)$ defined by the condition $V_p \nu_u(X) < \infty$, is a linear subspace. If $u \in F$ then $\nu_u$ has bounded semivariation, thus it defines a continuous linear form on $\mathscr{S}(\Sigma, E)$, which extends by continuity to the whole space $\mathscr{B}(\Sigma, E)$ [**15**]. Let $Tu \in \mathscr{C}(X, E)'$ be the restriction to $\mathscr{C}(X, E)$ of this linear form, i.e.

$$(Tu)(g) = \int g(x)\, d\nu_u(x) \tag{7}$$

for every $g \in \mathscr{C}(X, E)$.

4.3. LEMMA. *The map $T$: $F \to \mathscr{C}(X, E)'$ defined by* (7) *is a linear continuous operator, when $\mathscr{C}(X, E)'$ is endowed with the strong topology with respect to $\mathscr{C}(X, E)$.*

*Proof.* We have, for each $u \in F$,

$$(Tu)(g) = \int u(x)(g(x))\, d\tau(x) \tag{8}$$

for every $g \in \mathscr{C}(X, E)$. Indeed, the dominated convergence theorem and a standard density argument show that it suffices to see (8) when $g$ belongs to $\mathscr{S}(\Sigma, E)$, that is trivially true.

Let $H$ be a bounded subset of $\mathscr{C}(X, E)$. Then $B = \bigcup\{g(X): g \in H\}$ is a bounded subset of $E$. Hence, by (8), $|(Tu)(g)| \leq \int p_{B^0}(u(x))\, d\tau(x)$ and the lemma follows.

We are now ready to prove the sufficient condition:

4.4. THEOREM. *Let $X$ be a completely regular Hausdorff G-space and $E$ a Fréchet-Montel space. Then $\mathscr{C}(X, E)$ is a Grothendieck space.*

*Proof.* By 2.4 we can suppose, without loss of generality, that $X$ is compact.

Let $(g'_n)_n$ be an equicontinuous sequence in $\mathscr{C}(X, E)'$. By [**14**, III.4.5] there exists a continuous seminorm $p$ in $E$ such that $V_p \mu_n(X) \leq 1$, for every $n \in \mathbf{N}$, where $\mu_n$ is the representing measure of $g'_n$ [**14**, III].

Let $\tau = \Sigma_n 2^{-n} V_p \mu_n$. $\tau$ is a countably additive $[0,1]$-valued Borel measure, by [**14**, III.2.5]. Let $\Sigma$ be the completed $\sigma$-field of the Borel field of $X$ with respect to $\tau$. We shall denote also by $\tau$ and $\mu_n$ the natural extensions of the earlier measures to $\Sigma$.

Since $E$ is a Montel space, the measure $\mu_n\colon \Sigma \to E'_\beta$ is countably additive. Clearly $V_p\mu_n \leq 2^n$, thus $\mu_n$ has bounded variation and is $\tau$-absolutely continuous (when it is considered as an $E'_\beta$-valued measure).

We apply Theorem B, obtaining, for each $n \in \mathbf{N}$, a function $u_n \in L^1(\tau, E'_\beta)$ such that $\mu_n$ is the vector measure of density $u_n$ with respect to $\tau$.

Clearly $u_n \in F$ and $Tu_n = g'_n$, for every $n \in \mathbf{N}$.

Fix $g'' \in \mathscr{C}(X, E)''$. By Lemma 4.3 and Theorem 4.2, there exists $v \in L^\infty(\tau, E)$ such that $g''(g'_n) = \int u_n(x)(v(x))\, d\tau(x)$ for every $n \in \mathbf{N}$.

Let $Z$ be a set in $\Sigma$ with $\tau(Z) = 0$ and $v(X \setminus Z)$ bounded. The function $v_1 = \chi_{X\setminus Z} v$ is totally measurable, because $E$ is Montel and metrizable.

Given $\varepsilon > 0$, we can choose $v_2 \in \mathscr{S}(\Sigma, E)$ such that $p(v_3(x)) \leq \varepsilon/2$, for every $x \in X$, if $v_3 = v_1 - v_2$. Hence,

$$\left|\int u_n(x)(v_3(x))\, d\tau(x)\right| = \left|\int v_3(x)\, d\mu_n(x)\right| \leq \varepsilon/2 \tag{9}$$

for every $n \in \mathbf{N}$, because $V_p\mu_n(X) \leq 1$.

On the other hand, if $(g'_n)$ is $\sigma(\mathscr{C}(X, E)', \mathscr{C}(X, E))$-null, then $(\mu_n(A)(e))$ is a null sequence, for every $e \in E$ and $A \in \Sigma$. Indeed, since $X$ is a $G$-space, for each $e \in E$, the weak-star null sequence $(\mu_n(\cdot)(e))$ in $\mathscr{C}(X)'$, is also weak null, hence $(\mu_n(A)(e))$ is null for every Borel subset $A$ of $X$, and so for every $A \in \Sigma$.

Since $v_2$ is simple, it follows that

$$\lim_{n\to\infty} \int u_n(x)(v_2(x))\, d\tau(x) = 0. \tag{10}$$

By (9) and (10), $(g''(g'_n))$ is a null sequence, and we have shown that $(g'_n)$ is $\sigma(\mathscr{C}(X, E)', \mathscr{C}(X, E)'')$-null.

4.5. COROLLARY. *Let $X$ be a completely regular Hausdorff G-space and $E$ the inductive limit of the sequence $(E_n)$ of Fréchet-Montel spaces, such that every bounded subset of $E$ is localized in some $E_n$. Then $\mathscr{C}(X, E)$ is a Grothendieck space.*

*Proof*. We can again suppose $X$ compact. By **[10]**, the inductive limit of the sequence $(\mathscr{C}(X, E_n))$ is a dense topological subspace of $\mathscr{C}(X, E)$. By Proposition 2.3 (a) and (c), and Theorem 4.4, it follows that $\mathscr{C}(X, E)$ is a Grothendieck space.

4.6. COROLLARY. *Let $E$ be a Fréchet separable space. The following conditions are equivalent*:

(a) *$E$ is a Montel space.*

(b) *There exists a non-pseudofinite completely regular Hausdorff space $X$ such that $\mathscr{C}(X, E)$ is a Grothendieck space.*

(c) *For every completely regular Hausdorff G-space $X$, $\mathscr{C}(X, E)$ is a Grothendieck space.*

*Proof.* Use 4.4 and 3.3.

**5. Application to spaces of totally measurable functions.** Let $X$ be a nonempty set and $\Sigma$ a field of subsets of $X$. We will say that a subset $B$ of $X$ is open if for every $x \in B$ there is $A \in \Sigma$ with $x \in A$ and $A \subset B$. Endowed $X$ with this topology, let $X^*$ be the Hausdorff space associated to $X$, $\pi\colon X \to X^*$ the quotient map, and $\Sigma^* = \{\pi(A)\colon A \in \Sigma\}$.

The following lemma is easily established:

5.1. LEMMA (a) *$X^*$ is a completely regular Hausdorff zero-dimensional topological space.*

(b) *The map $A \in \Sigma \to \pi(A) \in \Sigma^*$ is a Boolean isomorphism.*

(c) *The map $g \in \mathscr{B}(\Sigma^*, E) \to g \circ \pi \in \mathscr{B}(\Sigma, E)$ is a topological isomorphism, and its restriction to $\mathscr{S}(\Sigma^*, E)$ so is onto $\mathscr{S}(\Sigma, E)$.*

(d) *The map $x^* \in X^* \to \{B^* \in \Sigma^*\colon x^* \in B^*\} \in \mathscr{P}(\Sigma^*)$ is one-to-one.*

By using 5.1, when one studies the linear topological properties of $\mathscr{B}(\Sigma, E)$, it can be supposed that $X$ is a dense subspace of a Hausdorff compact zero-dimensional topological space $K$ (namely, the Stone space of the Boolean algebra $\Sigma$), and $\Sigma$ is the trace in $X$ of the Boolean algebra of open and closed subsets of $K$. In this context we have the following theorem:

5.2. THEOREM. *There exists a subspace of $\mathscr{B}(\Sigma, E)$, containing $\mathscr{S}(\Sigma, E)$, that is topologically isomorphic to $\mathscr{C}(K, E)$.*

*Proof.* It is easy to check that the set of restrictions to $X$ of all elements of $\mathscr{C}(K, E)$ is such a subspace.

By Proposition 2.3 (a), it follows that $\mathscr{B}(\Sigma, E)$ is a Grothendieck space if and only if $\mathscr{C}(K, E)$ so is. Hence we can apply to $\mathscr{B}(\Sigma, E)$ the results of §§3 and 4.

5.3. REMARK. The question of when $\mathcal{B}(\Sigma)$ (equivalently, $\mathcal{C}(K)$) is a Grothendieck space is related to the validity of the Vitali-Hahn-Saks theorem for finitely additive scalar measures on $\Sigma$, of bounded variation. For instance, if $\Sigma$ is $\sigma$-complete, or more generally, $\Sigma$ has the subsequential interpolation property, then $\mathcal{B}(\Sigma)$ is a Grothendieck space (see **[13]** and **[3]**).

Finally, we show that the following result of Mendoza **[8]**, can be easily deduced from their earlier results in **[9]** and our Theorem 5.2.

5.4. THEOREM. *Suppose $\Sigma$ infinite. Then $\mathcal{B}(\Sigma, E)$ is infrabarrelled (resp. barrelled) if and only if $E'_\beta$ has property* (B) *of Pietsch* [**12**, 1.5.8], *and $E$ is infrabarrelled (resp. barrelled).*

*Proof*. Let us observe that $\mathcal{S}(\Sigma, E)$ is a large dense subspace of $\mathcal{B}(\Sigma, E)$. Indeed, if $H$ is a bounded subset of $\mathcal{B}(\Sigma, E)$, then the set of all $g$ in $\mathcal{S}(\Sigma, E)$ for which there exists $h \in H$ with $g(X) \subset h(X)$, is a bounded subset of $\mathcal{S}(\Sigma, E)$ whose closure in $\mathcal{B}(\Sigma, E)$ contains $H$.

Thus Theorem 5.2 implies that $\mathcal{B}(\Sigma, E)$ is infrabarrelled whenever $\mathcal{C}(K, E)$ so is, hence we have the first equivalence of the theorem, by **[9]**.

If $\mathcal{B}(\Sigma, E)$ is barrelled, then $E$ is barrelled and $\mathcal{B}(\Sigma, E)$ is infrabarrelled, so $E'_\beta$ has property (B). The converse follows easily because $\mathcal{C}(K, E)$ is topologically isomorphic to a dense subspace of $\mathcal{B}(\Sigma, E)$, by 5.2.

5.5. REMARK. We have also shown in 5.4 that, if $\Sigma$ is infinite, $\mathcal{S}(\Sigma, E)$ is infrabarrelled if and only if $E'_\beta$ has property $(B)$ and $E$ is infrabarrelled, a result of Mendoza **[8]**. In **[2]** we prove that $\mathcal{S}(\Sigma, E)$ is barrelled if and only if $\mathcal{S}(\Sigma)$ and $E$ so are, and $E$ is nuclear.

## REFERENCES


[1] G. Y. H. Chi, *On the Radon-Nikodým theorem and locally convex spaces with the Radon-Nikodým property*, Proc. Amer. Math. Soc., **61** (1977), 245–253.

[2] F. J. Freniche, *Barrelledness of the space of vector valued and simple functions*, Math. Ann., **267** (1984), 479–486.

[3] ______, *The Vitali-Hahn-Saks theorem for Boolean algebras with the subsequential interpolation property*, Proc. Amer. Math. Soc., **92** (1984), 362–366.

[4] R. Haydon, *Trois exemples dans la théorie des espaces de fonctions continues*, C. R. Acad. Sci. Paris, **276A**, (1973), 685–687.

[5] R. Hollstein, *Permanence properties of $\mathcal{C}(X, E)$*, Manuscripta Math., **38** (1982), 41–58.

[6] R. Isbell and Z. Semadeni, *Projections constants and spaces of continuous functions*, Trans. Amer. Math. Soc., **107** (1963), 38–48.
[7] H. Jarchow, *Locally Convex Spaces*, Stuttgart, B. G. Teubner 1981.
[8] J. Mendoza, *Barrelledness conditions on* $\mathscr{S}(\Sigma, E)$ *and* $\mathscr{B}(\Sigma, E)$, Math. Ann., **261** (1982), 11–22.
[9] ———, *Necessary and sufficient conditions for* $C(X, E)$ *to be barrelled or infrabarrelled*, Simon Stevin, **57** (1983), 103–123.
[10] J. Mûjica, *Spaces of continuous functions with values in an inductive limit*, Lecture Notes in Pures and Applied Math., **83** (1983).
[11] A. Nissenzweig, *$w^*$-sequential convergence*, Israel J. Math., **22** (1975), 266–272.
[12] A. Pietsch, *Nuclear Locally Convex Spaces*, Berlin-Heidelberg-New York, Springer 1972.
[13] W. Schachermayer, *On some classical measure-theoretic theorems for non-sigma-complete Boolean algebras*, Johannes Kepler Universität Linz, Linz-Auhof 1980.
[14] J. Schmets, *Spaces of Vector-Valued Continuous Functions*, Lecture Notes in Math. 1003, Berlin-Heidelberg-New York-Tokyo, Springer 1983.
[15] A. H. Suchat, *Integral representations theorems in topological vector spaces*, Trans. Amer. Math. Soc., **172** (1972), 373–397.
[16] S. Warner, *The topology of compact convergence on continuous functions spaces*, Duke Math. J., **25** (1958), 265–282.
[17] A. Wilansky, *Modern Methods in Topological Vector Spaces*, New York, McGraw-Hill 1978.

Received May 10, 1984.

UNIVERSIDAD DE SEVILLA
C / TARFIA SN.
SEVILLA-12
ESPAÑA