Homotopy in Digital Spaces*

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Abstract The main contribution of this paper is a new "extrinsic" digital fundamental group that can be readily generalized to define higher homotopy groups for arbitrary digital spaces. We show that the digital fundamental group of a digital object is naturally isomorphic to the fundamental group of its continuous analogue. In addition, we state a digital version of the Seifert–Van Kampen theorem.

Key words: Digital homotopy, digital fundamental group, lighting functions, Seifert–Van Kampen theorem.

1 Introduction

Thinning is an important pre–processing operation in pattern recognition whose goal is to reduce a digital image into a "topologically equivalent skeleton". In particular, thinning algorithms must preserve "tunnels" when processing three–dimensional digital images. As it was pointed out in [5], this requirement can be correctly established by means of an appropriate digital counterpart of the classical fundamental group in algebraic topology; see [16].

The first notion of a digital fundamental group (and even of higher homotopy groups) is due to Khalimsky [4]. He gave an "extrinsic" definition of this notion for a special class of digital spaces based on a topology on the set $\mathbb{Z}^n$, for every positive integer n. However, this approach is not suitable for other kinds of digital spaces often used in image processing, as the (α,β)-connected spaces, where $(\alpha,\beta)\in\{(4,8),(8,4)\}$ if $n=2$ and $(\alpha,\beta)\in\{(6,26),(26,6),(6,18),(18,6)\}$ if $n=3$. Within the graph-theoretical approach to Digital Topology, Kong solved partially this problem in [5] by defining a digital fundamental group for the class of *strongly normal digital picture spaces* (SN-DPS), which includes the (α,β)-connected spaces and the 2- and 3-dimensional Khalimsky's spaces. Nevertheless, Kong's definition seems not be general enough to give higher homotopy groups.

The goal of this paper is to introduce a new notion of digital fundamental group, denoted by π_1^d, using an "extrinsic" setting that can be readily generalized to define higher digital homotopy groups (see Remark 3.13), as Khalimsky's notion. But, in

*This work has been partially supported by the projects DGICYT PB96-1374 and DGES PB96-0098C04-01.

addition, this group is available on larger class of digital spaces than both Khalimsky's and Kong's digital fundamental groups.

The group π_1^d is defined, in Section 3, within the framework of the multilevel architecture for Digital Topology given in [2]. That architecture provides a link between the discrete world of digital pictures, which is represented by a polyhedral complex, and a Euclidean space through several other intermediate levels. More precisely, this framework involves a general method to associate each digital object, in an arbitrary digital space, with an Euclidean polyhedron called its *continuous analogue*, which naturally represents the "continuous perception" that an observer may take on that object. This architecture is one of the main contributions of our approach in relation to other cell–complexes approaches to Digital Topology. Another relevant contribution is an axiomatic notion of digital space, that allows us determine some categories of digital spaces with "nice" properties.

The multilevel architecture and, particularly, continuous analogues of objects can be applied in two different ways. Firstly, they can be used to check that a new digital notion is an accurate counterpart of the usual continuous one. So, we show in Section 4 that the digital fundamental group π_1^d of a digital object is naturally isomorphic to the classical fundamental group of its continuous analogue. And, secondly, they can be also used to obtain new results in Digital Topology, by translating the corresponding continuous results through the levels of the architecture. We use this technique, and the isomorphism above, to obtain a digital version of the Seifert–Van Kampen Theorem (see Section 5). Another relevant example, that shows the power of this technique, is the general Digital Index Theorem obtained in [3] for digital manifolds of arbitrary dimension which, in particular, generalizes the well–known result of Morgenthaler and Rosenfeld ([11]) to all types of (α, β)-surfaces ([6]) and to the strong 26-surfaces ([10]).

Although the digital Seifert–Van Kampen Theorem provides a powerful theoretical tool to obtain the group π_1^d for certain digital objects, it remains as an open question to find an algorithm that computes this group for arbitrary objects; that is, to resemble in our framework the well–known algorithm for the fundamental group of polyhedra ([12]). This problem could be tackled by adapting to our multilevel architecture the algorithm recently developed by Malgouyres in [9], which computes a presentation of the digital fundamental group of an object embedded in an arbitrary graph.

2 The multilevel architecture

In this section we briefly summarize the basic notions of the multilevel architecture for digital topology developed in [2] as well as the notation that will be used through all the paper.

In that architecture, the spatial layout of pixels in a digital image is represented by a *device model*, which is a homogeneously n-dimensional locally finite polyhedral complex K. Each n-cell in K is representing a pixel, and so the digital object displayed in a digital image is a subset of the set $\text{cell}_n(K)$ of n-cells in K; while the other lower dimensional cells in K are used to describe how the pixels could be linked to each other. A *digital space* is a pair (K, f), where K is a device model and f is *weak lighting* *function* defined on K. The function f is used to provide a continuous interpretation, called *continuous analogue*, for each digital object $O \subseteq \text{cell}_n(K)$.

By a homegeneously n-dimensional locally finite polyhedral complex we mean a set K of polytopes, in some Euclidean space $\mathbb{R}^d$, provided with the natural reflexive, antisimetric and transitive binary relationship "to be face of", that in addition satisfies the four following properties:

1. If $\sigma \in K$ and τ is a face of σ then $\tau \in K$.
2. If $\sigma, \tau \in K$ then $\sigma \cap \tau$ is a face of both σ and τ.
3. For each point x in the underlying polyhedron $|K| = \cup\{\sigma; \sigma \in K\}$ of K, there exists a neigbourhood of x which intersects only a finite number of polytopes in K; in particular, each polytope of K is face of a finite number of other polytopes in K.
4. Each polytope $\sigma \in K$ is face of some n-dimensional polytope in K.

These complexes are particular cases of cellular complexes, as they are usually defined in polyhedral topology. So, for simplicity, we will usually call a *cell* to any polytope in K, and K itself will be simply called a polyhedral complex. Next paragraph recalls some elementary notions from polyhedral topology used in this paper. We refer to [15] for further notions on this subject.

Given a polyhedral complex (i.e., a device model) K and two cells $\gamma, \sigma \in K$, we shall write $\gamma \leq \sigma$ if γ is a face of σ, and $\gamma < \sigma$ if in addition $\gamma \neq \sigma$. A centroid-map on K is a map $c : K \to |K|$ such that $c(\sigma)$ belongs to the interior of σ; that is, $c(\sigma) \in \mathring{\sigma} = \sigma - \partial\sigma$, where $\partial\sigma = \cup\{\gamma; \gamma < \sigma\}$ stands for the boundary of σ.

Example 2.1 In this paper it will be essential the role played by the archetypical device model R^n, termed the *standard cubical decomposition* of the Euclidean n-space $\mathbb{R}^n$. The device model R^n is the complex determined by the collection of unit n-cubes in $\mathbb{R}^n$ whose edges are parallel to the coordinate axes and whose centers are in the set $\mathbb{Z}^n$. The centroid-map we will consider in R^n associates to each cube σ its barycenter $c(\sigma)$, which is a point in the set $\mathcal{Z}^n$. Here, $\mathcal{Z} = \frac{1}{2}\mathbb{Z}$ stands for the set of points $\{z \in \mathbb{R}; z = y/2, y \in \mathbb{Z}\}$. In particular, if $\dim \sigma = n$ then $c(\sigma) \in \mathbb{Z}^n$, where $\dim \sigma$ denotes the dimension of σ. So that, every digital object O in R^n can be identified with a subset of points in $\mathbb{Z}^n$. Henceforth we shall use this identification without further comment.

Before to proceed with the definition of weak lighting function, we need some notions, which are illustrated in Fig. 1 for an object O in the device model R^2.

The first two notions formalize two types of "digital neighbourhoods" of a cell $\alpha \in K$ in a given digital object $O \subseteq \text{cell}_n(K)$. Indeed, we call the *star of* α *in* O to the set $\text{st}_n(\alpha; O) = \{\sigma \in O; \alpha \leq \sigma\}$ of n-cells (pixels) in O having α as a face. Similarly, the *extended star of* α *in* O is the set $\text{st}^*_n(\alpha; O) = \{\sigma \in O; \alpha \cap \sigma \neq \emptyset\}$ of n-cells (pixels) in O intersecting α.

The third notion is the *support* of a digital object O which is defined as the set $\text{supp}(O)$ of cells of K (not necessarily pixels) that are the intersection of n-cells (pixels) in O. Namely, $\alpha \in \text{supp}(O)$ if and only if $\alpha = \cap\{\sigma; \sigma \in \text{st}_n(\alpha; O)\}$. In particular, if α is a pixel in O then $\alpha \in \text{supp}(O)$. Notice also that, among all the lower dimensional cells of K, only those in $\text{supp}(O)$ are joining pixels in O.

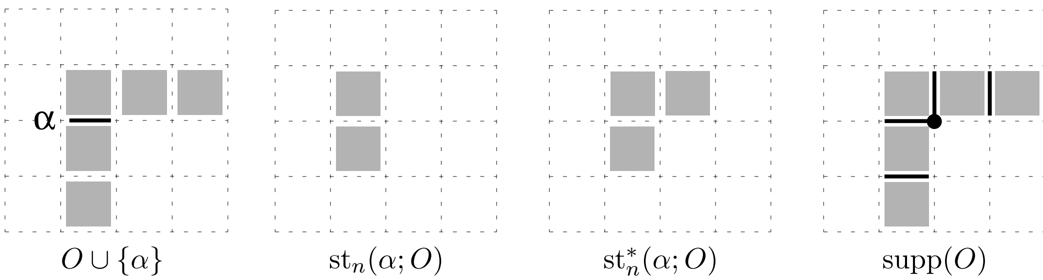


Fig. 1. The physical support of an object O and two types of digital neighbourhoods in O for a cell α. The cells in O together with the bold edges and dots are the elements in $\mathrm{supp}(O)$.

To ease the writing, we shall use the following notation: $\mathrm{supp}(K) = \mathrm{supp}(\mathrm{cell}_n(K))$, $\mathrm{st}_n(\alpha; K) = \mathrm{st}_n(\alpha; \mathrm{cell}_n(K))$ and $\mathrm{st}^*_n(\alpha; K) = \mathrm{st}^*_n(\alpha; \mathrm{cell}_n(K))$. Finally, we shall write $\mathcal{P}(A)$ for the family of all subsets of a given set A.

Definition 2.2 Given a device model K, a *weak lighting function* (w.l.f.) on K is a map $f : \mathcal{P}(\mathrm{cell}_n(K)) \times K \to \{0, 1\}$ satisfying the following five properties for all $O \in \mathcal{P}(\mathrm{cell}_n(K))$ and $\alpha \in K$:

1. if $\alpha \in O$ then $f(O, \alpha) = 1$;
2. if $\alpha \notin \mathrm{supp}(O)$ then $f(O, \alpha) = 0$;
3. $f(O, \alpha) \leq f(\mathrm{cell}_n(K), \alpha)$;
4. $f(O, \alpha) = f(\mathrm{st}^*_n(\alpha; O), \alpha)$; and,
5. if $O' \subseteq O \subseteq \mathrm{cell}_n(K)$ and $\alpha \in K$ are such that $\mathrm{st}_n(\alpha; O) = \mathrm{st}_n(\alpha; O')$, $f(O', \alpha) = 0$ and $f(O, \alpha) = 1$, then: (a) the set of cells $\alpha(O'; O) = \{\beta < \alpha; f(O', \beta) = 0, f(O, \beta) = 1\}$ is not empty; (b) the set $\cup\{\mathring{\beta}; \beta \in \alpha(O'; O)\}$ is connected in $\partial\alpha$; and, (c) if $O \subseteq \overline{O} \subseteq \mathrm{cell}_n(K)$, then $f(\overline{O}, \beta) = 1$ for every $\beta \in \alpha(O'; O)$.

If $f(O, \alpha) = 1$ we say that f *lights* the cell α for the object O.

A w.l.f. f is said to be *strongly local* if $f(O, \alpha) = f(\mathrm{st}_n(\alpha; O), \alpha)$ for all $\alpha \in K$ and $O \subseteq \mathrm{cell}_n(K)$. Notice that this strong local condition implies both properties (4) and (5) above.

From each weak lighting function f on an arbitrary device model K, we shall derive a "continuous interpretation" for any digital object $O \subseteq \mathrm{cell}_n(K)$ called its continuous analogue. The ideas underlying properties (1)-(5) in the previous definition are quite natural. In fact, they have been chosen to avoid continuous analogues which are contradictory with our usual interpretation of objects. So, we postpone their explanation until the end of this section, once the continuous analogue of an object has been defined. For this we need to introduce several other intermediate models as follows.

The *device level* of O is the subcomplex $K(O) = \{\alpha \in K; \alpha \leq \sigma,\ \sigma \in O\}$ of K induced by the cells in O. Notice that the map f_O given by $f_O(O', \alpha) = f(O, \alpha) f(O', \alpha)$, for all $O' \subseteq O$ and $\alpha \in K(O)$, is a w.l.f. on $K(O)$, and we call the pair $(K(O), f_O)$ the *digital subspace* of (K, f) induced by O.

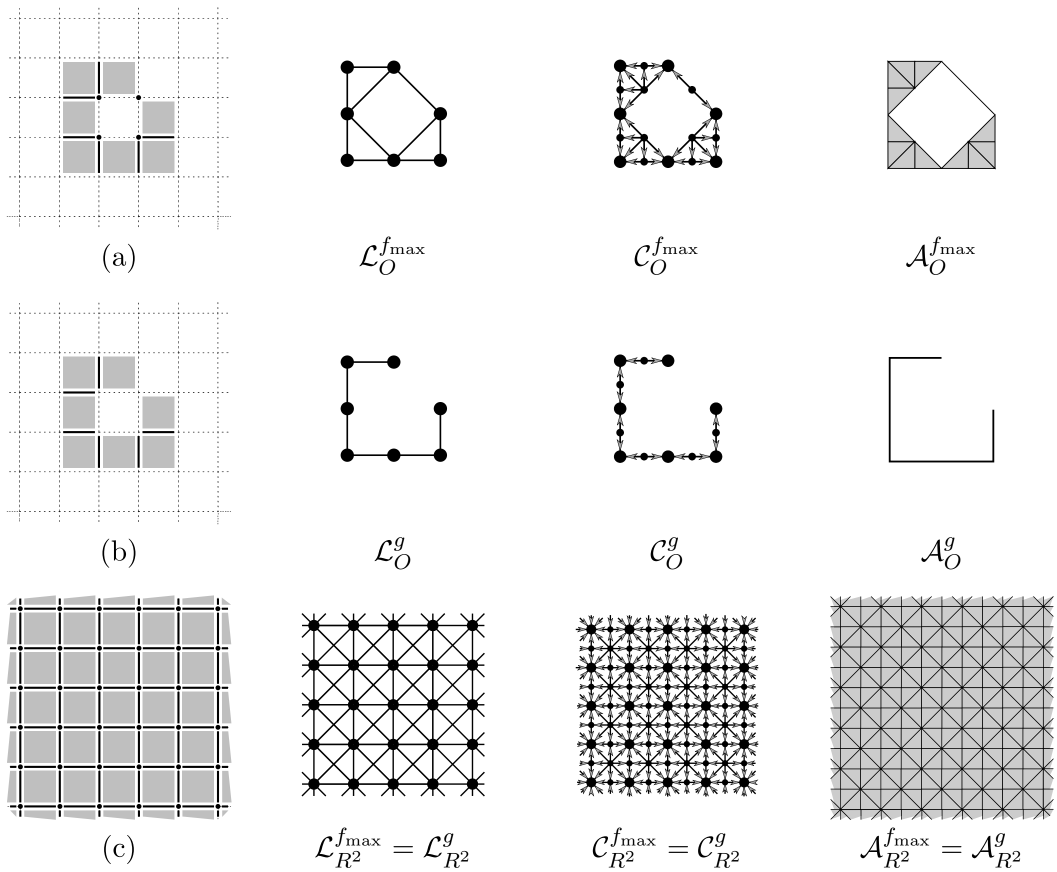


Fig. 2. Levels of the objects O and $\mathrm{cell}_2(R^2)$ for the w.l.f.'s $f_{\max}$ and g in Example 2.3.

The *logical level* of O is an undirected graph, $\mathcal{L}_O^f$, whose vertices are the centroids of n-cells in O and two of them $c(\sigma)$, $c(\tau)$ are adjacent if there exists a common face $\alpha \leq \sigma \cap \tau$ such that $f(O, \alpha) = 1$.

The *conceptual level of* O is the directed graph $\mathcal{C}_O^f$ whose vertices are the centroids $c(\alpha)$ of all cells $\alpha \in K$ with $f(O, \alpha) = 1$, and its directed edges are $(c(\alpha), c(\beta))$ with $\alpha < \beta$.

The *simplicial analogue* of O is the order complex $\mathcal{A}_O^f$ associated to the directed graph $\mathcal{C}_O^f$. That is, $\langle c(\alpha_0), c(\alpha_1), \dots, c(\alpha_m)\rangle$ is an m-simplex of $\mathcal{A}_O^f$ if $c(\alpha_0), c(\alpha_1), \dots, c(\alpha_m)$ is a directed path in $\mathcal{C}_O^f$; or, equivalently, if $f(O, \alpha_i) = 1$, for $0 \leq i \leq m$, and $\alpha_0 < \alpha_1 < \cdots < \alpha_m$. This simplicial complex defines the simplicial level for the object O in the architecture and, finally, the *continuous analogue* of O is the underlying polyhedron $|\mathcal{A}_O^f|$ of $\mathcal{A}_O^f$.

For the sake of simplicity, we will usually drop "f" from the notation of the levels of an object. Moreover, for the whole object $\mathrm{cell}_n(K)$ we will simply write $\mathcal{L}_K$, $\mathcal{C}_K$ and $\mathcal{A}_K$ for its levels.

Example 2.3 Every device model $K \neq \emptyset$ admits the weak lighting functions $f_{\max}$ and g given respectively by:

(a) $f_{\max}(O, \alpha) = 1$ if and only if $\alpha \in \mathrm{supp}(O)$
(b) $g(O, \alpha) = 1$ if and only if $\alpha \in \mathrm{supp}(O)$ and $\mathrm{st}_n(\alpha; K) \subseteq O$

In Figure 2 are shown two objects, O and $\mathrm{cell}_2(R^2)$, in the device model R^2, and their levels for these lighting functions. More precisely, Figures 2(a) and 2(b) show the 2-cells (grey squares) of the object O and the low-dimensional cells (bold edges and vertices) that the w.l.f.'s $f_{\max}$ and g light, respectively, for O. As the sets $\{\alpha \in R^2; f_{\max}(O,\alpha) = 1\}$ and $\{\alpha \in R^2; g(O,\alpha) = 1\}$ do not agree, all the levels of O in the digital spaces $(R^2, f_{\max})$ and (R^2, g) are distinct; in particular $|\mathcal{A}_O^{f_{\max}}| \neq |\mathcal{A}_O^g|$. On the other hand, $\{\alpha \in R^2; f_{\max}(\mathrm{cell}_2(R^2), \alpha) = 1\} = \{\alpha \in R^2; g(\mathrm{cell}_2(R^2), \alpha) = 1\}$ (see Figure 2(c)), and so all the levels of the object $\mathrm{cell}_2(R^2)$ are the same in these two digital spaces.

Actually, the family of digital spaces (R^n, g), for every positive integer n, and more precisely a particular class of digital subspaces (called *windows*) of these spaces, are the key that allows us to introduce in next Section an "extrinsic" notion of digital fundamental group. At this point, it is worth to point out that g induces in R^n the $(2n, 3^n - 1)$–connectivity (see [1, Def. 11]); that is, the generalization to arbitrary dimension of the $(4, 8)$–connectivity on $\mathbb{Z}^2$. On the other hand, $f_{\max}$ induces in R^n the $(3^n - 1, 2n)$-connectivity (see Figure 2).

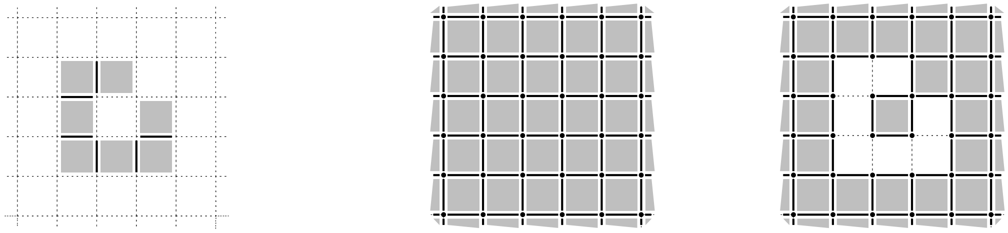
We finish this Section giving some intuitive ideas underlying properties (1)-(5) in the definition of weak lighting function (Definition 2.2). For this, keep in mind that the continuous analogue $|\mathcal{A}_O|$ of a digital object O intendes to be a continuous interpretation of O. So that, the (topological) properties of $|\mathcal{A}_O|$ should match our usual visual perception of that object, which is made of relations on the only visible elements: the pixels in O.

Let O be a digital object in an arbitrary digital space (K, f). Then property (1) says that the n-cells in O are always represented in its continuous analogue $|\mathcal{A}_O|$; that is, we can obviously see the pixels of O whenever we look at that object. In addition to these pixels, only cells from the set $\mathrm{supp}(O)$, but not necessarily all of them, can appear in $|\mathcal{A}_O|$ by property (2). Actually, the lighting function f determines what cells in $\mathrm{supp}(O)$ are perceived. Two important consequences are immediately derived from this property. Firstly, no other n-dimensional cell of K, but the pixels in O, are represented in $|\mathcal{A}_O|$. And, secondly, the lower dimensional cells of K which are represented in $|\mathcal{A}_O|$ are always connecting at least two pixels of O; in particular, and according with our usual perception, this prevent two isolated pixels of the object O from being connected in $|\mathcal{A}_O|$ by a sequence of lower dimensional cells of K that are not faces of pixels in O.

Then, property (3) ensures that whenever we perceive a cell joining pixels of some digital object O, the same cell must connect pixels in the object $\mathrm{cell}_n(K)$ consisting of all the pixels in the device model K; that is, the continuous analogue $|\mathcal{A}_O|$ of any object is always a subspace of the continuous analogue $|\mathcal{A}_K|$ of the digital space. Moreover, it is straightforwardly checked from property (3) that the simplicial complex $\mathcal{A}_O$ is a full subcomplex of $\mathcal{A}_K$.

Property (4), and the strong local property as well, state that our perception of objects is local: whether a cell α is represented or not in $|\mathcal{A}_O|$ depends on the pixels of a "digital neighbourhood" of α in O.

Finally, the rather intricate property (5) is needed to guarantee that our continuous analogue provides a right interpretation of the connectivity of complements of



(a) $O \cup \{\alpha \in R^2; g(O, \alpha) = 1\}$ (b) $\{\alpha \in R^2; g(\mathrm{cell}_2(R^2), \alpha) = 1\}$ (c) $\mathrm{Lb}(R^2/O)$

Fig. 3. The light body of the digital space (R^2, g) shaded by an object O.

objects. For example, let (K, f) be an arbitrary digital space, and let $O \subseteq \mathrm{cell}_n(K)$ be an object such that its background contains two isolated pixels σ_1, σ_2; that is, σ_i $(i = 1, 2)$ is an n-cell in $\mathrm{cell}_n(K) - O$ such that any other n-cell in K intersecting σ_i belongs to O. Following a natural interpretation, σ_1 and σ_2 define two distinct components of the background of O. Hence σ_1 and σ_2 should be represented in different connected components of the complement $|\mathcal{A}_K| - |\mathcal{A}_O|$ of the continuous analogue of O; and we use property (5) to prove this fact.

3 A digital fundamental group

The fundamental group of a topological space X, $\pi_1(X, x_0)$, is usually defined to be the set of homotopy classes of paths $\xi : I = [0, 1] \to X$ that send 0 and 1 to some fixed point x_0 (*loops* at x_0), where an homotopy between two paths ξ_1, ξ_2 is a continuous map $H : I \times I \to X$ such that

1. $H(x, 0) = \xi_1(x)$ and $H(x, 1) = \xi_2(x)$; and
2. $H(0, t) = H(1, t) = x_0$.

In this section we give digital counterparts of the notions of continuous loop and continuous homotopy that will enable us to introduce an "extrinsic" digital fundamental group for arbitrary digital spaces, which readily generalizes to higher digital homotopy groups. Actually, these digital loops and digital homotopies, as defined in Defs. 3.8 and 3.10 respectively, are particular cases of digital maps (Def. 3.3), whose definition makes use of the following notion.

Definition 3.1 Let $S \subseteq \mathrm{cell}_n(K)$ be a digital object in a digital space (K, f). The *light body of* (K, f) *shaded with* S is the set of cells $\mathrm{Lb}(K/S)$ that the w.l.f. f does not light for the object S but are lighted for $\mathrm{cell}_n(K)$; that is,

$$\begin{aligned}\mathrm{Lb}(K/S) &= \{\alpha \in K; f(\mathrm{cell}_n(K), \alpha) = 1, f(S, \alpha) = 0\} \\ &= \{\alpha \in K; c(\alpha) \in |\mathcal{A}_K| - |\mathcal{A}_S|\}.\end{aligned}$$

Figure 3(c) depicts the light body $\mathrm{Lb}(R^2/O)$ of the digital space (R^2, g) shaded by O, where $O \subseteq \mathrm{cell}_2(R^2)$ is the digital object shown in Figure 2 and g is the w.l.f. given in Example 2.3(2). For readability, we reproduce in Figures 3(a) and 3(b) the sets of cells $\{\alpha \in R^2; g(O, \alpha) = 1\}$ and $\{\alpha \in R^2; g(\text{cell}_2(R^2), \alpha) = 1\}$, which are part of Figure 2.

Notice that the light body of a digital space (K, f) shaded by an object S consists of the n-cells in $\text{cell}_n(K) - S$ together with the set of lower dimensional of K through which it is allowed to connect n-cells in $\text{cell}_n(K) - S$ without crossing the obstacle S. Suitable light bodies will be used to define the digital fundamental groups of an object and its complement. More precisely, given a digital object O in a digital space (K, f), we use $\text{Lb}(K(O)/\emptyset)$ —i.e., the light body of the digital subspace $(K(O), f_O)$, that O induces in (K, f), shaded by the empty object $S = \emptyset$— to define the fundamental group of O; while the light body of the whole space (K, f) shaded by the object $S = O$, $\text{Lb}(K/O)$, will be used to define the digital fundamental group of the complement of O; see Remark 3.14. Notice that $\text{Lb}(K(O)/\emptyset)$ agrees with the set of cells of K that the lighting function f lights for the digital object O.

Remark 3.2 (1) In general, the light body of a digital space shaded by a digital object is not always a polyhedral complex.

(2) From property (1) in Definition 2.2 it is immediate to check that $\text{cell}_n(K) - S \subseteq \text{Lb}(K/S)$ and $S \cap \text{Lb}(K/S) = \emptyset$, for any digital object S in a digital space (K, f).

(3) Finally, notice that the light body of an arbitrary digital space (K, f) shaded by the empty object is $\text{Lb}(K/\emptyset) = \{\alpha \in K; f(\text{cell}_n(K), \alpha) = 1\}$. In this sense, it can be understood that Figure 3(b) depicts the set $\text{Lb}(\text{cell}_2(R^2)/\emptyset)$. Moreover, for the object $\text{cell}_n(K)$ consisting of all the pixels in the device model K, $\text{Lb}(K/\text{cell}_n(K)) = \emptyset$.

Next we introduce a formal notion of digital map between digital spaces. Since the device model of a digital space is a polyhedral complex, a possible choice could be to define a digital map from a digital space (K_1, f_1) into another (K_2, f_2) as a cellular map between the device models K_1 and K_2, satisfying certain restrictions. However, this kind of definition is not convenient for our purposes as then the domain of such a digital map would be the whole set of cells in K_1, and not only those cells lighted by the w.l.f. f_1; that is, the cells of K_1 which are relevant in the digital space (K_1, f_1).

Definition 3.3 Let $(K_1, f_1), (K_2, f_2)$ be two digital spaces, with $\dim K_i = n_i$ $(i = 1, 2)$, and let $S_1 \subset \text{cell}_{n_1}(K_1)$ and $S_2 \subset \text{cell}_{n_2}(K_2)$ be two digital objects. A *digital* (S_1, S_2)-*map* (or, simply, a *d-map*) $\Phi_{S_1,S_2} : (K_1, f_1) \to (K_2, f_2)$ from (K_1, f_1) into (K_2, f_2) is a map $\phi : \text{Lb}(K_1/S_1) \to \text{Lb}(K_2/S_2)$ satisfying the two following properties:

1. $\phi(\text{cell}_{n_1}(K_1) - S_1) \subseteq \text{cell}_{n_2}(K_2) - S_2$; and,
2. for $\alpha, \beta \in \text{Lb}(K_1/S_1)$ with $\alpha < \beta$ then $\phi(\alpha) \leq \phi(\beta)$.

That is, ϕ carries top dimensional cells in $\text{Lb}(K_1/S_1)$ to top dimensional cells in $\text{Lb}(K_2/S_2)$ and preserves the face relations (although ϕ needs not be dimension preserving).

Example 3.4 (1) Let $S' \subset S \subseteq \text{cell}_n(K)$ be two digital objects and let $(K(S), f_S)$ be the digital subspace of (K, f) induced by S. Then, the inclusion $\text{Lb}(K(S)/S') \subseteq \text{Lb}(K/S')$ is a (S', S')-map from $(K(S), f_S)$ into (K, f). And, similarly, the inclusion $\text{Lb}(K/S') \subseteq \text{Lb}(K/\emptyset)$ defines a $(S', \emptyset)$-map from (K, f) into itself.

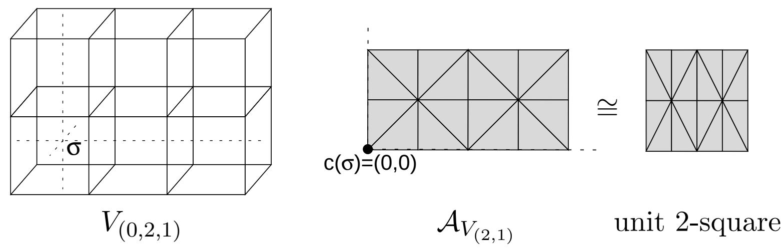


Fig. 4. A $(0, 2, 1)$-window in R^3 and its simplicial analogue, which is simplicially isomorphic to a triangulation of the unit 2-square.

(2) Let $S_1 \subset \text{cell}_{n_1}(K_1)$ and $\sigma \in \text{cell}_{n_2}(K_2)$. For any digital object $S_2 \subseteq \text{cell}_{n_2}(K_2) - \{\sigma\}$, the constant map $\phi^\sigma : \text{Lb}(K_1/S_1) \to \text{Lb}(K_2/S_2)$, given by $\phi^\sigma(\alpha) = \sigma$, for all $\alpha \in \text{Lb}(K_1/S_1)$, defines a (S_1, S_2)-map from (K_1, f_1) into (K_2, f_2) .

(3) The composition of digital maps is a digital map. Namely, if

$$\Phi_{S_1,S_2} : (K_1, f_1) \to (K_2, f_2) \text{ and } \Phi_{S_2,S_3} : (K_2, f_2) \to (K_3, f_3)$$

are d-maps, then the composite $\Phi_{S_2,S_3} \circ \Phi_{S_1,S_2}$ is also a d-map from (K_1, f_1) into (K_3, f_3).

Any d-map from (K_1, f_1) into (K_2, f_2) naturally induces a simplicial map between subcomplexes of the simplicial analogues of K_1 and K_2. More precisely, if $L_2 \subseteq L_1$ are simplicial complexes and $L_1 \setminus L_2 = \{\alpha \in L_1; \alpha \cap |\, L_2 \,| = \emptyset\}$ denotes the simplicial complement of L_2 in L_1, then it is straightforward to show

Proposition 3.5 *Any d-map $\Phi_{S_1,S_2} : (K_1, f_1) \to (K_2, f_2)$ induces a simplicial map $\mathcal{A}(\Phi_{S_1,S_2}) : \mathcal{A}_{K_1} \setminus \mathcal{A}_{S_1} \to \mathcal{A}_{K_2} \setminus \mathcal{A}_{S_2}$, which is defined on the vertices $c_1(\alpha)$ of $\mathcal{A}_{K_1} \setminus \mathcal{A}_{S_1}$ by $\mathcal{A}(\Phi_{S_1,S_2})(c_1(\alpha)) = c_2(\Phi_{S_1,S_2}(\alpha))$. Here c_i is a centroid-map on the device model K_i, for $i = 1, 2$.*

In this paper we are only interested in digital loops and homotopies, which are particular classes of digital maps whose domains are the light bodies of certain digital spaces, called *windows*. These windows play the same role as the unit interval, I, and the square, $I \times I$, in continuous homotopy. To introduce them we will use the following notation. Given two points $x = (x_1, \ldots, x_m), y = (y_1, \ldots, y_m) \in \mathbb{R}^m$, we write $x \preceq y$ if $x_i \leq y_i$, for all $1 \leq i \leq m$, while $x + y$ will stand for the usual vector addition $x + y = (x_1 + y_1, \ldots, x_m + y_m) \in \mathbb{R}^m$.

Definition 3.6 Given two points $r, x \in \mathbb{Z}^m$, with $r_i \geq 0$ for $1 \leq i \leq m$, we call a *window of size* r (or r*-window*) *of* R^m *based at* x to the digital subspace V_r^x of (R^m, g) induced by the digital object $O_r^x = \{\sigma \in \text{cell}_m(R^m); x \preceq c(\sigma) \preceq x + r\}$, where (R^m, g) is the digital space defined in Example 2.3. For the sake of simplicity, we shall write V_r to denote the r-window of R^m based at the point $x = (0, \ldots, 0) \in \mathbb{Z}^m$.

Notice that the simplicial analogue of an r-window V_r^x of R^m is simplicially isomorphic to a triangulation of a unit n-cube, where n is the number of non-zero coordinates in r (see Figure 4). Moreover, the set $\{y \in \mathcal{Z}^m; x \preceq y \preceq x + r\}$ are the centroids of the cells in $\text{Lb}(V_r^x/\emptyset)$ which actually span the simplicial analogue of V_r^x.

Remark 3.7 To ease the writing, given an r-window V_r of R^m, we will identify each cell $\alpha \in \mathrm{Lb}(V_r/\emptyset)$ with its centroid $c(\alpha)$. In particular, if V_r is an r-window of R^1, then $\mathrm{Lb}(V_r/\emptyset) = \{\sigma_0, \sigma_1, \ldots, \sigma_{2r-1}, \sigma_{2r}\}$ consists of $2r+1$ cells (points and segments) which will be identified with the numbers $c(\sigma_i) = i/2$, for $0 \leq i \leq 2r$. And, similarly, for a window $V_{(r_1,r_2)}$ of R^2, we identify each cell $\alpha \in \mathrm{Lb}(V_{(r_1,r_2)}/\emptyset)$ with a pair $c(\alpha) = (i/2, j/2)$, where $0 \leq i \leq 2r_1$ and $0 \leq j \leq 2r_2$.

With the notation above we are now ready to give "extrinsic" notions of walks and loops, in a digital object, which will lead us to the definition of a digital fundamental group.

Definition 3.8 Let $S, O \subseteq \mathrm{cell}_n(K)$ be two disjoint digital objects in a digital space (K, f), and σ, τ two n-cells in O. A *S-walk in O of length $r \in \mathbb{Z}$ from σ to τ* is a digital $(\emptyset, S)$-map $\phi_r : \mathrm{Lb}(V_r/\emptyset) \to \mathrm{Lb}(K(O \cup S)/S)$ such that $\phi_r(0) = \sigma$ and $\phi_r(r) = \tau$. A *S-loop in O based at σ* is a S-walk ϕ_r such that $\phi_r(0) = \phi_r(r) = \sigma$.

The *juxtaposition* of two given S-walks ϕ_r, ϕ_s in O, with $\phi_r(r) = \phi_s(0)$, is the S-walk $\phi_r * \phi_s : \mathrm{Lb}(V_{r+s}/\emptyset) \to \mathrm{Lb}(K(O \cup S)/S)$ of length $r+s$ given by

$$\phi_r * \phi_s(i/2) = \begin{cases} \phi_r(i/2) & \text{if } 0 \leq i \leq 2r \\ \phi_s(i/2 - r) & \text{if } 2r \leq i \leq 2(r+s) \end{cases}$$

Notice that the notion of a S-walk is compatible with the definition of S-path given in [1, Def. 5]. Recall that a S-path in O is a sequence $(\tau_i)_{i=0}^r$ of n-cells in O such that, for $1 \leq i \leq r$, there is a face $\alpha_i \leq \tau_{i-1} \cap \tau_i$ with $f(O \cup S, \alpha_i) = 1$ and $f(S, \alpha_i) = 0$; that is, $\alpha_i \in \mathrm{Lb}(K(O \cup S)/S)$. Actually, each S-walk ϕ_r defines a S-path given by the sequence $\varphi(\phi_r) = (\phi_r(i))_{i=0}^r$. And, conversely, a S-path $(\tau_i)_{i=0}^r$ in O yields a family Φ_r of S-walks such that $\phi_r \in \Phi_r$ if and only if $\phi_r(i) = \tau_i$, for $0 \leq i \leq r$, and $\phi_r(i - 1/2) \in \{\alpha \leq \tau_{2i-2} \cap \tau_{2i}; f(O \cup S, \alpha) = 1, f(S, \alpha) = 0\}$, for $1 \leq i \leq r$. However, this "extrinsic" notion of S-walk will be more suitable to define the digital fundamental group of an object since, together with the notion of r-window, it allows us to introduce the following definition of digital homotopy.

Definition 3.9 Let ϕ_r^1, ϕ_r^2 two S-walks in O of the same length $r \in \mathbb{Z}$ from σ to τ. We say that ϕ_r^1, ϕ_r^2 are *digitally homotopic* (or, simply, *d-homotopic*) *relative* $\{\sigma, \tau\}$, and we write $\phi_r^1 \simeq_d \phi_r^2$ rel. $\{\sigma, \tau\}$, if there exists an (r, s)-window $V_{(r,s)}$ in R^2 and a $(\emptyset, S)$-map $H : \mathrm{Lb}(V_{(r,s)}/\emptyset) \to \mathrm{Lb}(K(O \cup S)/S)$, called a *d-homotopy*, such that $H(i/2, 0) = \phi_r^1(i/2)$ and $H(i/2, s) = \phi_r^2(i/2)$, for $0 \leq i \leq 2r$, and moreover $H(0, j/2) = \sigma$ and $H(r, j/2) = \tau$, for $0 \leq j \leq 2s$. Here we use the identification $H(a_1, a_2) = H(\alpha)$, where $c(\alpha) = (a_1, a_2) \in \mathcal{Z}^2$ is the centroid of a cell $\alpha \in \mathrm{Lb}(V_{(r,s)}/\emptyset)$; see Remark 3.7.

The notion of S-walk in O above formalizes the idea of discrete path in a digital object O under the opposition of an obstacle S that cannot be traversed. In terms of our continuous analogue, each S-walk yields a continuous path in the polyhedron $|\mathcal{A}_{O \cup S}|$ that not intersect the continuous analogue of the obstacle, $|\mathcal{A}_S|$. In case $S = \emptyset$, the $\emptyset$-walks are, simply, discrete paths in the digital object O. And similarly, a d-homotopy between two S-walks ϕ_r^1, ϕ_r^2 is a discrete transformation, through adjacent pixels in O, from ϕ_r^1 to ϕ_r^2 that does not move across the obstacle S.

Clearly, the previous definition of d-homotopy induces an equivalence relation between the S-walks in O from σ to τ of the same length. Moreover, it is easy to show that the juxtaposition of S-walks is compatible with d-homotopies.

Definition 3.9 extends to S-walks of the arbitrary lengths as follows.

Definition 3.10 Let ϕ_r, ϕ_s two S-walks in O from σ to τ of lengths r and s respectively. We say that ϕ_r is d-homotopic to ϕ_s relative $\{\sigma, \tau\}$, and we write also $\phi_r \simeq_d \phi_s$ rel. $\{\sigma, \tau\}$, if there exist constant S-loops $\phi^{\tau}_{r'}$ and $\phi^{\tau}_{s'}$ such that $r + r' = s + s'$ and $\phi_r * \phi^{\tau}_{r'} \simeq_d \phi_s * \phi^{\tau}_{s'}$ rel. $\{\sigma, \tau\}$.

Proposition 3.11 *Let ϕ_r be a S-walk in O from σ to τ, and ϕ^{σ}_s, ϕ^{τ}_s two constant S-loops of the same length $s \in \mathbb{Z}$. Then, $\phi^{\sigma}_s * \phi_r \simeq_d \phi_r * \phi^{\tau}_s$ rel. $\{\sigma, \tau\}$.*

Proof. This result becomes trivial if ϕ_r is a constant S-walk. In particular, this is the case when $r = 0$. For the rest of cases we shall proceed by induction on the length r of the S-walk ϕ_r.

In case $r = 1$, we have that $\phi_r(0) = \sigma$, $\phi_r(1) = \tau$ and $\phi_r(1/2) = \alpha$ is a common face of σ and τ. Then, it is immediate to check that the map $H : \mathrm{Lb}(V_{(s+1,1)}/\emptyset) \to \mathrm{Lb}(K(O \cup S)/S)$ given by $H(i/2, 0) = \phi^{\sigma}_s * \phi_r(i/2)$ and $H(i/2, 1) = \phi_r * \phi^{\tau}_s(i/2)$, for $0 \leq i \leq 2s + 2$, and $H(0, 1/2) = \sigma$, $H(s + 1, 1/2) = \tau$ and $H(i/2, 1/2) = \alpha$, for $1 \leq i \leq 2s + 1$, is a $(\emptyset, S)$-map and hence a d-homotopy from $\phi^{\sigma}_s * \phi_r$ to $\phi_r * \phi^{\tau}_s$. Here we use again the identification $H(a_1, a_2) = H(\alpha)$, where $c(\alpha) = (a_1, a_2) \in \mathcal{Z}^2$ is the centroid of a cell $\alpha \in \mathrm{Lb}(V_{(s+1,1)}/\emptyset)$; see Remark 3.7.

Now, assume the result holds for S-walks of length less than or equal to $r - 1$, and notice that $\phi_r = \phi_{r-1} * \psi_1$, where ϕ_{r-1} and ψ_1 are the S-walks of length $r - 1$ and 1, respectively, given by $\phi_{r-1}(i/2) = \phi_r(i/2)$, for $0 \leq i \leq 2r - 2$, and $\psi_1(i/2) = \phi_r(r - 1 + i/2)$, for $0 \leq i \leq 2$. Then, we have

$$\begin{aligned} \phi^{\sigma}_s * \phi_r &= \phi^{\sigma}_s * \phi_{r-1} * \psi_1 \simeq_d \phi_{r-1} * \phi^{\phi_r(r-1)}_s * \psi_1 \text{ rel. } \{\sigma, \tau\} \\ &\simeq_d \phi_{r-1} * \psi_1 * \phi^{\tau}_s \text{ rel. } \{\sigma, \tau\} = \phi_r * \phi^{\tau}_s \end{aligned}$$

by induction hypothesis and the compatibility between d-homotopies and the juxtaposition of S-walks. ∎

Notice that Definition 3.10 induces an equivalence relation in the set of S-walks in O from σ to τ of arbitrary length. Moreover, from Proposition 3.11 it is not difficult to check that the juxtaposition is compatible with d-homotopies between S-walks of arbitrary length. Thus, the juxtaposition of S-loops naturally induces a product operation that endows the set of classes of S-loops in O based at a fixed, but arbitrary, n-cell $\sigma \in O$ with a group structure, for which the trivial element is the class of constant S-loops at σ, and the inverse of the class $[\phi_r]$ is represented by the S-loop ϕ_r^{-1} obtained by traversing ϕ_r backwards; that is, $\phi_r^{-1}(i/2) = \phi_r(r - i/2)$ for all $0 \leq i \leq 2r$. So, we next introduce the notion of digital fundamental group as follows.

Definition 3.12 Let S, O be two disjoint digital objects in a digital space (K, f), and σ an n-cell in O. The *digital fundamental group of O at σ shaded by S* is the set $\pi_1^d(O/S,\sigma)$ of d-homotopy classes of S-loops in O based at σ with the product operation $[\phi_r]\cdot[\psi_s]=[\phi_r*\psi_s]$. In case $S=\emptyset$, we will simply call to $\pi_1^d(O/\emptyset,\sigma)=\pi_1^d(O,\sigma)$ the *digital fundamental group of O at σ.*

Remark 3.13 The previous definition readily generalizes to give higher digital homotopy groups by replicating the same steps as above but starting with a suitable notion of m-dimensional S-loop. More explicitly, let $r\in\mathbb{Z}^m$ be a point with positive coordinates, and call boundary of an r-window V_r to the set of cells $\partial V_r=\{\alpha\in \mathrm{Lb}(V_r/\emptyset); c(\alpha)\in\partial\mathcal{A}_{V_r}\}$. Notice that the boundary $\partial\mathcal{A}_{V_r}$ is well-defined since $\mathcal{A}_{V_r}$ triangulates the unit m-cube. Then define an *m-dimensional S-loop* in O at σ of size r as any $(\emptyset,S)$-map $\phi_r:\mathrm{Lb}(V_r/\emptyset)\to\mathrm{Lb}(K(O\cup S)/S)$ such that the restriction of ϕ_r to the boundary ∂V_r is σ.

Remark 3.14 Definition 3.12 provides an entire family of digital fundamental groups for a given digital object O when the object S is allowed to range over the family of all subsets of $\mathrm{cell}_n(K)-O$. Particularly interesting are the groups $\pi_1^d(O/\emptyset,\sigma)=\pi_1^d(O,\sigma)$ and $\pi_1^d(O/(\mathrm{cell}_n(K)-O),\sigma)$ that, respectively, represents the digital fundamental group of the object O itself and the digital fundamental group of O as the complement of the object $\mathrm{cell}_n(K)-O$.

Next Section is aimed to show that the digital fundamental group $\pi_1^d(O,\sigma)$ of an object O is naturally isomorphic to the fundamental group $\pi_1(|\mathcal{A}_O|,c(\sigma))$ of its continuous analogue. The corresponding result for the complement of an object will be the subject of a future work.

4 Isomorphism with the continuous fundamental group

As it was quoted in the previous section, the fundamental group of a topological space X, $\pi_1(X,x_0)$, is defined to be the set of homotopy classes of loops at x_0. The set $\pi_1(X,x_0)$ is given the structure of a group by the operation $[\alpha]\cdot[\beta]=[\alpha*\beta]$, where $\alpha*\beta$ denotes the juxtaposition of paths. However, for a triangulated polyhedron $|L|$ there is an alternative definition of the fundamental group $\pi_1(|L|,x_0)$ that is more convenient for our purposes, so we next explain it briefly. Recall that an *edge–walk* in $|L|$ from a vertex v_0 to a vertex v_n is a sequence α of vertices $v_0,v_1,\dots,v_n$, such that for each $k=1,2,\dots,n$ the vertices v_{k-1},v_k span a simplex in L (possibly $v_{k-1}=v_k$). If $v_0=v_n$, α is called an *edge–loop based at* v_0.

Given another edge–walk $\beta=(v_j)_{j=n}^{m+n}$ whose first vertex is the same as the last vertex of α, the *juxtaposition* $\alpha*\beta=(v_i)_{i=0}^{m+n}$ is defined in the obvious way. The *inverse* of α is $\alpha^{-1}=(v_n,v_{n-1},\dots,v_0)$.

Two edge–walks α and β are said to be *equivalent* if one can be obtained from the other by a finite sequence of operations of the form:

(a) if $v_{k-1}=v_k$, replace $\dots,v_{k-1},v_k,\dots$ by $\dots,v_k,\dots$, or conversely replace $\dots,v_k,\dots$ by $\dots,v_{k-1},v_k,\dots$; or
(b) if v_{k-1},v_k,v_{k+1} span a simplex of L (not necessarily 2-dimensional), replace $\dots,v_{k-1},v_k,v_{k+1},\dots$ by $\dots,v_{k-1},v_{k+1},\dots$, or conversely.

This clearly sets up an equivalence relation between edge–walks, and the set of equivalence classes $[\alpha]$ of edge–loops α in L, based at a vertex v_0, forms a group $\pi_1(L, v_0)$ with respect to the juxtaposition of edge–loops. This group is called the *edge–group* of L.

Each edge–walk α in L defines in an obvious way a continuous path $\theta(\alpha)$ in the underlying polyhedron $|L|$; and so, we will identify henceforth the edge–walk α with the continuous path $\theta(\alpha)$. Actually this correspondence yields an isomorphism $\pi_1(|L|, v_0) \cong \pi_1(L, v_0)$. More precisely,

Theorem 4.1 ([12]; 3.3.9) *There exists an isomorphism $\Theta : \pi_1(L, v_0) \to \pi_1(|L|, v_0)$ which carries the class $[\alpha]$ to the class $[\theta(\alpha)]$.*

Corollary 4.2 *Let O, S be two disjoint digital objects in a digital space (K, f). Then $\pi_1(\mathcal{A}_{O\cup S} \setminus \mathcal{A}_S, c(\sigma)) \cong \pi_1(|\mathcal{A}_{O\cup S}| - |\mathcal{A}_S|, c(\sigma))$ for any $\sigma \in O$.*

Proof. From property (3) in Definition 2.2, it is immediate to check that the simplicial analogues $\mathcal{A}_{O\cup S}$ and $\mathcal{A}_S$ are both full subcomplexes of $\mathcal{A}_K$. Using this fact, Corollary 4.2 is a consequence of Theorem 4.1 and next lemma. ■

Lemma 4.3 *Let $K, L \subseteq J$ be two full subcomplexes. Then $|K \setminus L| = |K \setminus K \cap L|$ is a strong deformation retract of $|K| - |L| = |K| - |K \cap L|$.*

Proof. The lemma is actually Lemma 72.2 in [13] applied to the full subcomplex $K \cap L \subseteq K$. Notice that $K \cap L$ is full in K since L is full in J. ■

Given an arbitrary digital object O in a digital space (K, f) and any n-cell $\sigma \in O$, we next define a natural morphism, $h : \pi_1^d(O, \sigma) \to \pi_1(\mathcal{A}_O, c(\sigma))$, from the digital fundamental group of O at σ into the edge-group of its simplicial analogue $\mathcal{A}_O$ at the centroid $c(\sigma)$. For this, observe firstly that, for any $\emptyset$-loop ϕ_r in O based at σ, the sequence $c(\phi_r) = (c(\phi_r(i/2)))_{i=0}^{2r}$ defines an edge–loop in $\mathcal{A}_O$ based at $c(\sigma)$. So we simply set $h([\phi_r]) = [c(\phi_r)]$.

Remark 4.4 We have the two following properties

1. if ϕ_r and ϕ_s are two $\emptyset$-loops in O based at σ, then $c(\phi_r * \phi_s) = c(\phi_r) * c(\phi_s)$; and,
2. if ϕ_r is a constant $\emptyset$-loop in O, then $c(\phi_r)$ is also a constant edge–loop;

which are immediate from the definition of the edge–loop $c(\phi_r)$.

Lemma 4.5 *The correspondence h above is well defined and it yields a group homomorphism $h : \pi_1^d(O, \sigma) \to \pi_1(\mathcal{A}_O, c(\sigma))$*

Proof. Assume that $\phi_r \simeq_d \phi_s$ rel. σ are two equivalent $\emptyset$-loops in O. Then, by Definition 3.10, there exist two constant $\emptyset$-loops $\phi_{r'}^\sigma$ and $\phi_{s'}^\sigma$, such that $r + r' = s + s'$, and a d-homotopy $H : \mathrm{Lb}(V_{(r+r',t)}/\emptyset) \to \mathrm{Lb}(K(O)/\emptyset)$ from $\phi_r * \phi_{r'}^\sigma$ to $\phi_s * \phi_{s'}^\sigma$. By Proposition 3.5 we know that H induces a simplicial map $\mathcal{A}(H) : \mathcal{A}_{V_{(r+r',t)}} \to \mathcal{A}_O$. In addition, $\mathcal{A}_{V_{(r+r',t)}}$ is simplicially isomorphic to a triangulation of the unit square (see Figure 4) and, moreover, $\mathcal{A}(H)$ restricted to the top and the bottom of that unit square define $c(\phi_r * \phi^\sigma_{r'})$ and $c(\phi_s * \phi^\sigma_{s'})$, respectively. From these facts it is not difficult to show that $c(\phi_r * \phi^\sigma_{r'})$ and $c(\phi_s * \phi^\sigma_{s'})$ are equivalent edge–loops. Now, by properties in Remark 4.4, and using equivalence transformations of type (a), we derive that $c(\phi_r * \phi^\sigma_{r'}) = c(\phi_r) * c(\phi^\sigma_{r'})$ is an equivalent edge–loop to $c(\phi_r)$, and similarly $c(\phi_s * \phi^\sigma_{s'})$ is also equivalent to $c(\phi_s)$, from which it follows that h is well-defined. Finally, notice that h is an homomorphism of groups is straightforwardly proved from the two properties in Remark 4.4. ∎

We are now ready to state the main result of this section. Namely,

Theorem 4.6 *Let O be a digital object in the digital space (K, f). Then, the homomorphism h in Lemma 4.5 is an isomorphism. Hence the composite*

$$\Theta h : \pi_1^d(O, \sigma) \to \pi_1(|\mathcal{A}_O|, c(\sigma))$$

is also an isomorphism by Theorem 4.1.

The proof of this theorem relies in the construction of a particular family $F(\gamma)$ of $\emptyset$-loops, called the digital representatives of γ, for each edge–loop γ in $\mathcal{A}_O$ which is based at a vertex $c(\sigma)$, with $\sigma \in O$. In order to define $F(\gamma)$, we introduce the following notions.

Definition 4.7 A vertex $c(\gamma_i)$ of an edge–loop $\gamma = (c(\gamma_i))_{i=0}^t$ in $\mathcal{A}_O$ is said to be *reducible in* γ if $i > 0$ and one of the two following properties holds:

(a') $\gamma_{i-1} = \gamma_i$; or

(b') there exists a vertex $c(\gamma_k)$, with $i < k \leq t$, such that $\gamma_k \neq \gamma_i$, and either $\gamma_{i-1} < \gamma_i < \gamma_j$ or $\gamma_{i-1} > \gamma_i > \gamma_j$, $j = \min\{k; i < k \leq t, \gamma_i \neq \gamma_k\}$.

We also call *reducible* to any edge–loop γ that contains some reducible vertex; otherwise we say that γ is *irreducible*.

Remark 4.8 Notice that, if $\gamma = (c(\gamma_i))_{i=0}^t$ is an irreducible edge–loop in $\mathcal{A}_O$, then either $\gamma_{i-1} < \gamma_i > \gamma_{i+1}$ or $\gamma_{i-1} > \gamma_i < \gamma_{i+1}$, for all $0 < i < t$. Moreover, in case γ is based at a vertex $c(\sigma)$ with $\sigma \in O$, then the length of γ must be an even number $t = 2r$, and thus $\gamma_{2i-2} > \gamma_{2i-1} < \gamma_{2i}$ for all $0 < i \leq r$. Notice also that, for an arbitrary edge–loop $\gamma = (c(\gamma_i))_{i=0}^t$ in $\mathcal{A}_O$, the vertex $c(\gamma_0)$ is never reducible.

Lemma 4.9 *Given an edge–loop γ in $\mathcal{A}_O$ based at $c(\sigma)$, with $\sigma \in O$, its reducible vertices may be removed from γ in any order to yield a unique irreducible edge–loop $\overline{\gamma} = (c(\overline{\gamma}_i))_{i=0}^{2r}$, based at the same vertex $c(\sigma)$, which is equivalent to γ.*

Proof. First we observe that an equivalent edge–loop γ' is obtained whenever a reducible vertex $c(\gamma_{i_0})$ is dropped from γ. Indeed, if property (a') holds for a vertex $c(\gamma_{i_0})$, its deletion from γ is an equivalence transformation of type (a) above. And if $c(\gamma_{i_0})$ satisfies property (b'), then either $\gamma_{i_0-1} < \gamma_{i_0} \leq \gamma_{i_0+1}$ or $\gamma_{i_0-1} > \gamma_{i_0} \geq \gamma_{i_0+1}$. In any case, the vertices $c(\gamma_{i_0-1}), c(\gamma_{i_0}), c(\gamma_{i_0+1})$ span a simplex in $\mathcal{A}_O$, and hence removing $c(\gamma_{i_0})$ is a transformation of type (b).

Next we show that another reducible vertex $c(\gamma_{i_1})$ in γ (if any) remains a reducible vertex in γ' and then the lemma will easily follow by induction.

Assume $i_1 < i_0$ (the case $i_0 < i_1$ is similar). In such a case if $c(\gamma_{i_1})$ is reducible of type (a') in γ, it is obvious that $c(\gamma_{i_1})$ is also reducible of the same type in γ'. So, let us assume that $c(\gamma_{i_1})$ is reducible of type (b') in γ via the face relations $\gamma_{i_1-1} < \gamma_{i_1} < \gamma_j$, for $j = \min\{k; i_1 < k \leq t, \gamma_k \neq \gamma_{i_1}\}$ (the other possibility $\gamma_{i_1-1} > \gamma_{i_1} > \gamma_j$ is similar). Moreover, if $j \neq i_0$ then $c(\gamma_{i_1})$ is clearly reducible of type (b') in γ'. Otherwise we have $\gamma_{i_1-1} < \gamma_{i_1} = \cdots = \gamma_{i_0-1} < \gamma_{i_0}$ and hence $c(\gamma_{i_0})$ is necessarily a reducible vertex of type (b') in γ associated to the face relations $\gamma_{i_0-1} < \gamma_{i_0} \leq \gamma_{i_0+1}$. Therefore $\gamma_{i_1-1} < \gamma_{i_1} < \gamma_{i_0+1}$, and $c(\gamma_{i_1})$ is a reducible vertex in γ'. ∎

For an edge–loop γ as in the hypothesis of Lemma 4.9, we use $\overline{\gamma}$ to define the family $F(\gamma)$ of $\emptyset$-loops at σ of length r as follows.

Definition 4.10 The set $F(\gamma)$ of *digital representatives* of γ consists of all $\emptyset$-loops ϕ_r for which $\phi_r(0) = \phi_r(r) = \sigma$, $\phi_r(i - 1/2) = \overline{\gamma}_{2i-1}$, and $\phi_r(i) \in \mathrm{st}_n(\overline{\gamma}_{2i}; O)$, for $1 \leq i \leq r$.

Notice that $\mathrm{st}_n(\overline{\gamma}_{2i}; O) = \{\overline{\gamma}_{2i}\}$ if and only if $\overline{\gamma}_{2i} \in O$, while $\mathrm{st}_n(\overline{\gamma}_{2i}; O)$ contains at least two elements otherwise. This is clear since $c(\overline{\gamma}_{2i}) \in \mathcal{A}_O$ yields $\overline{\gamma}_{2i} \in \mathrm{supp}(O)$ by property (2) in Definition 2.2. Thus, in any case, $F(\gamma) \neq \emptyset$ is a non-empty set; and moreover, $F(\gamma) = \{\overline{\gamma}\}$ if and only if $\overline{\gamma}_{2i} \in O$ for all $0 \leq i \leq r$.

Remark 4.11 (1) Let γ be an edge–loop in $\mathcal{A}_O$ based at $c(\sigma)$, with $\sigma \in O$. If the edge–loop δ is obtained by removing from γ anyone of its reducible vertices, then $\overline{\delta} = \overline{\gamma}$ and hence $F(\delta) = F(\gamma)$.

(2) Let ϕ_r be an $\emptyset$-loop in O based at σ such that $\phi_r(i - 1/2) \neq \phi_r(i)$, for $1 \leq i \leq 2r$. Then, the edge–loop $c(\phi_r) = (c(\phi_r(i/2)))_{i=0}^{2r}$ is irreducible and, moreover, $F(c(\phi_r)) = \{\phi_r\}$ since $\phi_r(i) \in O$ for $0 \leq i \leq r$.

Proposition 4.12 *Let γ be an edge–loop in $\mathcal{A}_O$ based at $c(\sigma)$, with $\sigma \in O$. Then, the two following properties hold:*

1. *For each $\emptyset$-loop $\phi_r \in F(\gamma)$, $h([\phi_r]) = [\gamma]$.*
2. *Any two $\emptyset$-loops $\phi_r^1, \phi_r^2 \in F(\gamma)$ are d-homotopic.*

Proof. Part (1) will follow if we show that the edge–loop $c(\phi_r) = (c(\phi_r(i/2)))_{i=0}^{2r}$ defined by ϕ_r is equivalent to γ. For this, let us consider the set $\overline{F}(\gamma)$ of edge–loops $\alpha = (c(\alpha_i))_{i=0}^{2r}$ at $c(\sigma)$ such that $\alpha_0 = \alpha_{2r} = \sigma$, $\alpha_{2i-1} = \overline{\gamma}_{2i-1}$ for $1 \leq i \leq r$, and $\alpha_{2i} \in \mathrm{st}_n(\overline{\gamma}_{2i}; O) \cup \{\overline{\gamma}_{2i}\}$ for $0 \leq i \leq r$. Notice that $\{c(\phi_r); \phi_r \in F(\gamma)\} \cup \{\overline{\gamma}\} \subseteq \overline{F}(\gamma)$. Since $\overline{\gamma}$ was obtained from γ by transformations of types (a) and (b), they are equivalent edge–loops. So, it will suffice to show that any $\alpha \in \overline{F}(\gamma)$ is equivalent to $\overline{\gamma}$. This will be done by induction on the number $t(\alpha)$ of vertices $c(\alpha_{2i})$ in α for which $\alpha_{2i} \neq \overline{\gamma}_{2i}$.

For $t(\alpha) = 0$ we get $\alpha = \overline{\gamma}$. Assume that $\alpha \in \overline{F}(\gamma)$ is equivalent to $\overline{\gamma}$ if $t(\alpha) \leq t-1$. Then, for an edge–loop $\alpha \in \overline{F}(\gamma)$ with $t(\alpha) = t$, let $c(\alpha_{2i})$ be any vertex in α such that $\alpha_{2i} \neq \overline{\gamma}_{2i}$ (notice that $0 \neq i \neq r$). The definition of $\overline{\gamma}$ yields $\overline{\gamma}_{2i+1}, \overline{\gamma}_{2i-1} < \overline{\gamma}_{2i} < \alpha_{2i} \in O$. So, we obtain a new edge–loop $\tilde{\alpha} \in \overline{F}(\gamma)$ with $t(\tilde{\alpha}) = t - 1$ by setting $c(\tilde{\alpha}_j) = c(\alpha_j)$ if $j \neq 2i$ and $c(\tilde{\alpha}_{2i}) = c(\overline{\gamma}_{2i})$. Moreover, α is an edge–loop equivalent to $\tilde{\alpha}$ (by two equivalence transformations of type (b)) and hence α is an edge–loop equivalent to $\overline{\gamma}$ by induction hypothesis.

To show part (2) it is enough to observe that the map

$$H : \mathrm{Lb}(V_{(r,1)}/\emptyset) \to \mathrm{Lb}(K(O)/\emptyset) ,$$

given by $H(i/2, 0) = \phi_r^1(i/2)$, $H(i/2, 1) = \phi_r^2(i/2)$ and $H(i/2, 1/2) = \overline{\gamma}_i$, for $0 \leq i \leq 2r$, is a d-homotopy relating ϕ_r^1 and ϕ_r^2. Here we use once more the identification of a cell $\alpha \in \mathrm{Lb}(V_{(r,1)}/\emptyset)$ with its centroid $c(\alpha) = (a_1, a_2) \in \mathcal{Z}^2$; see Remark 3.7. ■

Proposition 4.13 *Let γ and δ be two edge-loops in $\mathcal{A}_O$ based at $c(\sigma) \in \mathcal{A}_O$ such that they are related by an equivalence transformation relation of type (a) or (b). Then all $\emptyset$-loops in $F(\gamma) \cup F(\delta)$ are d-homotopic rel. σ.*

Proof. It is an immediate consequence of Lemma 4.14 below and Proposition 4.12(2). ■

Proof of Theorem 4.6. Corollary 4.2 and Proposition 4.12(1) show that the group homomorphism $h : \pi_1^d(O, \sigma) \to \pi_1(\mathcal{A}_O, c(\sigma))$ is onto. So, it will suffice to prove that any two $\emptyset$-loops, ϕ_r and ϕ_s, in O define the same element in $\pi_1^d(O, \sigma)$ provided $h([\phi_r]) = [c(\phi_r)] = [c(\phi_s)] = h([\phi_s])$.

Since $c(\phi_r)$ and $c(\phi_s)$ are equivalent edge–loops, there exists a sequence $\alpha_0, \alpha_1, \ldots, \alpha_k$ of edge–loops at $c(\sigma)$ in $\mathcal{A}_O$ such that $\alpha_0 = c(\phi_r)$, $\alpha_k = c(\phi_s)$ and α_{i-1}, α_i are related by an equivalence transformation of type (a) or (b). Then, Proposition 4.13 yields that every $\emptyset$-loop in $\cup_{i=0}^k F(\alpha_i)$ defines the same element in $\pi_i^d(O, \sigma)$. In particular, ϕ_r and ϕ_s are d-homotopic since $F(\alpha_0) = \{\phi_r\}$ and $F(\alpha_k) = \{\phi_s\}$; see Remark 4.11(2). Hence h is injective and the result follows. ■

Lemma 4.14 *Let $\gamma = (c(\gamma_i))_{i=0}^t$ and δ be two edge–loops in $\mathcal{A}_O$ based at $c(\sigma)$. Assume that δ is obtained by removing a vertex $c(\gamma_{i_0})$ from γ after an equivalence transformation of type (a) or (b). Then, for each $\emptyset$-loop $\phi \in F(\gamma)$ there exist $\psi \in F(\delta)$ and a d-homotopy $\phi \simeq_d \psi$ rel. σ.*

Proof. The hypothesis leads to one of the following cases

(1) $0 < i_0 < t$, the centroids $c(\gamma_{i_0-1}), c(\gamma_{i_0}), c(\gamma_{i_0+1})$ span a simplex in $\mathcal{A}_O$ and $\gamma_{i_0-1} < \gamma_{i_0} > \gamma_{i_0+1}$.
(2) $0 < i_0 < t$, the centroids $c(\gamma_{i_0-1}), c(\gamma_{i_0}), c(\gamma_{i_0+1})$ span a simplex in $\mathcal{A}_O$ and $\gamma_{i_0-1} > \gamma_{i_0} < \gamma_{i_0+1}$.
(3) $c(\gamma_{i_0})$ is a reducible vertex in γ.
(4) $\gamma_{i_0} = \gamma_{i_0+1}$, and hence the vertex $c(\gamma_{i_0+1})$ is reducible.

In cases (3) and (4) the edge–loop δ is obtained by dropping a reducible vertex from γ, so $F(\gamma) = F(\delta)$ by Remark 4.11(1) and the result follows from Proposition 4.12(2). Therefore we concentrate our efforts in proving the lemma for the case (1) since case (2) is settled in a similar way.

We start by considering the number $n(\gamma)$ of reducible vertices of γ in the set

$$V_\gamma = \{c(\gamma_j); 0 \leq j \leq i_0 - 2\} \cup \{c(\gamma_j); i_0 + 2 \leq j \leq t\}.$$

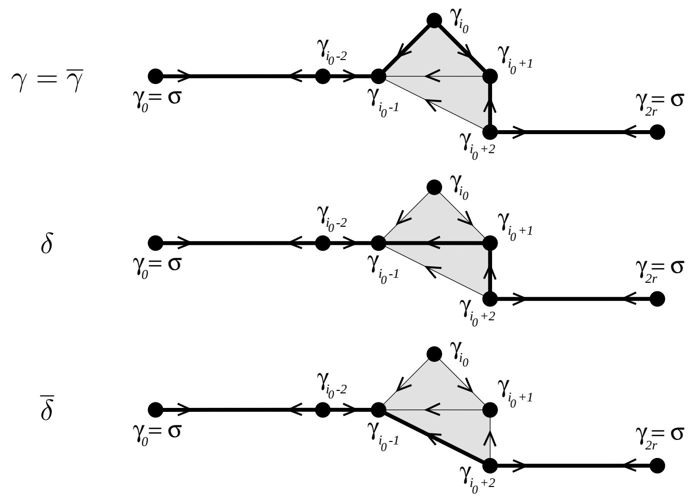


Fig. 5.

Since any reducible vertex in V_γ is also a reducible vertex of δ we can remove all of them from both γ and δ. This way we replace γ and δ by two new edge–loops γ' and δ' respectively such that $n(\gamma') = 0$. Moreover, by Remark 4.11(1), $F(\gamma) = F(\gamma')$ and $F(\delta) = F(\delta')$. Hence, by Proposition 4.12(2), there is no loss of generality in assuming $\gamma = \gamma'$ and $\delta = \delta'$.

Next we consider all possible face relations among the pairs of cells $(\gamma_{i_0-2}, \gamma_{i_0-1})$, $(\gamma_{i_0+1}, \gamma_{i_0+2})$ and $(\gamma_{i_0-1}, \gamma_{i_0+1})$. Notice that the two elements in each pair may be equal, and in case (2) it is also possible that $i_0 = 1$ or $i_0 = t - 1$. The proof requires in general the four steps below whatever are the face relations we consider. For illustrating the proof we give a detailed account of these steps for the case (1) and the face relations

$$\gamma_{i_0-2} > \gamma_{i_0-1} < \gamma_{i_0} > \gamma_{i_0+1} < \gamma_{i_0+2} \tag{I}$$

and

$$\gamma_{i_0-1} < \gamma_{i_0+1} \ . \tag{II}$$

Step A. *Describe the irreducible edge–loops* $\overline{\gamma}$ *and* $\overline{\delta}$.

The face relations (I) and (II) yield that γ has not reducible vertices, so that $\gamma = \overline{\gamma}$ is an edge–loop of even length $t = 2r$ by Remark 4.8. In addition, the irreducible edge–loop $\overline{\delta}$ associated to δ is

$$\overline{\delta} = (c(\gamma_0), \ldots, c(\gamma_{i_0-2}), c(\gamma_{i_0-1}), c(\gamma_{i_0+2}), \ldots, c(\gamma_{2r}))$$

since $c(\gamma_{i_0+1})$ is reducible in δ by the face relations (I) and (II); see Figure 5. Therefore, any digital representative of γ is an $\emptyset$-loop of length r, while digital representatives of δ have length $r - 1$.

Notice that under a different set of face relations γ and $\overline{\gamma}$ may be distinct. In any case, the length of $\overline{\gamma}$ is always greater than or equal to the length of $\overline{\delta}$, and the same happens for the digital representatives of γ and δ.

Step B. *Given a digital representative* $\phi \in F(\gamma)$ *of* γ*, derive a digital representative* $\psi \in F(\delta)$ *of* δ.

Given $\phi = \phi_r \in F(\gamma)$, it is not difficult to check from Step A that the $\emptyset$-loop $\psi = \psi_{r-1}$, given by $\psi_{r-1}(j/2) = \phi_r(j/2)$, for $0 \leq j \leq i_0 - 1$, and $\psi_{r-1}(j/2) = \phi_r(j/2+1)$, for $i_0 \leq j \leq 2r-2$, is a digital representative of the edge–loop δ.

Step C. *Obtain a new $\emptyset$-loop $\overline{\psi}$ d-homotopic to ψ and such that $\overline{\psi}$ and ϕ have the same length.*

By Definition 3.10, the $\emptyset$-loops $\psi = \psi_{r-1}$ and $\psi_{r-1} * \psi_1^\sigma$ are d-homotopic, where ψ_1^σ is the constant $\emptyset$-loop of length 1 at $\sigma = \psi_{r-1}(0) = \psi_{r-1}(r-1)$. Then, Proposition 3.11 yields the following d-homotopy

$$\psi_{r-1} * \psi_1^\sigma \simeq_d \psi_{\frac{i_0}{2}} * \psi_1^\tau * \psi_{r-1-\frac{i_0}{2}} = \overline{\psi}_r \ ,$$

where $\psi_{\frac{i_0}{2}}$ and $\psi_{r-1-\frac{i_0}{2}}$ are the $\emptyset$-walks in O given by $\psi_{\frac{i_0}{2}}(j/2) = \psi_{r-1}(j/2)$, for $0 \leq j \leq i_0$ and $\psi_{r-1-\frac{i_0}{2}}(j/2) = \psi_{r-1}((j+i_0)/2)$, for $0 \leq j \leq 2r - i_0 - 2$, respectively, and moreover ψ_1^τ is the constant $\emptyset$-loops of length 1 at $\tau = \psi_{r-1}((i_0)/2)$.

In general, different constant $\emptyset$-loops may be required for other sets of face relations. Notice also that this step could be not necessary in case the original digital representatives ϕ and ψ have the same length.

Step D. *Describe a d-homotopy between ϕ and $\overline{\psi}$. As a consequence, the lemma follows.*

From the face relations (I) and (II) it is not difficult to show that the d-map given by

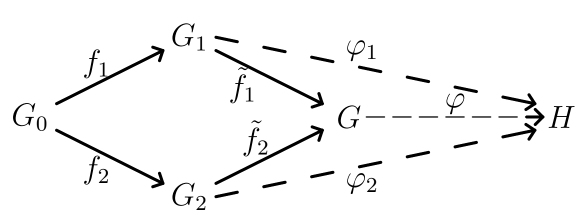
$$H(\frac{j}{2}, \frac{k}{2}) = \begin{cases} \phi_r(j/2) & \text{if } k = 0 \text{ and } 0 \leq j \leq 2r \\ \phi_r(j/2) & \text{if } k = 1 \text{ and } 0 \leq j \leq i_0 - 1 \text{ or } i_0 + 1 \leq j \leq 2r \\ \gamma_{i_0 - 1} & \text{if } k = 1 \text{ and } j = i_0 - 1 \\ \overline{\psi}_r(j/2) & \text{if } k = 2 \text{ and } 0 \leq j \leq 2r - 2 \end{cases}$$

is a d-homotopy between ϕ_r and the $\emptyset$-loop $\overline{\psi}_r \simeq_d \psi_{r-1}$.

Any other set of face relations leads to a possibly different d-homotopy, anyway of the same nature as H above. ■

5 A Digital Seifert–Van Kampen Theorem

The Seifert–Van Kampen Theorem is the basic tool for computing the fundamental group of a space which is built of pieces whose fundamental groups are known. The statement of the theorem involves the notion of push–out of groups, so we begin by explaining this bit of algebra. A group G is said to be the push–out of the solid arrow commutative diagram



if for any group H and homomorphisms φ_1, φ_2 with $\varphi_1 f_1 = \varphi_2 f_2$ there exists a unique homomorphism φ such that $\varphi \tilde{f_i} = \varphi_i$ $(i = 1, 2)$. Then, the Seifert–Van Kampen Theorem is the following

Theorem 5.1 (Th. 7.40 in [14]) *Let K be a simplicial complex having connected subcomplexes K_1 and K_2 such that $K = K_1 \cup K_2$ and $K_0 = K_1 \cap K_2$ is connected. If $v_0 \in K_0$ is a vertex then $\pi_1(K, v_0)$ is the push–out of the diagram*

$$\begin{array}{ccc} \pi_1(K_0, v_0) & \xrightarrow{i_{1*}} & \pi_1(K_1, v_0) \\ {\scriptstyle i_{2*}}\downarrow & & \downarrow{\scriptstyle j_{1*}} \\ \pi_1(K_2, v_0) & \xrightarrow{j_{2*}} & \pi_1(K, v_0) \end{array}$$

where i_{k*} and j_{k*} are the homomorphisms of groups induced by the obvious inclusions.

By using explicit presentations of the groups $\pi_1(K_i, v_0)$ $(i = 0, 1, 2)$ the Seifert–Van Kampen Theorem can be restated as follows. Suppose there are presentations $\pi_1(K_i, v_0) \cong (x_1^i, x_2^i, \dots; r_1^i, r_2^i, \dots)$, $i = 0, 1, 2$. Then, the fundamental group of K has the presentation

$$\begin{aligned} \pi_1(K, v_0) \ \cong \ & (x_1^1, x_2^1, \dots, x_1^2, x_2^2, \dots; \\ & r_1^1, r_2^1, \dots, r_1^2, r_2^2, \dots, i_{1*}(x_1^0) = i_{2*}(x_1^0), i_{1*}(x_2^0) = i_{2*}(x_2^0), \dots). \end{aligned}$$

In other words, one puts together the generators and relations from $\pi_1(K_1, v_0)$ and $\pi_1(K_2, v_0)$, plus one relation for each generator x_i^0 of $\pi_1(K_0, v_0)$ which says that its images in $\pi_1(K_1, v_0)$ and $\pi_1(K_2, v_0)$ are equal.

The digital analogue of the Seifert–Van Kampen Theorem is not always true as the following example shows.

Example 5.2 Let O_1, O_2 be the two digital objets in the digital space (R^2, g) shown in Figure 6. It is readily checked that both $\pi_1^d(O_1, \sigma)$ and $\pi_1^d(O_2, \sigma)$ are trivial groups, but $\pi_1^d(O_1 \cup O_2, \sigma) = \mathbb{Z}$ despite of O_1, O_2 and $O_1 \cap O_2$ are connected digital objects.

However we can easily derive a Digital Seifert–Van Kampen Theorem for certain objects in a quite large class of digital spaces. Namely, the locally strong digital spaces; that is, the digital spaces (K, f) for which the lighting function f satisfies $f(O, \alpha) = f(\mathrm{st}_n(\alpha; O), \alpha)$. We point out that all the (α, β)-connected digital spaces on $\mathbb{Z}^3$ defined within the graph-theoretical approach to Digital Topology, for $\alpha, \beta \in \{6, 18, 26\}$, are examples of locally strong digital spaces; see [1, Example 2].

Theorem 5.3 (Digital Seifert–Van Kampen Theorem) *Let (K, f) be a locally strong digital space, and let $O \subseteq cell_n(K)$ be a digital object in (K, f) such that $O = O_1 \cup O_2$, where O_1, O_2 and $O_1 \cap O_2$ are connected digital objects. Assume in addition that $\mathcal{A}_{O_1 \cap O_2} \subseteq \mathcal{A}_{O_1} \cap \mathcal{A}_{O_2}$ and $\mathcal{A}_{O_i} \subseteq \mathcal{A}_O$ $(i = 1, 2)$. Moreover assume that for each cell $\sigma \in O_1 - O_2$ any cell $\tau \in O$ which is adjacent to σ in O lies in O_1. Then, for $\sigma \in O_1 \cap O_2$, $\pi_1^d(O, \sigma)$ is the push–out of the diagram*

$$\begin{array}{ccc} \pi_1^d(O_1 \cap O_2, \sigma) & \longrightarrow & \pi_1^d(O_1, \sigma) \\ \downarrow & & \downarrow \\ \pi_1^d(O_2, \sigma) & \longrightarrow & \pi_1^d(O, \sigma) \end{array}$$

where the homomorphisms are induced by the obvious inclusions.

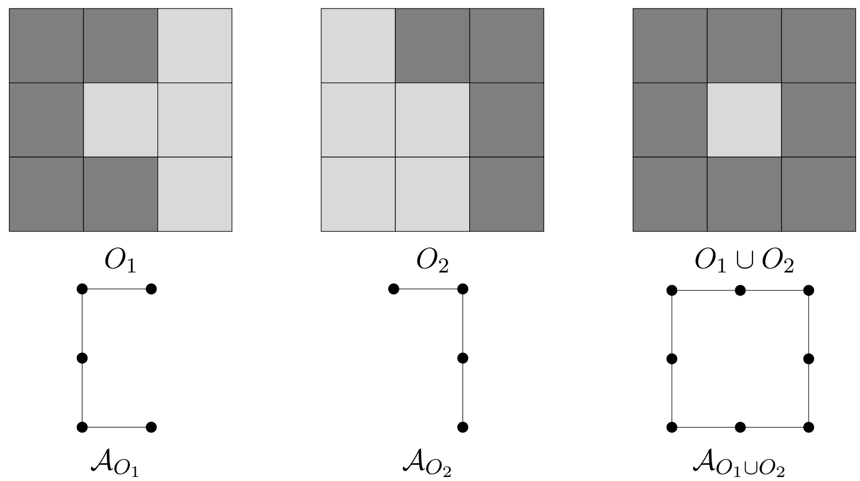


Fig. 6. A digital object for which the Digital Seifert–Van Kampen Theorem does not hold.

The proof of this theorem is immediate consequence of Theorem 4.6 and Theorem 5.1 if we have at hand the equalities $|\mathcal{A}_{O_1 \cap O_2}| = |\mathcal{A}_{O_1}| \cap |\mathcal{A}_{O_2}|$ and $|\mathcal{A}_O| = |\mathcal{A}_{O_1}| \cup |\mathcal{A}_{O_2}|$. We check these equalities in the two lemmas below.

Lemma 5.4 *If $f(O, \alpha) = 1$ then one of the following statements holds:*

1. $st_n(\alpha; O) = st_n(\alpha; O_1 \cap O_2) = st_n(\alpha; O_1) = st_n(\alpha; O_2)$*; or*
2. $st_n(\alpha; O) = st_n(\alpha; O_i)$ *and* $st_n(\alpha; O_j) = st_n(\alpha; O_1 \cap O_2)$, $\{i, j\} = \{1, 2\}$.

Proof. In case $\mathrm{st}_n(\alpha; O) = \mathrm{st}_n(\alpha; O_1 \cap O_2)$, we obtain (1) from the inclusions

$$\mathrm{st}_n(\alpha; O_1 \cap O_2) \subseteq \mathrm{st}_n(\alpha; O_i) \subseteq \mathrm{st}_n(\alpha; O), \ (i = 1, 2).$$

Otherwise, there exists $\sigma \in O - (O_1 \cap O_2)$ with $\alpha \leq \sigma$. Assume $\sigma \in O_1 - O_2$, then for all $\tau \in \mathrm{st}_n(\alpha; O)$ we have $\tau \in O_1$ by hypothesis and hence $\mathrm{st}_n(\alpha; O) = \mathrm{st}_n(\alpha; O_1)$. Moreover $\mathrm{st}_n(\alpha; O_2) \subseteq \mathrm{st}_n(\alpha; O) = \mathrm{st}_n(\alpha; O_1)$ yields $\mathrm{st}_n(\alpha; O_1 \cap O_2) = \mathrm{st}_n(\alpha; O_2) \subset \mathrm{st}_n(\alpha; O)$.

The case $\sigma \in O_2 - O_1$ is similar since then $\tau \in O_2$ ($\tau \notin O_2$ yields $\tau \in O_1 - O_2$ and hence $\sigma \in O_1$ by hypothesis). ■

Lemma 5.5 $\mathcal{A}_{O_1} \cap \mathcal{A}_{O_2} \subseteq \mathcal{A}_{O_1 \cap O_2}$ *and* $\mathcal{A}_O \subseteq \mathcal{A}_{O_1} \cup \mathcal{A}_{O_2}$*. And so the equalities follow by hypothesis.*

Proof. Let $c(\alpha) \in \mathcal{A}_{O_1} \cap \mathcal{A}_{O_2}$, then $f(O_i, \alpha) = 1$ for $i = 1, 2$ and hence $\mathrm{st}_n(\alpha; O_1 \cap O_2) = \mathrm{st}_n(\alpha; O_i)$ for some i by Lemma 5.4. Thus $f(O_1 \cap O_2, \alpha) = 1$ by the strong local condition of f, and so $c(\alpha) \in \mathcal{A}_{O_1 \cap O_2}$. Finally $\mathcal{A}_{O_1} \cap \mathcal{A}_{O_2} \subseteq \mathcal{A}_{O_1 \cap O_2}$ since $\mathcal{A}_{O_1 \cap O_2}$ is a full subcomplex by property (3) in Definition 2.2.

Now let $\gamma = \langle c(\gamma_0), \ldots, c(\gamma_k) \rangle \in \mathcal{A}_O$. Then $\mathrm{st}_n(\gamma_k; O) \subseteq \mathrm{st}_n(\gamma_{k-1}; O) \subseteq \cdots \subseteq \mathrm{st}_n(\gamma_0; O)$. By Lemma 5.4 and the strong local condition we easily obtain $\gamma \in \mathcal{A}_{O_i}$ whenever $\mathrm{st}_n(\gamma_0; O) = \mathrm{st}_n(\gamma_0; O_i)$ $(i = 1, 2)$. ■

Future work

The Digital Seifert–Van Kampen Theorem provides us with a theoretical tool that, under certain conditions, computes the digital fundamental group of an object. Nevertheless, the effective computation of the digital fundamental group requires an algorithm to compute a presentation of this group directly at the object's logical level. In a near future we will intend to develop such an algorithm for general digital spaces, as well as to compare the digital fundamental group in Def. 3.12 with those already introduced by Khalimsky [4] and Kong [5].

Acknowledgments

The authors are pleased to acknowledge the insightful review written by one of the referees of the manuscript.

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