Digital Homotopy with Obstacles*

R. Ayala[1], E. Domínguez[2], A.R. Francés[2], A. Quintero[1]

[1] Dpt. de Geometría y Topología. Facultad de Matemáticas. Universidad de Sevilla. Apto. 1160. E-41080 – Sevilla. Spain.
e-mail: quintero@cica.es
[2] Dpt. de Informática e Ingeniería de Sistemas. Facultad de Ciencias. Universidad de Zaragoza. E-50009 – Zaragoza. Spain.
e-mail: afrances@posta.unizar.es

Abstract As a sequel of [4], this paper is devoted to the computation of the digital fundamental group $\pi_1^d(O/S;\sigma)$ defined by loops in the digital object O for which the digital object S acts as an "obstacle". We prove that for arbitrary digital spaces the group $\pi_1^d(O/S;\sigma)$ maps onto the usual fundamental group of the difference of continuous analogues $|\mathcal{A}_{O\cup S}| - |\mathcal{A}_S|$. Moreover, we show that this epimorphism turns to be an isomorphism for a large class of digital spaces including most of the examples in digital topology.

Introduction

This paper deals with a notion of digital fundamental group for complements of objects in binary digital pictures. The interest of such a notion is found in the theory of 3d image thinning algorithms. After applying a 3d thinning algorithm, the "tunnels" in the input and output digital pictures must agree in number and position, and this can be correctly specified by saying that the algorithm preserves the digital fundamental groups of both the object displayed in the picture and its complement (see Criterion 2.3.2 in [7]).

The first notion of a digital fundamental group was given by Khalimsky [6] for a particular type of digital spaces, which are based on a topology on the set $\mathbb{Z}^n$ for every integer $n > 0$. This way, Khalimsky deals with sets of pixels regardless of considering them as digital objects themselves or as complements of other objects. However, Khalimsky's fundamental group is not suitable for other kinds of digital spaces often used in image processing, as the (α,β)–connected spaces defined on the grid $\mathbb{Z}^n$, where $(\alpha,\beta) \in \{(4,8),(8,4)\}$ if $n = 2$ and $(\alpha,\beta) \in \{(6,26),(26,6),(6,18),(18,6)\}$ if $n = 3$. Later on, Kong [7] gave a different notion of a digital fundamental group for a large class of graph–based digital spaces, including the (α,β)–connected spaces. As usual in the graph–theoretical approach to Digital Topology, Kong's digital fundamental group involves a different definition for objects and their complements in a given digital space. Namely, if $O \subseteq \mathbb{Z}^n$ is an object in the (α,β)-connected digital space, Kong defines the digital fundamental group of the complement of O in that space as the fundamental group of the object $\mathbb{Z}^n - O$ in the corresponding (β,α)-connected

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digital space. Nevertheless, this notion is restricted to 2d and 3d digital spaces and seems not generalize to give higher digital homotopy groups.

Recently, the authors [4] have introduced a fairly general notion of digital fundamental group that includes, as particular cases, the corresponding notions for both objects and their complements in a digital space. More precisely, in [4] we define the digital fundamental group $\pi_1^d(O/S,\sigma)$ of a set of pixels O regarding to an object S, which plays the role of an "obstacle" that the loops in O cannot cross; and then, for a digital object O in a digital space X, the particular cases $\pi_1^d(O,\sigma)=\pi_1^d(O/\emptyset,\sigma)$ and $\pi_1^d(X-O/O,\sigma)$ correspond to the digital fundamental groups of the object O and its complement in X, respectively. This approach presents, at least from a theoretical point of view, several advantages over the notions of Khalimsky and Kong. Firstly, it can be readily generalized to define higher digital homotopy groups (see [4]), as Khalimsky's notion; and, secondly, this group is available on a larger class of arbitrarily dimensional digital spaces than both Khalimsky's and Kong's digital fundamental groups.

The group $\pi_1^d(O/S,\sigma)$ was introduced within the framework of the multilevel architecture for Digital Topology given in [3]. That architecture provides a link between the discrete world of digital pictures and Euclidean spaces, where the "continuous perception" that an observer may take on a picture is represented *via* a polyhedron called its *continuous analogue*. In general, this link can be used to obtain new results in Digital Topology, by translating the corresponding continuous results (for instance, we use it in [2] to prove a general Digital Index Theorem for all (α,β)-connected spaces on $\mathbb{Z}^3$ and also for digital spaces defined on the grid $\mathbb{Z}^n$, for $n\geq 3$). Moreover, this link can be also used to check that a new digital notion is an accurate counterpart of the usual continuous one. So, we give in [4] an isomorphism from the digital fundamental group $\pi_1^d(O,\sigma)=\pi_1^d(O/\emptyset,\sigma)$ of an object O onto the classical fundamental group of its continuous analogue.

As a sequel, we extend in this paper the results in [4] to the more elaborate case of the digital fundamental group $\pi_1^d(X-O/O,\sigma)$ of an object's complement. More precisely, for an arbitrary obstacle $S\neq\emptyset$, we give in Section 3.1 an epimorphism from the digital fundamental group $\pi_1^d(O/S,\sigma)$ onto the fundamental group of the complement of the obstacle's continuous analogue. Although there is strong evidence that this epimorphism is not injective in general, we show in Section 3.2 that it is actually an isomorphism for a large class of digital spaces, including those most used in image processing. This supports also for complements of objects the suitability of our definition of the digital fundamental group π_1^d in [4].

For the convenience of the reader we review the basic notions of the multilevel architecture quoted above and the definition of the group $\pi_1^d(O/S,\sigma)$ in Sections 1 and 2, respectively.

1 Preliminaries

In this section we briefly summarize the basic notions of the multilevel architecture for digital topology developed in [3] as well as the notation that will be used through all the paper.

In that architecture, the spatial layout of pixels in a digital image is represented by a *device model*, which is a homogeneously n-dimensional locally finite polyhedral complex K. Each n-cell in K is representing a pixel, and so the digital object displayed in a digital image is a subset of the set $\mathrm{cell_n(K)}$ of n-cells in K; while the other lower dimensional cells in K are used to describe how the pixels could be linked to each other. A *digital space* is a pair (K, f), where K is a device model and f is *weak lighting function* defined on K. The function f is used to provide a continuous interpretation, called *continuous analogue*, for each digital object $O \subseteq \mathrm{cell_n(K)}$.

By a homegeneously n-dimensional locally finite polyhedral complex we mean a set K of polytopes, in some Euclidean space $\mathbb{R}^d$, provided with the natural ordering given by the relationship "to be face of", that in addition satisfies the four following properties:

1. If $\sigma \in K$ and τ is a face of σ then $\tau \in K$.
2. If $\sigma, \tau \in K$ then $\sigma \cap \tau$ is a face of both σ and τ.
3. For each point x in the underlying polyhedron $|K| = \cup\{\sigma; \sigma \in K\}$ of K, there exists a neigbourhood of x which intersects only a finite number of polytopes in K; in particular, each polytope of K is face of a finite number of other polytopes in K.
4. Each polytope $\sigma \in K$ is face of some n-dimensional polytope in K.

Given a device model K and two polytopes $\gamma, \sigma \in K$, we shall write $\gamma \leq \sigma$ if γ is a face of σ, and $\gamma < \sigma$ if in addition $\gamma \neq \sigma$. A centroid-map on K is a map $c : K \to |K|$ such that $c(\sigma)$ belongs to the interior of σ; that is, $c(\sigma) \in \mathring{\sigma} = \sigma - \partial\sigma$, where $\partial\sigma = \cup\{\gamma; \gamma < \sigma\}$ stands for the boundary of σ.

Remark 1.1 A homegeneously n-dimensional locally finite polyhedral complex K can be regarded as an *abstract cellular complex* whose cells are the polytopes in K. So, for simplicity, K will be called a polyhedral complex, and its polytopes will be simply referred to as cells in this paper. Moreover, the abstract complex K can be endowed with the structure of a locally finite topological $\mathrm{T_0}$ space with base $\mathcal{B} = \{\mathcal{U}_\alpha; \alpha \in \mathcal{K}\}$, where $U_\alpha = \{\beta \in K; \alpha \leq \beta\}$. Actually, this topological space $(K, \mathcal{B})$ is a quotient of the Euclidean polyhedron $|K|$ by the map $q : |K| \to K$ that assigns the cell σ to each point $x \in \mathring{\sigma}$.

Example 1.2 In this paper it will be essential the role played by the archetypical device model R^n, termed the *standard cubical decomposition* of the Euclidean n-space $\mathbb{R}^n$. The device model R^n is the complex determined by the collection of unit n-cubes in $\mathbb{R}^n$ whose edges are parallel to the coordinate axes and whose centers are in the set $\mathbb{Z}^n$. The centroid-map we will consider in R^n associates to each cube σ its barycenter $c(\sigma)$, which is a point in the set $\mathcal{Z}^n$. Here, $\mathcal{Z} = \frac{1}{2}\mathbb{Z}$ stands for the set of points $\{x \in \mathbb{R}; x = z/2, z \in \mathbb{Z}\}$. In particular, if $\dim\sigma = n$ then $c(\sigma) \in \mathbb{Z}^n$, where $\dim\sigma$ denotes the dimension of σ. So that, every digital object O in R^n can be identified with a subset of points in $\mathbb{Z}^n$. Henceforth we shall use this identification without further comment.

Before to proceed with the definition of weak lighting function, we need some notions, which are illustrated in Fig. 1 for an object O in the device model R^2.

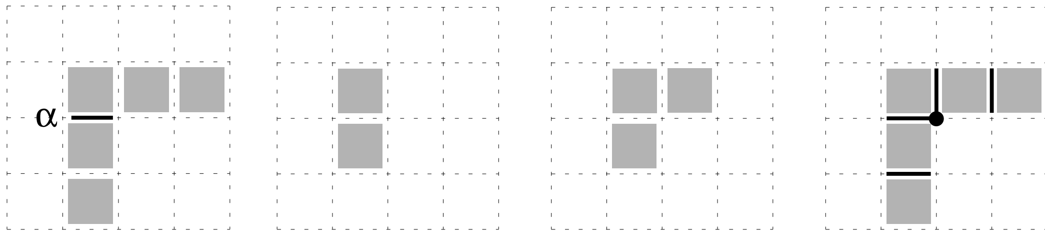

Fig. 1. The support of an object O and two types of digital neighbourhoods in O for a cell α. The cells in O together with the bold edges and dots are the elements in supp(O).

The first two notions formalize two types of "digital neighbourhoods" of a cell $\alpha \in K$ in a given digital object $O \subseteq \mathrm{cell_n(K)}$. Indeed, we call the *star of* α *in* O to the set $\mathrm{st_n}(\alpha; \mathrm{O}) = \{\sigma \in \mathrm{O}; \alpha \leq \sigma\}$ of n-cells (pixels) in O having α as a face. Similarly, the *extended star of* α *in* O is the set $\mathrm{st^*_n}(\alpha; \mathrm{O}) = \{\sigma \in \mathrm{O}; \alpha \cap \sigma \neq \emptyset\}$ of n-cells (pixels) in O intersecting α.

The third notion is the *support* of a digital object O which is defined as the set supp(O) of cells of K (not necessarily pixels) that are the intersection of n-cells (pixels) in O. Namely, $\alpha \in \mathrm{supp(O)}$ if and only if $\alpha = \cap\{\sigma; \sigma \in \mathrm{st_n}(\alpha; \mathrm{O})\}$. In particular, if α is a pixel in O then $\alpha \in \mathrm{supp(O)}$. Notice also that, among all the lower dimensional cells of K, only those in supp(O) are directly joining pixels in O.

To ease the writing, we shall use the following notation: $\mathrm{supp(K)} = \mathrm{supp(cell_n(K))}$, $\mathrm{st_n}(\alpha; \mathrm{K}) = \mathrm{st_n}(\alpha; \mathrm{cell_n(K)})$ and $\mathrm{st^*_n}(\alpha; \mathrm{K}) = \mathrm{st^*_n}(\alpha; \mathrm{cell_n(K)})$. Finally, we shall write $\mathcal{P}(A)$ for the family of all subsets of a given set A.

Definition 1.3 Given a device model K, a *weak lighting function* (w.l.f.) on K is a map $f : \mathcal{P}(\mathrm{cell_n(K)}) \times \mathrm{K} \to \{0, 1\}$ satisfying the following five axioms for all $O \in \mathcal{P}(\mathrm{cell_n(K)})$ and $\alpha \in K$:

(1) *object axiom*: if $\alpha \in O$ then $f(O, \alpha) = 1$;
(2) *support axiom*: if $\alpha \notin \mathrm{supp(O)}$ then $f(O, \alpha) = 0$;
(3) *weak monotone axiom*: $f(O, \alpha) \leq f(\mathrm{cell_n(K)}, \alpha)$;
(4) *weak local axiom*: $f(O, \alpha) = f(\mathrm{st^*_n}(\alpha; \mathrm{O}), \alpha)$; and,
(5) *complement connectivity axiom*: if $O' \subseteq O \subseteq \mathrm{cell_n(K)}$ and $\alpha \in K$ are such that $\mathrm{st_n}(\alpha; \mathrm{O}) = \mathrm{st_n}(\alpha; \mathrm{O'})$, $f(O', \alpha) = 0$ and $f(O, \alpha) = 1$, then: (a) the set of cells $\alpha(O'; O) = \{\omega < \alpha; f(O', \omega) = 0,\ f(O, \omega) = 1\}$ is not empty; (b) the set $\cup\{\mathring{\omega}; \omega \in \alpha(O'; O)\}$ is connected in $\partial\alpha$ (or, equivalently, the set $\alpha(O'; O)$ is connected in the topological space $(K, \mathcal{B})$ given in Remark 1.1); and, (c) if $O \subseteq \overline{O} \subseteq \mathrm{cell_n(K)}$, then $f(\overline{O}, \omega) = 1$ for every $\omega \in \alpha(O'; O)$.

If $f(O, \alpha) = 1$ we say that f *lights* the cell α for the object O.

A w.l.f. f is said to be *strongly local* if $f(O, \alpha) = f(\mathrm{st_n}(\alpha; \mathrm{O}), \alpha)$ for all $\alpha \in K$ and $O \subseteq \mathrm{cell_n(K)}$. Notice that this strong local axiom implies both axioms (4) and (5) above.

A weak lighting function f on a device model K can be regarded as a mapping that assigns a subset $\{\alpha \in K; f(O, \alpha) = 1\}$ of cells of K to each digital object $O \subseteq \mathrm{cell_n(K)}$.

In this sense, lighting functions are particular examples of "face membership rules" as introduced by Kovalevsky in [10]. Our contribution in this point are the axioms (1)–(5) in Def. 1.3. These axioms are intended for limiting the set of Kovalevsky's face membership rules to those that do not lead to topological properties which are contradictory with the natural perception of digital objects (see [5]). Indeed, the intuitive ideas underlying axioms (1)–(4) above are the following. Axiom (1) says that to display a digital object O on a computer screen all its pixels must be lighted. By axiom (2) only the lower dimensional cells in $\mathrm{supp(O)}$ can be lighted in order to connect immediately adjacent pixels of O. And axiom (3) states that a cell which is lighted for any digital object must be also lighted for the object $\mathrm{cell_n(K)}$ consisting of all the pixels in the digital space; that is, if a cell α connects some pixels in a given object, then α is connecting the same pixels in $\mathrm{cell_n(K)}$ too. Axiom (4) as well as the strong local axiom say that our perception of a digital object O is local, and so the lighting of a cell α depends on a "digital neighbourhood" of α in O. Finally, the rather complex axiom (5) is needed to avoid certain problems related to the connectivity of an object's complement (see [3, 4, 5] for details).

The following property, which will be used in Section 3.2, is immediate from Definition 1.3.

Lemma 1.4 *Any digital space (K, f) is strongly local at each vertex and at each top dimensional cell $\alpha \in K$; that is, $f(O, \alpha) = f(\mathrm{st_n}(\alpha; \mathrm{O}), \alpha)$ for any digital object $O \subseteq \mathrm{cell_n(K)}$ and $\dim \alpha = 0$ or $\dim \alpha = \dim K$.*

Given an arbitrary digital space (K, f), we shall derive from the lighting function f a "continuous interpretation" for any digital object $O \subseteq \mathrm{cell_n(K)}$, called its continuous analogue. For this we use a fixed, but arbitrary, centroid–map $c : K \to |\,K\,|$ on the device model K to introduce several other intermediate models for O as follows.

The *device level* of O is the subcomplex $K(O) = \{\alpha \in K; \alpha \leq \sigma \in O\}$ induced by O. Notice that $K(O)$ can be considered as a device model itself.

The *logical level* of O is an undirected graph, $\mathcal{L}^f_O$, whose vertices are the centroids of n-cells in O and two of them $c(\sigma)$, $c(\tau)$ are adjacent if there exists a common face $\alpha \leq \sigma \cap \tau$ such that $f(O, \alpha) = 1$.

The *conceptual level of* O is the directed graph $\mathcal{C}^f_O$ whose vertices are the centroids $c(\alpha)$ of all cells $\alpha \in K$ with $f(O, \alpha) = 1$, and its directed edges are $(c(\alpha), c(\beta))$ with $\alpha < \beta$.

The *simplicial analogue* of O is the order complex $\mathcal{A}^f_O$ associated to the directed graph $\mathcal{C}^f_O$. That is, $\langle c(\alpha_0), c(\alpha_1), \ldots, c(\alpha_m)\rangle$ is an m-simplex of $\mathcal{A}^f_O$ if $c(\alpha_0), c(\alpha_1), \ldots, c(\alpha_m)$ is a directed path in $\mathcal{C}^f_O$; or, equivalently, if $f(O, \alpha_i) = 1$, for $0 \leq i \leq m$, and $\alpha_0 < \alpha_1 < \cdots < \alpha_m$. That is, $\mathcal{A}^f_O$ is obtained by "filling in" all the triangles, tetrahedra, etc... in the conceptual level $\mathcal{C}^f_O$. Finally, the *continuous analogue* of O is the underlying polyhedron $|\,\mathcal{A}^f_O\,|$ of $\mathcal{A}^f_O$.

The following result is straightforwardly checked from the definitions.

Lemma 1.5 *For any digital object O in a digital space (K, f), the map*

$$f_O : \mathcal{P}(\mathrm{cell_n(K(O))}) \times \mathrm{K(O)} = \mathcal{P}(\mathrm{O}) \times \mathrm{K(O)} \to \{0, 1\}$$

given by $f_O(O',\alpha) = f(O,\alpha)f(O',\alpha)$, *for* $O' \subseteq O$ *and* $\alpha \in K(O)$, *is a w.l.f. on the device model* $K(O)$. *So, we call the pair* $(K(O), f_O)$ *the* digital subspace of (K,f) induced by O.

Remark 1.6 (1) Notice that the simplicial analogue $\mathcal{A}^f_O$ of any digital object $O \subseteq \mathrm{cell_n(K)}$ is always a full subcomplex of the first derived subdivision $K^{(1)}$ of K induced by the chosen centroid–map c. Moreover, axiom (3) in Def. 1.3 yields that $\mathcal{A}^f_O \subseteq \mathcal{A}^f_{\mathrm{cell_n(K)}}$, and so $\mathcal{A}^f_O$ is also a full subcomplex of $\mathcal{A}^f_{\mathrm{cell_n(K)}}$.

(2) Let $(K(O), f_O)$ be the digital subspace induced by a digital object O in a digital space (K,f). If $O' \subseteq O$, one easily checks the equality $\mathcal{A}^{f_O}_{O'} = \mathcal{A}^f_O \cap \mathcal{A}^f_{O'}$, since all these are full subcomplexes of $K^{(1)}$. In particular $\mathcal{A}^{f_O}_O = \mathcal{A}^f_O$; that is, the continuous analogue of an object does not change when it is considered as the ambient digital space.

(3) Given a locally finite topological T_0 space X, Kong and Khalimsky construct in [9] a *polyhedral analogue* $|K(X)|$ for X. It can be easily checked that, for any digital object O in a digital space (K,f), our continuous analogue $|\mathcal{A}_O|$ coincides with the polyhedral analogue $|K(X_O)|$ of the set $X_O = \{\alpha \in K; f(O,\alpha) = 1\}$ of cells which are lighted for O endowed with the relative topology of the abstract complex K in Remark 1.1.

For the sake of simplicity, we will usually drop "f" from the notation of the levels of an object. Moreover, for the whole object $\mathrm{cell_n(K)}$ we will simply write $\mathcal{L}_K$, $\mathcal{C}_K$ and $\mathcal{A}_K$ for its levels.

Example 1.7 (1) Every device model $K \neq \emptyset$ admits the weak lighting functions $f_{\max}$ and g given respectively by:

(a) $f_{\max}(O,\alpha) = 1$ if and only if $\alpha \in \mathrm{supp(O)}$
(b) $g(O,\alpha) = 1$ if and only if $\alpha \in \mathrm{supp(O)}$ and $\mathrm{st_n}(\alpha;\mathrm{K}) \subseteq \mathrm{O}$

In Fig. 2 are shown two objects, O and $\mathrm{cell_2(R^2)}$, in the device model R^2, and their levels for these lighting functions. More precisely, Figs. 2(a) and 2(b) show the 2-cells (grey squares) of the object O and the low-dimensional cells (bold edges and vertices) that the w.l.f.'s $f_{\max}$ and g light, respectively, for O. As these sets, $\{\alpha \in R^2; f_{\max}(O,\alpha) = 1\}$ and $\{\alpha \in R^2; g(O,\alpha) = 1\}$, do not agree, all the levels of O in the digital spaces $(R^2, f_{\max})$ and (R^2, g) are distinct, in particular $|\mathcal{A}^{f_{\max}}_O| \neq |\mathcal{A}^g_O|$. On the other hand,

$$\{\alpha \in R^2; f_{\max}(\mathrm{cell_2(R^2)},\alpha) = 1\} = \{\alpha \in \mathrm{R^2}; \mathrm{g}(\mathrm{cell_2(R^2)},\alpha) = 1\}$$

(see Figure 2(c)), and so all the levels of the object $\mathrm{cell_2(R^2)}$ are the same in these two digital spaces.

(2) Both the w.l.f.'s $f_{\max}$ and g given above satisfy the strong local axiom in Definition 1.3. Next we give an example of a non strongly local digital space (R^n, h). For any integer $n > 0$, the w.l.f. h is defined on the device model R^n by $h(O,\alpha) = 1$ if and only if: (a) $\dim\alpha = n$ and $\alpha \in O$; (b) $\dim\alpha \leq n-2$ and $\mathrm{st_n}(\alpha;\mathrm{R^n}) \subseteq \mathrm{O}$; and, (c) $\dim\alpha = n-1$, $\alpha \in \mathrm{supp(O)}$, and either $\mathrm{st^*_n}(\alpha;\mathrm{R^n}) \subseteq \mathrm{O}$ or there exist $\sigma,\tau \in \mathrm{st^*_n}(\alpha;\mathrm{R^n}) - \mathrm{O}$ such that $\sigma \cap \tau = \emptyset$.

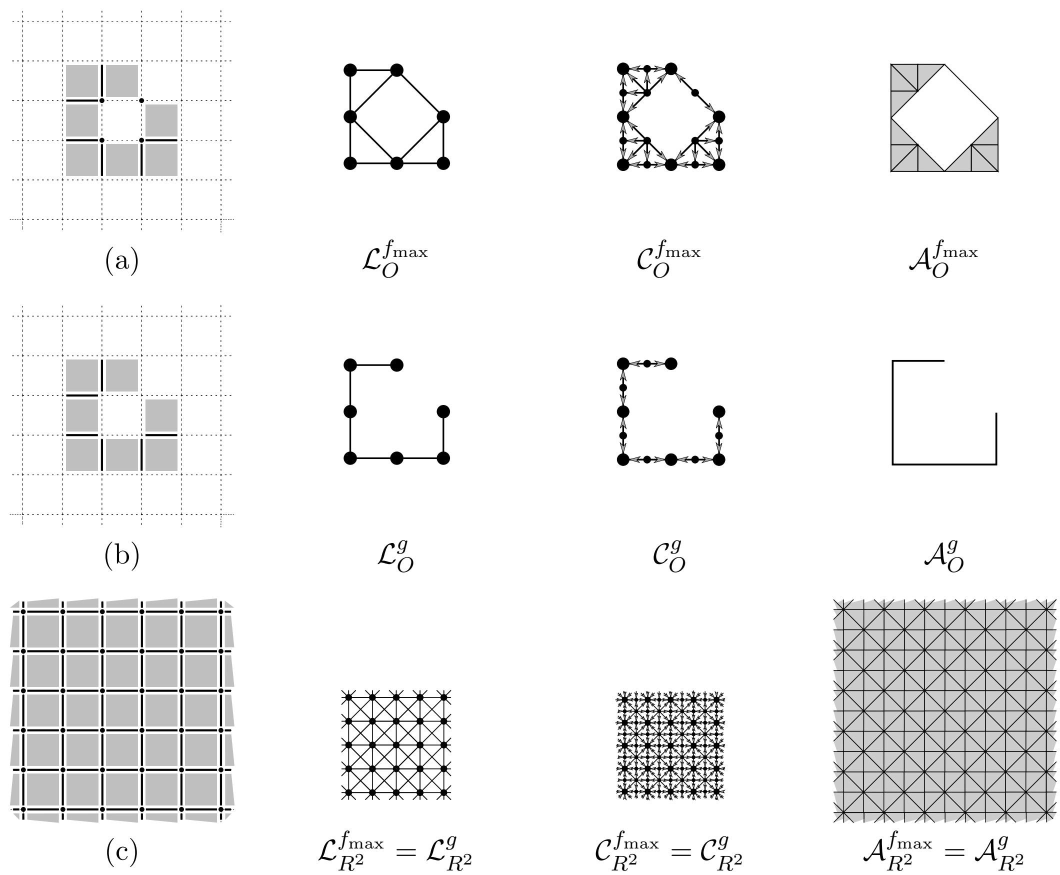

Fig. 2. Levels of the objects O and $\mathrm{cell}_2(\mathrm{R}^2)$ for the w.l.f.'s $f_{\max}$ and g in Example 1.7(1).

Actually, the family of digital spaces (R^n, g) with $n \geq 1$, for g the w.l.f. in Example 1.7, and more precisely a particular class of digital subspaces (called *windows*) of these spaces, are the key that allows us to introduce in next Section a notion of digital fundamental group. At this point, it is worth to point out that g induces in R^n the $(2n, 3^n - 1)$–connectivity (see [1, Def. 11]); that is, the generalization to arbitrary dimension of the $(4, 8)$–connectivity on $\mathbb{Z}^2$. On the other hand, $f_{\max}$ induces in R^n the $(3^n - 1, 2n)$-connectivity (see Fig. 2).

2 A digital fundamental group

The fundamental group of a topological space X, $\pi_1(X, x_0)$, is usually defined to be the set of homotopy classes of loops based at fixed point x_0 (i.e., maps $\xi : I = [0, 1] \to X$ with $\xi(0) = \xi(1) = x_0$), where an homotopy between two loops ξ_1, ξ_2 is a continuous map $H : I \times I \to X$ such that $H(x, 0) = \xi_1(x)$, $H(x, 1) = \xi_2(x)$ and $H(0, t) = H(1, t) = x_0$.

In this section we collect the definitions and basic facts involved in the notion of digital fundamental group as introduced in [4]. These definitions provide suitable digital analogues for the notions of continuous loop and continuous homotopy, which are in fact particular examples of digital maps (see Def. 2.2). To define them we need the following technical notion. We refer to [4] for more details.

Definition 2.1 Let $S \subseteq \mathrm{cell_n(K)}$ be a digital object in a digital space (K, f). The *light body of* (K, f) *shaded with* S is the set of cells $\mathrm{Lb(K/S)}$ not lighted for the object S but lighted for $\mathrm{cell_n(K)}$; that is,

$$\begin{aligned}\mathrm{Lb(K/S)} &= \{\alpha \in K; f(\mathrm{cell_n(K)}, \alpha) = 1, \mathrm{f(S}, \alpha) = 0\} \\ &= \{\alpha \in K; c(\alpha) \in |\mathcal{A}_K| - |\mathcal{A}_S|\}.\end{aligned}$$

Actually, the notion of light body plays a key role in defining the digital fundamental group of an object O with respect to an arbitrary obstacle S (see Def. 2.10). Indeed, this group will be defined as the set of homotopy classes of digital loops in $\mathrm{Lb(K(O \cup S)/S)}$; that is, the light body, shaded by the obstacle S, of the subspace $(K(O \cup S), f_{O \cup S})$ that the set of pixels $O \cup S$ induces in the digital space (K, f).

Definition 2.2 Let $(K_1, f_1), (K_2, f_2)$ be two digital spaces, with $\dim K_i = n_i$ $(i = 1, 2)$, and let $S_1 \subset \mathrm{cell_{n_1}(K_1)}$ and $S_2 \subset \mathrm{cell_{n_2}(K_2)}$ be two digital objects. A *digital* (S_1, S_2)*-map* (or, simply, a *d-map*) $\Phi_{S_1,S_2} : (K_1, f_1) \to (K_2, f_2)$ from (K_1, f_1) into (K_2, f_2) is a map $\phi : \mathrm{Lb(K_1/S_1)} \to \mathrm{Lb(K_2/S_2)}$ satisfying the two following properties:

1. $\phi(\mathrm{cell_{n_1}(K_1) - S_1}) \subseteq \mathrm{cell_{n_2}(K_2) - S_2}$; and,
2. for $\alpha, \beta \in \mathrm{Lb(K_1/S_1)}$ with $\alpha < \beta$ then $\phi(\alpha) \leq \phi(\beta)$.

That is, ϕ carries top dimensional cells in $\mathrm{Lb(K_1/S_1)}$ to top dimensional cells in $\mathrm{Lb(K_2/S_2)}$ and preserves the face relations (although ϕ needs not be dimension preserving).

Notice that any d-map is a continuous map if we consider $\mathrm{Lb(K_1/S_1)}$ and $\mathrm{Lb(K_2/S_2)}$ as subspaces of the abstract complexes K_1 and K_2 topologized as in Remark 1.1. The following result also holds.

Proposition 2.3 *Any d-map* $\Phi_{S_1,S_2} : (K_1, f_1) \to (K_2, f_2)$ *induces a simplicial map* $\mathcal{A}(\Phi_{S_1,S_2}) : \mathcal{A}_{K_1} \backslash \mathcal{A}_{S_1} \to \mathcal{A}_{K_2} \backslash \mathcal{A}_{S_2}$*, which is defined on the vertices* $c_1(\alpha)$ *of* $\mathcal{A}_{K_1} \backslash \mathcal{A}_{S_1}$ *by* $\mathcal{A}(\Phi_{S_1,S_2})(c_1(\alpha)) = c_2(\Phi_{S_1,S_2}(\alpha))$*. Here* c_i *is a centroid-map on the device model* K_i*, for* $i = 1, 2$.

In the previous proposition $L_1 \backslash L_2 = \{\alpha \in L_1; \alpha \cap |L_2| = \emptyset\}$ denotes the simplicial complement of the subcomplex $L_2 \subseteq L_1$.

In order to define digital loops and digital homotopies as particular types of digital maps, next definition provides us with a particular class of digital spaces, called *windows*, that will play the same role as the unit interval, I, and the unit square, $I \times I$, in continuous homotopy. For this, we will use the following notation. Given two points $x = (x_1, \ldots, x_m), y = (y_1, \ldots, y_m) \in \mathbb{R}^m$, we write $x \preceq y$ if $x_i \leq y_i$ for all $1 \leq i \leq m$, while $x + y$ will stand for the usual vector addition $x + y = (x_1 + y_1, \ldots, x_m + y_m) \in \mathbb{R}^m$.

Definition 2.4 Given two points $r, x \in \mathbb{Z}^m$, with $r_i \geq 0$ for $1 \leq i \leq m$, we call a *window of size* r (or r*-window*) *of* R^m *based at* x to the digital subspace V_r^x of (R^m, g) induced by the digital object $O_r^x = \{\sigma \in \mathrm{cell_m(R^m)}; \mathrm{x} \preceq \mathrm{c}(\sigma) \preceq \mathrm{x + r}\}$, where (R^m, g) is the digital space defined in Example 1.7. For the sake of simplicity, we shall write V_r to denote the r-window of R^m based at the point $x = (0, \ldots, 0) \in \mathbb{Z}^m$.

Remark 2.5 To ease the writing, given an r-window V_r of R^m, we will identify each cell $\alpha \in \mathrm{Lb(V_r/\emptyset)}$ with its centroid $c(\alpha) \in \mathcal{Z}^m$ (see Example 1.2). In particular, if V_r is an r-window of R^1, then $\mathrm{Lb(V_r/\emptyset)} = \{\sigma_0, \sigma_1, \ldots, \sigma_{2r-1}, \sigma_{2r}\}$ consists of $2r+1$ cells (points and segments) which will be identified with the numbers $c(\sigma_i) = i/2$, for $0 \leq i \leq 2r$. And, similarly, for a window $V_{(r_1,r_2)}$ of R^2, we identify each cell $\alpha \in \mathrm{Lb(V_{(r_1,r_2)}/\emptyset)}$ with a pair $c(\alpha) = (i/2, j/2)$, where $0 \leq i \leq 2r_1$ and $0 \leq j \leq 2r_2$.

We are now ready to define digital loops and digital homotopies as follows.

Definition 2.6 Let $S, O \subseteq \mathrm{cell_n(K)}$ be two disjoint digital objects in a digital space (K, f), and σ, τ two n-cells in O. A *S-walk in O of length $r \in \mathbb{Z}$ from σ to τ* is a digital $(\emptyset, S)$-map $\phi_r : \mathrm{Lb(V_r/\emptyset)} \to \mathrm{Lb(K(O \cup S)/S)}$ such that $\phi_r(0) = \sigma$ and $\phi_r(r) = \tau$. A *S-loop in O based at σ* is a S-walk ϕ_r such that $\phi_r(0) = \phi_r(r) = \sigma$.

The *juxtaposition* of two given S-walks ϕ_r, ϕ_s in O, with $\phi_r(r) = \phi_s(0)$, is the S-walk $\phi_r * \phi_s : \mathrm{Lb(V_{r+s}/\emptyset)} \to \mathrm{Lb(K(O \cup S)/S)}$ of length $r+s$ given by

$$\phi_r * \phi_s(i/2) = \begin{cases} \phi_r(i/2) & \text{if } 0 \leq i \leq 2r \\ \phi_s(i/2 - r) & \text{if } 2r \leq i \leq 2(r+s) \end{cases}$$

Notice that a S–loop ϕ_r in O is actually a sequence $(\phi_r(i))_{i=0}^r$ of adjacent pixels in O such that each pair $\phi_r(i-1), \phi_r(i)$ of successive pixels have a common face $\phi_r(i-\frac{1}{2})$ which is not lighted for the object S. In this sense ϕ_r does not cross the obstacle S. Similarly, a digital homotopy, as defined below, transforms a S-loop ϕ^1 to ϕ^2 using adjacent pixels but, in the same way, it is not allowed to cross the obstacle S.

Definition 2.7 Let ϕ_r^1, ϕ_r^2 two S-walks in O of the same length $r \in \mathbb{Z}$ from σ to τ. We say that ϕ_r^1, ϕ_r^2 are *digitally homotopic* (or, simply, *d-homotopic*) *relative* $\{\sigma, \tau\}$, and we write $\phi_r^1 \simeq_d \phi_r^2$ rel. $\{\sigma, \tau\}$, if there exists an (r, s)-window $V_{(r,s)}$ in R^2 and a $(\emptyset, S)$-map $H : \mathrm{Lb(V_{(r,s)}/\emptyset)} \to \mathrm{Lb(K(O \cup S)/S)}$, called a *d-homotopy*, such that $H(i/2, 0) = \phi_r^1(i/2)$ and $H(i/2, s) = \phi_r^2(i/2)$, for $0 \leq i \leq 2r$, and moreover $H(0, j/2) = \sigma$ and $H(r, j/2) = \tau$, for $0 \leq j \leq 2s$. Here we use the identification $H(a_1, a_2) = H(\alpha)$, where $c(\alpha) = (a_1, a_2) \in \mathcal{Z}^2$ is the centroid of a cell $\alpha \in \mathrm{Lb(V_{(r,s)}/\emptyset)}$; see Remark 2.5.

Definition 2.8 Let ϕ_r, ϕ_s two S-walks in O from σ to τ of lengths r and s respectively. We say that ϕ_r is d-homotopic to ϕ_s relative $\{\sigma, \tau\}$, $\phi_r \simeq_d \phi_s$ rel. $\{\sigma, \tau\}$, if there exist constant S-loops $\phi_{r'}^\tau$ and $\phi_{s'}^\tau$ such that $r + r' = s + s'$ and $\phi_r * \phi_{r'}^\tau \simeq_d \phi_s * \phi_{s'}^\tau$ rel. $\{\sigma, \tau\}$.

The following result, whose proof can be found in [4], will be needed in the sequel.

Proposition 2.9 *Let ϕ_r be a S-walk in O from σ to τ, and ϕ_s^σ, ϕ_s^τ two constant S-loops of the same length $s \in \mathbb{Z}$. Then, $\phi_s^\sigma * \phi_r \simeq_d \phi_r * \phi_s^\tau$ rel. $\{\sigma, \tau\}$.*

Notice that d-homotopies induce an equivalence relation in the set of S–walks in O from σ to τ. Moreover, from Proposition 2.9 it is not difficult to check that the juxtaposition is compatible with d-homotopies between S–walks. Thus, the juxtaposition of S-loops naturally induces a product operation that endows the set of classes of S-loops in O based at an n-cell $\sigma \in O$ with a group structure, for which the trivial element is the class of constant S-loops at σ, and the inverse of the class $[\phi_r]$ is represented by the S-loop ϕ_r^{-1} obtained by traversing ϕ_r backwards; that is, $\phi_r^{-1}(i/2) = \phi_r(r - i/2)$ for all $0 \leq i \leq 2r$. So, we next introduce the notion of digital fundamental group as follows.

Definition 2.10 Let S, O be two disjoint digital objects in a digital space (K, f), and σ an n-cell in O. The *digital fundamental group of O at σ with obstacle at S* is the set $\pi_1^d(O/S, \sigma)$ of d-homotopy classes of S-loops in O based at σ with the product operation $[\phi_r] \cdot [\psi_s] = [\phi_r * \psi_s]$. In case $S = \emptyset$, we will simply call $\pi_1^d(O/\emptyset, \sigma) = \pi_1^d(O, \sigma)$ the *digital fundamental group of O at σ*.

Remark 2.11 Definition 2.10 provides an entire family of digital fundamental groups for a given digital object O when the object S is allowed to range over the family of all subsets of $\mathrm{cell_n(K) - O}$. Particularly interesting are the groups $\pi_1^d(O/\emptyset, \sigma) = \pi_1^d(O, \sigma)$ and $\pi_1^d(O/(\mathrm{cell_n(K)} - \mathrm{O}), \sigma)$ that, respectively, represents the digital fundamental group of the object O itself and the digital fundamental group of O as the complement of the object $\mathrm{cell_n(K) - O}$.

3 The relationship with the continuous fundamental group

In [4] we show that the digital fundamental group $\pi_1^d(O, \sigma)$ of a digital object coincides with the classical fundamental group of its continuous analogue $|\,\mathcal{A}_O\,|$. In this Section we tackle the problem of computing the digital fundamental group $\pi_1^d(O/S, \sigma)$ of O with a disjoint object S acting as an "obstacle" for the loops in O. The Section is divided into two parts, in 3.1 we deal with the general case and we produce a epimorphism

$$h : \pi_1^d(O/S, \sigma) \to \pi_1(|\,\mathcal{A}_{O \cup S}\,| - |\,\mathcal{A}_S\,|, c(\sigma))$$

onto the classical fundamental group of the complement of the obstacle's continuous analogue. The second part 3.2 provides us with a large class of digital spaces for which the above homomorphism yields an isomorphism.

We recall that, for a triangulated polyhedron $|\,L\,|$, there is an alternative definition of the fundamental group $\pi_1(|\,L\,|, x_0)$ that will be more convenient for our purposes. So we next explain it briefly. For this, we call an *edge–walk* in L from a vertex v_0 to a vertex v_n to a sequence α of vertices $(v_0, v_1, \ldots, v_n)$, such that for each $k = 1, 2, \ldots, n$ the vertices v_{k-1}, v_k span a simplex in L (possibly $v_{k-1} = v_k$). If $v_0 = v_n$, α is called an *edge–loop based at* v_0.

Given another edge–walk $\beta = (v_j)_{j=n}^{m+n}$ whose first vertex is the same as the last vertex of α, the *juxtaposition* $\alpha * \beta = (v_i)_{i=0}^{m+n}$ is defined in the obvious way. The *inverse* of α is $\alpha^{-1} = (v_n, v_{n-1}, \ldots, v_0)$.

Two edge–walks α and β are said to be *equivalent* if one can be obtained from the other by a finite sequence of operations of the form:

(a) if $v_{k-1} = v_k$, replace $\ldots, v_{k-1}, v_k, \ldots$ by $\ldots, v_k, \ldots$, or conversely replace $\ldots, v_k, \ldots$ by $\ldots, v_{k-1}, v_k, \ldots$; or

(b) if v_{k-1}, v_k, v_{k+1} span a simplex of L (not necessarily 2-dimensional), replace $\dots, v_{k-1}, v_k, v_{k+1}, \dots$ by $\dots, v_{k-1}, v_{k+1}, \dots$, or conversely.

This clearly sets up an equivalence relation between edge–walks, and the set $\pi_1(L, v_0)$ of equivalence classes $[\alpha]$ of edge–loops α in L, based at a vertex v_0, is given the structure of group by the operation $[\alpha] \cdot [\beta] = [\alpha * \beta]$. This group is called the *edge–group* of L.

Each edge–walk α in L defines in an obvious way a continuous path $\theta(\alpha)$ in the underlying polyhedron $|L|$; and so, we will identify henceforth the edge–walk α with the continuous path $\theta(\alpha)$. Actually this correspondence yields an isomorphism $\pi_1(|L|, v_0) \cong \pi_1(L, v_0)$. More precisely,

Theorem 3.1 ([12]; 3.3.9) *There exists an isomorphism $\Theta : \pi_1(L, v_0) \to \pi_1(|L|, v_0)$ which carries the class $[\alpha]$ to the class $[\theta(\alpha)]$.*

Corollary 3.2 *Let O, S be two disjoint digital objects in a digital space (K, f). Then $\pi_1(\mathcal{A}_{O\cup S} \setminus \mathcal{A}_S, c(\sigma)) \cong \pi_1(|\mathcal{A}_{O\cup S}| - |\mathcal{A}_S|, c(\sigma))$ for any $\sigma \in O$.*

Proof. By Remark 1.6(1) we know that both $\mathcal{A}_{O\cup S}$ and $\mathcal{A}_S$ are full subcomplexes of $\mathcal{A}_K$. Then Lemma 72.2 in [13] yields that $|\mathcal{A}_{O\cup S} \setminus \mathcal{A}_S| = |\mathcal{A}_{O\cup S} \setminus (\mathcal{A}_{O\cup S} \cap \mathcal{A}_S)|$ is a strong deformation retract of $|\mathcal{A}_{O\cup S}| - |\mathcal{A}_S| = |\mathcal{A}_{O\cup S}| - |\mathcal{A}_{O\cup S} \cap \mathcal{A}_S|$ and the result follows by Theorem 3.1. ■

Let (K, f) be an arbitrary digital space. Given two disjoint digital objects $O, S \subseteq \mathrm{cell_n(K)}$ and any n-cell $\sigma \in O$ we next define a natural homomorphism,

$$h : \pi_1^d(O/S, \sigma) \to \pi_1(\mathcal{A}_{O\cup S} \setminus \mathcal{A}_S, c(\sigma)) \ , \qquad (1)$$

from the digital fundamental group of O based at σ and with obstacle at the object S into the edge–group of the simplicial complex $\mathcal{A}_{O\cup S} \setminus \mathcal{A}_S$ based at the centroid $c(\sigma)$, as follows. Let ϕ_r be any S-loop in O. Then, we just observe that the sequence $c(\phi_r) = (c(\phi_r(i/2)))_{i=0}^{2r}$ defines an edge–loop in $\mathcal{A}_{O\cup S} \setminus \mathcal{A}_S$. So that, we simply set $h([\phi_r]) = [c(\phi_r)]$. Notice that h is the generalization to the case $S \neq \emptyset$ of the homomorphism used in [4] to show the isomorphism $\pi_1^d(O, \sigma) = \pi_1^d(O/\emptyset, \sigma) \cong \pi_1(\mathcal{A}_O, c(\sigma))$.

Remark 3.3 The following properties are easily checked

(1) If ϕ_r and ϕ_s are two S-loops at σ, then $c(\phi_r) * c(\phi_s) = c(\phi_r * \phi_s)$.
(2) If ϕ_r is a constant S-loop at σ then $c(\phi_r)$ is a constant edge–loop at $c(\sigma)$.

Lemma 3.4 *The correspondence h, given in (1) above, is well defined. Moreover h is a group homomorphism.*

Proof. Assume $\phi_r \simeq_d \phi_s$ rel. σ are two equivalent S-loops in O. Then there exist two constant S-loops $\phi_{r'}^\sigma$ and $\phi_{s'}^\sigma$ such that $r + r' = s + s'$ and a d-homotopy $H : \phi_r * \phi_{r'}^\sigma \simeq_d \phi_s * \phi_{s'}^\sigma$ rel. σ. That is, H is an $(\emptyset, S)$-map $H : (V_{(r+r',t)}, g) \to (K(O\cup S), f_{O\cup S})$, where $V_{(r+r',t)}$ is a window in Def. 2.4 and $(K(O\cup S), f_{O\cup S})$ is the digital subspace of (K, f) induced by $O\cup S$; see Lemma 1.5. Therefore, by Proposition 2.3 and Remark 1.6(2) we get a simplicial map $\mathcal{A}(H) : \mathcal{A}_{V_{(r+r',t)}} \to \mathcal{A}_{O\cup S}\setminus\mathcal{A}_S$. Notice that from the definition of the w.l.f. g in Example 1.7(1) it readily follows that $\mathcal{A}_{V_{(r+r',t)}}$ is simplicially isomorphic to a triangulation of the unit square $I\times I$, and hence $\mathcal{A}(H)$ yields a homotopy between the edge–loops $c(\phi_r * \phi^\sigma_{r'})$ and $c(\phi_s * \phi^\sigma_{s'})$. Finally, the properties in Remark 3.3 and suitable equivalence transformations of type (a) yield that $c(\phi_r * \phi^\sigma_{r'}) = c(\phi_r) * c(\phi^\sigma_{r'})$ is equivalent to $c(\phi_r)$, and similarly $c(\phi_s * \phi^\sigma_{s'})$ is also equivalent to $c(\phi_s)$. Notice also that h is an homomorphism of groups as an immediate consequence of property (1) in Remark 3.3. ■

3.1 The general case: epimorphism onto the classical fundamental group.

This Section is aimed to show that, for arbitrary disjoint digital objects $O, S \subseteq \mathrm{cell_n(K)}$ in a digital space (K, f), the homomorphism of groups

$$h : \pi^d_1(O/S, \sigma) \to \pi_1(\mathcal{A}_{O\cup S} \setminus \mathcal{A}_S, c(\sigma))$$

is always an epimorphism. For $S = \emptyset$, we proved in [4] that the homomorphism h above is actually an isomorphism of groups $\pi^d_1(O/\emptyset, \sigma) \cong \pi_1(\mathcal{A}_O, c(\sigma))$. For this we associate to each edge–loop Γ in $\mathcal{A}_O$ a family of *digital representatives* $F(\Gamma)$ such that for each digital $\emptyset$–loop $\phi_r \in F(\Gamma)$ the edge–loop $c(\phi_r)$ is equivalent to Γ. In this section we show that this procedure can be generalized to get a non–empty family $\mathcal{D}(\Gamma)$ of S-loops in O of digital representatives for any edge–loop Γ in $\mathcal{A}_{O\cup S}\setminus\mathcal{A}_S$. This immediately yields that the homomorphism h is onto with full generality. However, the construction of the family $\mathcal{D}(\Gamma)$ suggests that the homomorphism h need not to be injective in general. In any case, Section 3.2 provides a large class of digital spaces, including those often used in image processing, for which h is in fact an isomorphism.

In order to define the family $\mathcal{D}(\Gamma)$ we start generalizing the notion of irreducible edge–loop introduced in [4].

Definition 3.5 A vertex $c(\gamma_i)$, of and edge–walk $\Gamma = (c(\gamma_i))^t_{i=0}$ in $\mathcal{A}_{O\cup S}\setminus\mathcal{A}_S$, is said to be *reducible in* Γ if $i > 0$ and one of the following properties holds

(1) $\gamma_{i-1} = \gamma_i$
(2) there exits a vertex $c(\gamma_k)$ with $i < k \le t$ such that $\gamma_k \neq \gamma_i$ and either $\gamma_{i-1} < \gamma_i < \gamma_j$ or $\gamma_{i-1} > \gamma_i > \gamma_j$, where $j = \min\{k; i < k \le t, \gamma_i \neq \gamma_k\}$.

An edge–walk is said to be *reducible* if it contains a reducible vertex; otherwise we say that Γ is *irreducible*.

The proof of the next lemma is similar to that of Lemma 4.7 in [4] with the obvious changes.

Lemma 3.6 *Any edge–walk* Γ *in* $\mathcal{A}_{O\cup S}\setminus\mathcal{A}_S$ *is equivalent to an irreducible edge–walk,* $\overline{\Gamma} = (c(\overline{\gamma}_i))^k_{i=0}$, *obtained by deleting all reducible vertices in* Γ.

Remark 3.7 (a) If $\Gamma = (c(\gamma_i))^t_{i=0}$ is an irreducible edge–walk in $\mathcal{A}_{O\cup S} \setminus \mathcal{A}_S$ then either $\gamma_{i-1} < \gamma_i > \gamma_{i+1}$ or $\gamma_{i-1} > \gamma_i < \gamma_{i+1}$ for all $0 < i < t$. Moreover, in case both

γ_0 and γ_t are n-cells in O then the length of Γ is an even number, $t = 2r$, and so $\gamma_{2i-2} > \gamma_{2i-1} < \gamma_{2i}$, for $1 \leq i \leq r$. In particular, this property holds for any edge–loop Γ in $\mathcal{A}_{O\cup S} \setminus \mathcal{A}_S$ which is based at a vertex $c(\sigma)$ with $\sigma \in O$.

(b) Notice also that for an arbitrary edge–walk $\Gamma = (c(\gamma_i))_{i=0}^t$ in $\mathcal{A}_{O\cup S} \setminus \mathcal{A}_S$ the vertex $c(\gamma_0)$ is never reducible. And, if $\overline{\Gamma} = (c(\overline{\gamma}_i))_{i=0}^k$ is the irreducible edge–walk obtained from Γ by deleting all its reducible vertices, then $\gamma_t = \overline{\gamma}_k$.

(c) Let $c(\phi_r) = (c(\phi_r(i/2)))_{i=0}^{2r}$ be the edge–loop defined by a given S-loop ϕ_r in O. It is not difficult to show that the irreducible edge–loop $\overline{c(\phi_r)}$ is, in fact, $c(\psi_s)$ for some S-loop ψ_s $(s \leq r)$ d-homotopic to ϕ_r.

For arbitrary digital spaces it may happen, for a cell $\alpha \in K$ with $c(\alpha) \in \mathcal{A}_{O\cup S}\setminus\mathcal{A}_S$, that the set $\mathrm{st_n}(\alpha; \mathrm{O}) = \emptyset$ is empty. This fact makes the search of digital representatives for an arbitrary edge–loop Γ in $\mathcal{A}_{O\cup S} \setminus \mathcal{A}_S$ much more intricate than the case $S = \emptyset$ in [4]. In order to obtain such digital representatives for the edge–loop Γ we first set the following

Definition 3.8 Let $O, S \subseteq \mathrm{cell_n(K)}$ be two disjoint digital objects in a digital space (K, f). We say that a cell $\alpha \in K$ is a *singular cell* for the pair (O, S), or simply an (O, S)-*singular cell*, if $c(\alpha) \in \mathcal{A}_{O\cup S} \setminus \mathcal{A}_S$ but $\mathrm{st_n}(\alpha; \mathrm{O}) = \emptyset$ (or, equivalently, $\mathrm{st_n}(\alpha; \mathrm{S}) = \mathrm{st_n}(\alpha; \mathrm{O} \cup \mathrm{S})$). Otherwise, if $c(\alpha) \in \mathcal{A}_{O\cup S} \setminus \mathcal{A}_S$ and $\mathrm{st_n}(\alpha; \mathrm{O}) \neq \emptyset$, α is called an (O, S)-*regular cell.*

We will also call (O, S)-regular to any edge–loop $\Omega = (c(\omega_i))_{i=0}^t$ in $\mathcal{A}_{O\cup S}\setminus\mathcal{A}_S$ whose vertices correspond to (O, S)-regular cells; that is, ω_i is (O, S)-regular for $0 \leq i \leq t$.

Remark 3.9 (a) Notice that all cells $\alpha \in O$ are (O, S)-regular for any digital object S such that $O \cap S = \emptyset$. And, similarly, if α is a vertex of K such $c(\alpha) \in \mathcal{A}_{O\cup S} \setminus \mathcal{A}_S$, then α is (O, S)-regular by Lemma 1.4.

(b) If α is an (O, S)-singular cell then axiom (5) in the definition of w.l.f. applies. So, the set $\alpha(S; O \cup S) = \{\beta < \alpha; c(\beta) \in \mathcal{A}_{O\cup S} \setminus \mathcal{A}_S\}$ is not empty and connected in $\partial\alpha$. Moreover, from Lemma 4.5 in [3] it is derived the existence of (O, S)-regular cells in the set $\alpha(S; O \cup S)$.

Despite the difficulties above, it is still not hard to define the digital representatives for the family of irreducible (O, S)-regular edge–loops in $\mathcal{A}_{O\cup S} \setminus \mathcal{A}_S$. We proceed as follows.

Definition 3.10 Let $\Omega = (c(\omega_i))_{i=0}^{2r}$ be an irreducible (O, S)-regular edge–loop in $\mathcal{A}_{O\cup S} \setminus \mathcal{A}_S$ based at $c(\sigma)$, with $\sigma \in O$. The set $\mathcal{D}(\Omega)$ of *digital representatives of* Ω consists of all S-loops ϕ_r in O for which $\phi_r(0) = \phi_r(r) = \sigma$, $\phi_r(i - \frac{1}{2}) = \omega_{2i-1}$, and $\phi_r(i) \in \mathrm{st_n}(\omega_{2i}; \mathrm{O})$, for $1 \leq i \leq r$.

Remark 3.11 For any S-loop ϕ_r in O, the edge–loop $c(\phi_r) = (c(\phi_r(i/2)))_{i=0}^{2r}$ is (O, S)-regular since $\phi_r(i) \in O$ for $0 \leq i \leq r$. In addition, $c(\phi_r)$ is irreducible in case $\phi_r(i - \frac{1}{2}) \neq \phi_r(i)$, for $1 \leq i \leq 2r$, and thus $\mathcal{D}(c(\phi_r)) = \{\phi_r\}$.

Next we state the crucial property of the digital representatives of an irreducible (O, S)-regular edge–loop in relation with the homomorphism $h : \pi_1^d(O/S, \sigma) \to \pi_1(\mathcal{A}_{O\cup S} \setminus \mathcal{A}_S, c(\sigma))$ above; compare with Proposition 4.12 in [4].

Proposition 3.12 *Let $\Omega = (c(\omega_i))_{i=0}^{2t}$ be any irreducible (O,S)-regular edge–loop in $\mathcal{A}_{O\cup S} \setminus \mathcal{A}_S$ based at $c(\sigma)$, with $\sigma \in O$. For any S-loop $\phi_t \in \mathcal{D}(\Omega)$ the equality $h([\phi_t]) = [\Omega]$ holds. Moreover, any two S-loops in $\mathcal{D}(\Omega)$ are d-homotopic.*

Proof. First we show that the edge–loop $c(\phi_t) = (c(\phi_t(i/2)))_{i=0}^{2t}$ defined by ϕ_t is equivalent to Ω. For this, let $\overline{\mathcal{D}}(\Omega)$ be the set of edge–loops $\Lambda = (c(\lambda_i))_{i=0}^{2t}$ at $c(\sigma)$ such that $\lambda_0 = \lambda_{2t} = \sigma$, $\lambda_{2i-1} = \omega_{2i-1}$ for $1 \leq i \leq t$, and $\lambda_{2i} \in \mathrm{st_n}(\omega_{2i}; \mathrm{O}) \cup \{\omega_{2i}\}$ for $1 < i < t$. Notice that $\overline{\mathcal{D}}(\Omega)$ contains the set of edge–loops $\{c(\phi_t); \phi_t \in \mathcal{D}(\Omega)\} \cup \{\Omega\}$. Moreover, any $\Lambda \in \overline{\mathcal{D}}(\Omega)$ is equivalent to Ω. This will be proved by induction on the number $k(\Lambda)$ of vertices $c(\lambda_{2i})$ in Λ for which $\lambda_{2i} \neq \omega_{2i}$. For $k(\Lambda) = 0$ we get $\Lambda = \Omega$. Assume that all $\Lambda \in \overline{\mathcal{D}}(\Omega)$ are equivalent to Ω for $k(\Lambda) \leq k-1$, and let Λ be any edge–loop with $k = k(\Lambda)$. Given any vertex $c(\lambda_{2i})$ in Λ with $\lambda_{2i} \neq \omega_{2i}$ (notice that $0 \neq i \neq t$) we get $\omega_{2i-1}, \omega_{2i+1} < \omega_{2i} < \lambda_{2i}$ since Ω is irreducible. Therefore we obtain a new edge–loop $\tilde{\Lambda} \in \overline{\mathcal{D}}(\Omega)$, with $k(\tilde{\Lambda}) = k-1$, by setting $c(\tilde{\lambda}_j) = c(\lambda_j)$ for $j \neq 2i$, and $c(\tilde{\lambda}_{2i}) = c(\omega_{2i})$. Moreover, $\tilde{\Lambda}$ is equivalent to Λ (by two equivalence transformations of type (b)) and hence Λ is equivalent to Ω by the induction hypothesis.

For the second property, we simply observe that the S-loops $\phi_t^1, \phi_t^2 \in \mathcal{D}(\Omega)$ are d-homotopic rel. σ by the $(\emptyset, S)$-map $H : \mathrm{Lb}(\mathrm{V}_{(r,1)}/\emptyset) \to \mathrm{Lb}(\mathrm{K}(\mathrm{O} \cup \mathrm{S})/\mathrm{S})$ given by $H(i/2, 0) = \phi_t^1(i/2)$, $H(i/2, 1) = \phi_t^2(i/2)$ and $H(i/2, 1/2) = \omega_i$, for $0 \leq i \leq 2t$. Here, we are using the identification of a cell $\alpha \in \mathrm{Lb}(\mathrm{V}_{(r,1)}/\emptyset)$ with its centroid $c(\alpha) = (a_1, a_2) \in \mathcal{Z}^2$ (see Remark 2.5). ∎

In order to obtain a family $\mathcal{D}(\Gamma)$ of digital representatives for an arbitrary edge–loop Γ, we construct an auxiliary family $\mathrm{pre}^2\mathcal{D}(\Gamma)$ of irreducible (O,S)-regular edge–loops. For this we shall use of another family of edge–loops $\mathrm{pre}\mathcal{D}(\Gamma)$. This two–step process starts at the irreducible edge–loop $\overline{\Gamma} = (c(\overline{\gamma}_i))_{i=0}^{2r}$ obtained from Γ by deleting all its reducible vertices; see Lemma 3.6. Then, the edge–loops in $\mathrm{pre}\mathcal{D}(\Gamma)$ are chosen by diverting $\overline{\Gamma}$ off the vertices $c(\overline{\gamma}_{2i-1})$, with an odd index, corresponding to (O,S)-singular cells. And $\mathrm{pre}^2\mathcal{D}(\Gamma)$ consists of the edge–loops obtained from each $\Delta = (c(\delta_{2i}))_{i=0}^{2t} \in \mathrm{pre}\mathcal{D}(\Gamma)$ bypassing each vertex $c(\delta_{2i})$, with δ_{2i} an (O,S)-singular cell, along a new edge–walk whose vertices correspond to (O,S)-regular cells in $\partial\delta_{2i}$.

Indeed, the elements in $\mathrm{pre}\mathcal{D}(\Gamma)$ are the family of edge–loops $\Delta = (c(\delta_i))_{i=0}^{2r}$ in $\mathcal{A}_{O\cup S} \setminus \mathcal{A}_S$ with the same length as $\overline{\Gamma}$ and such that $\delta_{2i} = \overline{\gamma}_{2i}$, for $0 \leq i \leq r$. Moreover, $\delta_{2i-1} = \overline{\gamma}_{2i-1}$ whenever $\overline{\gamma}_{2i-1}$ is an (O,S)-regular cell; and, otherwise, we choose $\delta_{2i-1} \in \{\alpha < \overline{\gamma}_{2i-1}; \alpha \text{ is an } (O,S)\text{-regular cell}\}$, which is a non–empty set by Remark 3.9(b).

Notice that any $\Delta \in \mathrm{pre}\mathcal{D}(\Gamma)$ is irreducible. Moreover, the following lemma is immediate

Lemma 3.13 *Any $\Delta \in \mathrm{pre}\mathcal{D}(\Gamma)$ is equivalent to $\overline{\Gamma}$, and hence to Γ.*

Proof. Just observe that the substitution of any cell $\overline{\gamma}_{2i-1}$ by one of its faces induces two equivalence transformations of type (b) between $\overline{\Gamma}$ and Δ. ∎

If we write $\mathrm{pre}\mathcal{D}(\Gamma) = \{\Delta_{\mathrm{k}}\}_{\mathrm{k}\in \mathrm{J}_\Gamma}$, a new family of irreducible edge–loops $\mathrm{pre}^2\mathcal{D}(\Delta_{\mathrm{k}})$ is defined for each $\Delta_k = (c(\delta_i))_{i=0}^{2r}$ as follows. An irreducible edge–loop $\Omega \in \mathrm{pre}^2\mathcal{D}(\Delta_{\mathrm{k}})$ is obtained by removing the reducible vertices from the juxtaposition of edge–walks

$\Omega = \Omega_0 * \Omega_1 * \cdots * \Omega_r$, where $\Omega_0 = (c(\delta_0), c(\delta_1))$, $\Omega_r = (c(\delta_{2r-1}), c(\delta_{2r}))$ and the component Ω_j, for $1 \leq j \leq r-1$, is the constant edge–loop $\Omega_j = (c(\delta_{2j-1}))$ if $\delta_{2j-1} = \delta_{2j+1}$. Otherwise, if δ_{2j} is an (O,S)-regular cell, in particular if $\delta_{2j} \in O$ (see Remark 3.9(a)), then $\Omega_j = (c(\delta_{2j-1}), c(\delta_{2j}), c(\delta_{2j+1}))$. Finally, if δ_{2j-1} is an (O,S)-singular cell we pick Ω_j out the edge–walks obtained from the following lemma for the (O,S)-regular cells $\beta_1 = \delta_{2j-1}$ and $\beta_2 = \delta_{2j+1}$.

Lemma 3.14 (cf. Lemma 4.8 in [3]) *Let $O, S \subseteq \mathrm{cell_n(K)}$ be two disjoint digital objects in a digital space (K, f), and let $\alpha \in K$ be an (O,S)-singular cell. Then, for any two distinct (O,S)-regular cells*

$$\beta_1, \beta_2 \in \alpha(S; O \cup S) = \{\beta < \alpha; f(O \cup S, \beta) = 1, f(S, \beta) = 0\}$$

there exist irreducible edge–walks $\Theta = (c(\theta_i))_{i=0}^m$ in $\mathcal{A}_{O\cup S} \setminus \mathcal{A}_S$ from $c(\beta_1)$ to $c(\beta_2)$ such that

(1) for $0 \leq i \leq m$, $\theta_i < \alpha$ and it is an (O,S)-regular cell; and,
(2) Θ is equivalent to the edge–walk $(c(\beta_1), c(\alpha), c(\beta_2))$.

Proof. By axiom (5) in Def. 1.3 we know that the set $\alpha(S; O \cup S)$ is connected and so we can choose an edge–walk $\Phi = (c(\phi_i))_{i=0}^t$ in $\alpha(S; O \cup S)$ from $c(\beta_1)$ to $c(\beta_2)$. By deleting the reducible vertices we can assume that Φ is irreducible (see Lemma 3.6). Notice that Φ need not have an even length since β_1 and β_2 may have arbitrary dimensions. In any case, it is obvious that Φ is equivalent to the edge–walk $(c(\beta_1), c(\alpha), c(\beta_2))$.

We derive the walk Θ from Φ as follows. First we observe that $\dim \alpha \geq 2$ by Lemma 1.4 and axiom (5) in Def. 1.3. Then we argue inductively on $l = \dim \alpha$. For $l = 2$ we have necessarily $\dim \phi_i \leq 1$. Moreover, if $\dim \phi_i = 0$ then ϕ_i is an (O,S)-regular cell by Remark 3.9(a). If $\dim \phi_i = 1$ with $0 < i < t$, the cells ϕ_{i-1} and ϕ_{i+1} are necessarily vertices of the edge $\phi_i \in K$. If, in addition, $\phi_{i-1} \neq \phi_{i+1}$ it follows that ϕ_i is also an (O,S)-regular cell by axiom (5) in Def. 1.3. Otherwise, if $\phi_{i-1} = \phi_{i+1}$, then we can delete the vertices $c(\phi_i)$ and $c(\phi_{i+1})$ from Φ to get a new irreducible edge–walk from $c(\beta_1)$ to $c(\beta_2)$ which is equivalent to Φ by two equivalence transformations of type (b). By deleting all the pairs $(c(\phi_i), c(\phi_{i+1}))$, for which ϕ_i is an edge in K and $\phi_{i-1} = \phi_{i+1}$, we obtain the desired edge–walk Θ.

Assume now that Θ can be derived from Φ for any cell α with $\dim \alpha < l$, and let $\dim \alpha = l$. Following the construction of the family pre$\mathcal{D}(\Gamma)$ we define the edge–walk $\Phi' = (c(\phi'_i))_{i=0}^t$ by $\phi'_{2j-1} = \phi_{2j-1}$ and also $\phi'_{2j} = \phi_{2j}$, for $0 \leq 2j \leq t$, if ϕ_{2j} is an (O,S)-regular cell. Otherwise we choose $\phi'_{2j} \in \{\alpha < \phi_{2j}; \alpha \text{ is } (O,S)\text{-regular}\}$. It is easily checked that Φ' is an irreducible edge–walk equivalent to Φ with its same length. Moreover, $\dim \phi'_{2j-1} < \dim \alpha$, for $0 \leq 2j-1 \leq t$, and $\phi'_0 = \beta_1$ and $\phi'_t = \beta_2$.

We define Θ by the juxtaposition $\Theta = \Theta_1 * \cdots * \Theta_k$ defined as follows. The index k is the largest integer with $2k-1 \leq t$, and the edge–walks Θ_j, for $1 \leq j \leq k$, are given by the next conditions:

1. $\Theta_j = (c(\phi'_{2j-2}))$ if $\phi'_{2j-2} = \phi'_{2j}$;
2. $\Theta_j = (c(\phi'_{2j-2}), c(\phi'_{2j-1}), c(\phi'_{2j}))$ if $\phi'_{2j-2} \neq \phi'_{2j}$ and ϕ'_{2j-1} is (O,S)-regular;

3. Θ_j is any of the edge–walks given by the induction hypothesis applied to ϕ'_{2j-1} and its faces ϕ'_{2j-2}, ϕ'_{2j} whenever $\phi'_{2j-2} \neq \phi'_{2j}$ and ϕ'_{2j-1} is an (O,S)-singular cell.

By construction one readily checks that Θ satisfies properties (1) and (2) in the lemma. Moreover, after deleting the reducible vertices (if any) in Θ we can assume that Θ is also an irreducible edge–walk. ■

Remark 3.15 Observe that, given $\Delta \in \mathrm{pre}\mathcal{D}(\Gamma)$, any edge–loop $\Omega \in \mathrm{pre}^2\mathcal{D}(\Delta)$ is, by construction, equivalent to Δ, and hence to Γ by Lemma 3.13. Moreover, it is irreducible and (O,S)-regular.

Finally, we define the family $\mathcal{D}(\Gamma)$ of digital representatives of Γ as follows

Definition 3.16 Let Γ be an arbitrary edge–loop in $\mathcal{A}_{O\cup S} \setminus \mathcal{A}_S$ based at $c(\sigma)$, with $\sigma \in O$. We define the set $\mathcal{D}(\Gamma)$ of *digital representatives* of Γ by

$$\mathcal{D}(\Gamma) = \bigcup_{\Delta \in \mathrm{pre}\mathcal{D}(\Gamma)} \left(\bigcup_{\Omega \in \mathrm{pre}^2\mathcal{D}(\Delta)} \mathcal{D}(\Omega) \right) .$$

Remark 3.17 (1) Let ∇ be and edge–loop in $\mathcal{A}_{O\cup S} \setminus \mathcal{A}_S$ obtained by removing from Γ any of its reducible vertices. Then $\overline{\nabla} = \overline{\Gamma}$ and hence $\mathcal{D}(\Gamma) = \mathcal{D}(\nabla)$. In particular, $\mathcal{D}(\Gamma) = \mathcal{D}(\overline{\Gamma})$, where $\overline{\Gamma}$ is the irreducible edge–loop obtained from Γ by removing all its reducible vertices.

(2) If Γ is an (O,S)-regular edge–loop in $\mathcal{A}_{O\cup S} \setminus \mathcal{A}_S$ then the irreducible edge–loop $\overline{\Gamma}$ is also (O,S)-regular. Thus, $\mathrm{pre}^2\mathcal{D}(\Gamma) = \{\overline{\Gamma}\}$, and all the digital representatives in $\mathcal{D}(\Gamma)$ are d-homotopic by Proposition 3.12.

(3) If ϕ_r is a S-loop in O, the family $\mathcal{D}(c(\phi_r))$ of digital representatives of the (O,S)-regular edge–loop $c(\phi_r) = (c(\phi_r(i/2)))_{i=0}^{2r}$ consists of a single element ψ_s, with $s \leq r$, by Remark 3.11. Moreover, ψ_s and ϕ_r are d-homotopic by Remark 3.7(c).

We are now ready to prove

Theorem 3.18 *Let (K,f) be an arbitrary digital space. For any two disjoint digital objects $O, S \subseteq \mathrm{cell_n(K)}$ the homomorphism*

$$h : \pi_1^d(O/S, \sigma) \to \pi_1(\mathcal{A}_{O\cup S} \setminus \mathcal{A}_S, c(\sigma))$$

is onto.

Proof. Given any edge–loop Γ in $\mathcal{A}_{O\cup S} \setminus \mathcal{A}_S$ based at $c(\sigma)$, we consider any edge–loop $\Omega \in \mathrm{pre}^2\mathcal{D}(\Gamma)$ which is equivalent to Γ by Remark 3.15. Then the result follows from Proposition 3.12. ■

Remark 3.19 To show that the homomorphism h is injective it is required, as a necessary condition, that $\phi^1 \simeq_d \phi^2$ rel. σ for any pair $\phi^1, \phi^2 \in \mathcal{D}(\Gamma)$ of digital representatives of an arbitrary edge–loop Γ in $\mathcal{A}_{O\cup S} \setminus \mathcal{A}_S$ (see Proposition 3.12). The construction of the family $\mathcal{D}(\Gamma)$ suggests that this fact may not be true in general. The main problem is that, from the available data, we cannot derive a d-homotopy between $\phi^1 \in \mathcal{D}(\Omega^1)$ and $\phi^2 \in \mathcal{D}(\Omega^2)$ whenever $\Omega^1 \neq \Omega^2$ in $\mathrm{pre}^2\mathcal{D}(\Gamma)$. However, we conjecture that this d-homotopy will be found if, for each (O,S)-singular cell α, the set $\cup\{\mathring{\omega}; \omega \in \alpha(S; O\cup S)\}$ is required to be simply connected instead of just connected as we require in Def. 1.3(5b).

3.2 A case of isomorphism.

For important cases, the family of digital representatives $\mathcal{D}(\Gamma)$ in Def. 3.16 is dramatically simplified. Recall that, in general, the family $\mathcal{D}(\Gamma)$ is obtained by a three–steps procedure that involves the definition of two auxiliary families of edge–loops pre$\mathcal{D}(\Gamma)$ and pre$^2\mathcal{D}(\Gamma)$. In this Section we will give a large class of digital spaces (K, f) for which the families pre$\mathcal{D}(\Gamma)$ and pre$^2\mathcal{D}(\Gamma)$ are reduced to singletons; so that, the difficulties pointed out in Remark 3.19 vanish. This will allow us to show that the epimorphism h in Theorem 3.18 is an isomorphism for a large class of digital spaces, which includes those most used in image processing. Namely, we will prove below

Theorem 3.20 *Let (K, f) be any digital space which is strongly local except possibly in 1-cells; that is, for any digital object $O \subseteq \mathrm{cell_n(K)}$ and any cell $\alpha \in K$ with $\dim\alpha \neq 1$, $f(O, \alpha) = f(\mathrm{st_n(\alpha; O)}, \alpha)$. Then the homomorphism $h : \pi_1^d(O/S, \sigma) \to \pi_1(\mathcal{A}_{O\cup S}\backslash\mathcal{A}_S, c(\sigma))$ is an isomorphism for any pair of disjoint objects $O, S \subseteq \mathrm{cell_n(K)}$.*

Corollary 3.21 *Let (K, f) be a strongly local digital space. For disjoint digital objects $O, S \subseteq \mathrm{cell_n(K)}$ the homomorphism h is an isomorphism.*

Recall that a digital space (K, f) is said to be strongly local if $f(O, \alpha) = f(\mathrm{st_n(\alpha; O)}, \alpha)$ for all $\alpha \in K$ and $O \subseteq \mathrm{cell_n(K)}$; see Def. 1.3. For each pair $(p, q) \neq (6, 6)$, with $p, q \in \{6, 18, 26\}$, it can be found a strongly local lighting function $f_{p,q}$ on the device model R^3 such that the digital space $(R^3, f_{p,q})$ represents the (p, q)-connectivity on the grid $\mathbb{Z}^3$; and, moreover, all the (p, q)-surfaces, as defined by Kong and Roscoe in [8], are digital surfaces in $(R^3, f_{p,q})$; see Theorem 13 in [1]. Notice also that, for an arbitrary device model K, the digital spaces $(K, f_{\max})$ and (K, g) given in Example 1.7(1) are strongly local. Hence, for these relevant examples, Corollary 3.21 holds.

Moreover, as an immediate consequence of Lemma 1.4, we have also

Corollary 3.22 *The homomorphism h is an isomorphism for digital spaces (K, f) with $\dim K \leq 2$.*

For non strongly local three–dimensional digital spaces we have the following

Lemma 3.23 *Let (R^3, f) be any digital space with R^3 the standard cubical decomposition of the Euclidean space $\mathbb{R}^3$. Moreover, assume $|\mathcal{A}_{R^3}| = \mathbb{R}^3$. Then the two following statements are equivalent.*

(i) *For each $O \subseteq \mathrm{cell_3(R^3)}$ and $\alpha \in R^3$ with $\dim\alpha = 2$, $f(O, \alpha) = 1$ if and only if $\alpha \in \mathrm{supp(O)}$.*
(ii) *(R^3, f) is strongly local except possibly for 1-cells.*

Proof. (i) implies (ii). It is clear that $\alpha \in \mathrm{supp(O)}$ if and only if $\alpha \in \mathrm{supp(st_3(\alpha; O))}$. Therefore, for $\dim\alpha = 2$ and $\alpha \in \mathrm{supp(O)}$ we have $f(O, \alpha) = f(\mathrm{st_3(\alpha; O)}, \alpha) = 1$ by (i). Otherwise, in case $\alpha \notin \mathrm{supp(O)}$, then $f(O, \alpha) = 0$ and $f(\mathrm{st_3(\alpha; O)}, \alpha) = 0$ by axiom (2) in Def. 1.3. For cells $\alpha \in R^3$ with $\dim\alpha \in \{0, 3\}$ the result follows from Lemma 1.4.

(ii) implies (i). For any object $O \subseteq \mathrm{cell}_3(\mathrm{R}^3)$ and any 2-dimensional cell $\alpha \in R^3$ one easily checks that $\alpha \in \mathrm{supp(O)}$ if and only if $\mathrm{st}_3(\alpha; \mathrm{O}) = \mathrm{st}_3(\alpha; \mathrm{R}^3)$. Hence $f(O, \alpha) = 0$ whenever $\alpha \notin \mathrm{supp(O)}$ by axiom (2) in Def. 1.3 while $f(O, \alpha) = f(\mathrm{st}_3(\alpha; \mathrm{R}^3), \alpha) = \mathrm{f}(\mathrm{cell}_3(\mathrm{R}^3), \alpha) = 1$ if $\alpha \in \mathrm{supp(O)}$. Here we use that $|\mathcal{A}_{R^3}| = \mathbb{R}^3$. ■

Then, we easily derive from Theorem 3.20 and Lemma 3.23 the following

Theorem 3.24 *The homomorphism h is an isomorphism for the non strongly local digital space (R^3, f^{BM}) given in [3].*

We point out that the digital surfaces in (R^3, f^{BM}) coincide with the strong 26-surfaces defined by Malgouyres and Bertrand [11].

The rest of this section is devoted to the proof of Theorem 3.20. We start with the following result, whose proof is immediate from definitions.

Lemma 3.25 *Let (K, f) be a digital space which is strongly local at the cell $\alpha \in K$. Then this cell is (O, S)-regular for any pair of disjoint digital objects $O, S \subseteq \mathrm{cell_n(K)}$ for which $c(\alpha) \in \mathcal{A}_{O\cup S} \setminus \mathcal{A}_S$.*

Lemma 3.26 *Let (K, f) be any digital space which is strongly local except possibly in 1-cells, and let O, S be two disjoint digital objects in (K, f). For any edge–loop Γ in $\mathcal{A}_{O\cup S} \setminus \mathcal{A}_S$ based at a vertex $c(\sigma)$, with $\sigma \in O$, the set $\mathrm{pre}^2\mathcal{D}(\Gamma) = \{\Omega_\Gamma\}$ is a singleton. In particular, all the digital representatives of Γ are d-homotopic by Proposition 3.12.*

Proof. Notice that any cell $\alpha \in K$ with $c(\alpha) \in \mathcal{A}_{O\cup S} \setminus \mathcal{A}_S$ and $\dim \alpha \neq 1$ is (O, S)-regular by Lemma 3.25. So, the construction of the family $\mathcal{D}(\Gamma)$ is determined by the vertices $c(\alpha)$ with $\dim \alpha = 1$. More explicitly, if $\overline{\Gamma} = (c(\overline{\gamma}_i))_{i=0}^{2r}$ is the irreducible edge–loop in Lemma 3.6 then the family $\mathrm{pre}\mathcal{D}(\Gamma) = \{\Delta_\Gamma\}$ consists of a unique (irreducible) edge–loop $\Delta_\Gamma = (c(\delta_i))_{i=0}^{2r}$ obtained by setting $\delta_{2i} = \overline{\gamma}_{2i}$ and replacing each vertex $c(\overline{\gamma}_{2i-1})$, with $\overline{\gamma}_{2i-1}$ an (O, S)-singular 1-cell, by $c(\delta_{2i-1})$ where $\delta_{2i-1} < \overline{\gamma}_{2i-1}$ is the unique vertex of $\overline{\gamma}_{2i-1}$ which is a (O, S)-regular cell. Here we use axiom (5) in Def. 1.3. Moreover, $\mathrm{pre}^2\mathcal{D}(\Gamma) = \mathrm{pre}^2\mathcal{D}(\Delta_\Gamma) = \{\Omega_\Gamma\}$ is also a singleton since, for any vertex $c(\delta_{2i})$ in Δ_Γ, with $0 < i < r$, such that $\delta_{2i} = \overline{\gamma}_{2i}$ is an (O, S)-singular 1-cell, axiom (5) in Def. 1.3 yields $\delta_{2i-1} = \delta_{2i+1}$. Hence Ω_Γ is determined by replacing the edge–walk $(c(\delta_{2i-1}), c(\delta_{2i}), c(\delta_{2i+1}))$ by the constant edge–walk $(c(\delta_{2i-1}))$. ■

Lemma 3.27 *Let (K, f) be a digital space which is strongly local except possibly in 1-cells, and let O, S be two disjoint digital objects in (K, f). Then any edge–loop $\Gamma = (c(\gamma_i))_{i=0}^k$ in $\mathcal{A}_{O\cup S} \setminus \mathcal{A}_S$ based at a $c(\sigma)$, with $\sigma \in O$, is equivalent to an (O, S)-regular edge–loop $\Gamma^* = (c(\gamma_i^*))_{i=0}^k$ called the* regularization *of Γ. Moreover, if Σ is another edge–loop obtained from Γ by removing a vertex $c(\gamma_{i_0})$ via an equivalence transformation of type (a) or (b), then the regularization of Σ, Σ^*, can be derived from Γ^* after an equivalence transformation of the same type.*

Proof. We construct the edge–loop Γ^* as follows. If γ_i is (O, S)-regular we set $\gamma_i^* = \gamma_i$. Otherwise, $\dim \gamma_i = 1$ by Lemma 3.25 and we take γ_i^* to be the unique vertex $\gamma_i^* < \gamma_i$ which is (O, S)-regular. Here we use Axiom 5 in Def. 1.3. In order to show that $\Gamma^* = (c(\gamma_i^*))_{i=0}^k$ is an edge–loop equivalent to Γ we consider the set $\mathcal{R}\text{eg}(\Gamma)$ consisting of finite sequences $\Lambda = (c(\lambda_i))_{i=0}^k$ such that $\lambda_i = \gamma_i$ if γ_i is (O,S)-regular and $\lambda_i \in \{\gamma_i, \gamma_i^*\}$ otherwise. Notice that $\{\Gamma, \Gamma^*\} \subseteq \mathcal{R}\text{eg}(\Gamma)$. Next we show inductively that each $\Lambda \in \mathcal{R}\text{eg}(\Gamma)$ is an edge–loop in $\mathcal{A}_{O\cup S} \setminus \mathcal{A}_S$ based at $c(\sigma)$ which is equivalent to Γ. For this, let $t(\Lambda)$ be the number of vertices $c(\lambda_i)$ with $\lambda_i \neq \gamma_i$. If $t(\Lambda) = 0$ then $\Lambda = \Gamma$ and the result is obvious. Assume the result holds for $t(\Lambda) \leq t-1$ and take $\Lambda \in \mathcal{R}\text{eg}(\Gamma)$ with $t(\Lambda) = t$. Given any vertex $c(\lambda_i) \in \Lambda$ with $\lambda_i = \gamma_i^* < \gamma_i$ we consider the sequence $\tilde{\Lambda} \in \mathcal{R}\text{eg}(\Gamma)$ with $\tilde{\lambda}_j = \lambda_j$ if $j \neq i$ and $\tilde{\lambda}_i = \gamma_i$. Notice that $0 < i < n$ since $\gamma_0 = \gamma_k = \sigma = \lambda_0 = \lambda_k \in O$ is an (O,S)-regular cell by Remark 3.9(a). By the induction hypothesis $\tilde{\Lambda}$ is an edge-loop in $\mathcal{A}_{O\cup S} \setminus \mathcal{A}_S$ based at $c(\sigma)$ which is equivalent to Γ. Moreover, we have the following possible face relations between $\tilde{\lambda}_j = \lambda_j$ $(j = i-1, i+1)$ and $\tilde{\lambda}_i = \gamma_i$: (1) $\tilde{\lambda}_j < \tilde{\lambda}_i$, or (2) $\tilde{\lambda}_j \geq \tilde{\lambda}_i$. In the first case we have that $\lambda_j = \gamma_i^* = \lambda_i$, while in the second $\lambda_j \geq \gamma_i > \gamma_i^* = \lambda_i$. In any case, Λ is an edge–loop equivalent to $\tilde{\Lambda}$, and hence to Γ, via two transformations of type (b).

Let us now assume that $c(\gamma_{i_0})$ can be removed from Γ by an equivalence transformation. Then one of the following cases necessarily occurs

1. $0 < i_0 < t$; $\gamma_{i_0-1} < \gamma_{i_0} > \gamma_{i_0+1}$
2. $0 < i_0 < t$; $\gamma_{i_0-1} > \gamma_{i_0} < \gamma_{i_0+1}$
3. $\gamma_{i_0-1} = \gamma_{i_0}$
4. $\gamma_{i_0-1} < \gamma_{i_0} < \gamma_{i_0+1}$
5. $\gamma_{i_0-1} > \gamma_{i_0} > \gamma_{i_0+1}$
6. $\gamma_{i_0} = \gamma_{i_0+1}$

Let $(1)^* \dots (6)^*$ denote the corresponding statements for the vertices in Γ^*. The reader can easily check that $(i) \Rightarrow (i)^*$ if no (O,S)-singular cell is involved. In case γ_{i_0-1} is singular then both (2) and (5) yield $(3)^*$, and for the rest of statements we get $(i) \Rightarrow (i)^*$. If γ_{i_0} is singular the (1) yields $\gamma_{i_0-1}^* = \gamma_i^* = \gamma_{i_0+1}^*$, while $(4) \Rightarrow (3)^*$ and $(5) \Rightarrow (6)^*$, and $(i) \Rightarrow (i)^*$ for the other cases. Finally if γ_{i_0+1} is singular we derive $(6)^*$ from both (2) and (4), while for the remaining cases $(i) \Rightarrow (i)^*$. ∎

Proof of Theorem 3.20. We already know that h is onto by Theorem 3.18. So, it will suffice to prove that any two S-loops, ϕ, ψ, in O define the same element in $\pi_1^d(O/S, \sigma)$ provided $h([\phi]) = [c(\phi)] = [c(\psi)] = h([\psi])$.

Since $c(\phi)$ and $c(\psi)$ are equivalent edge–loops, there exists a sequence $\alpha_0, \alpha_1, \dots, \alpha_k$ of edge–loops in $\mathcal{A}_{O\cup S} \setminus \mathcal{A}_S$ such that $\alpha_0 = c(\phi)$, $\alpha_k = c(\psi)$ and α_{i-1}, α_i are related by an equivalence transformation of type (a) or (b). Moreover, by Remark 3.11, $c(\phi)$ and $c(\psi)$ are (O,S)-regular, and the regularized edge–loops $c(\phi) = \alpha_0^*, \alpha_1^*, \dots, \alpha_k^* = c(\psi)$ define also a sequence of equivalent edge–loops by Lemma 3.27. Then, Remark 3.17(3) and Lemma 3.28 below yields that every S-loop in $\cup_{i=0}^k \mathcal{D}(\alpha_i^*)$ defines the same element in $\pi_1^d(O/S, \sigma)$. Hence ϕ and ψ are d-homotopic by Remark 3.17(3). ∎

This lemma is an extension of Lemma 4.14 in [4] which corresponds to the special case $S = \emptyset$.

Lemma 3.28 *Let $O, S \subseteq \mathrm{cell_n(K)}$ be two disjoint digital objects in an arbitrary digital space (K, f), and let $\Gamma = (c(\gamma_i))_{i=0}^t$ be an (O,S)-regular edge–loop in $\mathcal{A}_{O\cup S} \setminus \mathcal{A}_S$ based* *at $c(\sigma)$, with $\sigma \in O$. Assume that an edge–loop Σ is obtained by removing a vertex $c(\gamma_{i_0})$ from Γ after an equivalence transformation of type (a) or (b). Then, for each S-loop $\phi \in \mathcal{D}(\Gamma)$ there exists a digital representative $\psi \in \mathcal{D}(\Sigma)$ and a d-homotopy $\phi \simeq_d \psi$ rel. σ.*

Proof. The hypothesis leads to one of the following cases

(1) $0 < i_0 < t$, the centroids $c(\gamma_{i_0-1}), c(\gamma_{i_0}), c(\gamma_{i_0+1})$ span a simplex in $\mathcal{A}_{O\cup S} \setminus \mathcal{A}_S$ and $\gamma_{i_0-1} < \gamma_{i_0} > \gamma_{i_0+1}$.
(2) $0 < i_0 < t$, the centroids $c(\gamma_{i_0-1}), c(\gamma_{i_0}), c(\gamma_{i_0+1})$ span a simplex in $\mathcal{A}_{O\cup S} \setminus \mathcal{A}_S$ and $\gamma_{i_0-1} > \gamma_{i_0} < \gamma_{i_0+1}$.
(3) $c(\gamma_{i_0})$ is a reducible vertex in γ.
(4) $\gamma_{i_0} = \gamma_{i_0+1}$, and hence the vertex $c(\gamma_{i_0+1})$ is reducible.

In cases (3) and (4) the edge–loop Σ is obtained by dropping a reducible vertex from Γ, so $\mathcal{D}(\Gamma) = \mathcal{D}(\Sigma)$ by Remark 3.17(1) and the result follows from Remark 3.17(2). Therefore we concentrate our efforts in proving the lemma for the case (1) since case (2) is settled in a similar way.

We start by considering the number $n(\Gamma)$ of reducible vertices of Γ in the set

$$V_\Gamma = \{c(\gamma_j); 0 \le j \le i_0 - 2\} \cup \{c(\gamma_j); i_0 + 2 \le j \le t\}.$$

Since any reducible vertex in V_Γ is also a reducible vertex of Σ we can remove all of them from both Γ and Σ. This way we replace Γ and Σ by two new edge–loops Γ' and Σ' respectively such that $n(\Gamma') = 0$. Moreover, by Remark 3.17(1), $\mathcal{D}(\Gamma) = \mathcal{D}(\Gamma')$ and $\mathcal{D}(\Sigma) = \mathcal{D}(\Sigma')$. Hence, by Remark 3.17(2), there is no loss of generality in assuming $\Gamma = \Gamma'$ and $\Sigma = \Sigma'$.

Next we consider all possible face relations among the pairs of cells $(\gamma_{i_0-2}, \gamma_{i_0-1})$, $(\gamma_{i_0+1}, \gamma_{i_0+2})$ and $(\gamma_{i_0-1}, \gamma_{i_0+1})$. Notice that the two elements in each pair may be equal, and in case (2) it is also possible that $i_0 = 1$ or $i_0 = t - 1$. The proof requires in general the four steps below whatever are the face relations we consider. For illustrating the proof we give a detailed account of these steps for the case (1) and the face relations

$$\gamma_{i_0-2} > \gamma_{i_0-1} < \gamma_{i_0} > \gamma_{i_0+1} < \gamma_{i_0+2} \tag{2}$$

and

$$\gamma_{i_0-1} < \gamma_{i_0+1} \ . \tag{3}$$

Step A. *Describe the irreducible edge–loops $\overline{\Gamma}$ and $\overline{\Sigma}$.*

The face relations (II) and (III) yield that Γ has not reducible vertices, so that $\Gamma = \overline{\Gamma}$ is a edge–loop of even length $t = 2r$ by Remark 3.7(a). In addition, the irreducible edge–loop $\overline{\Sigma}$ associated to Σ is

$$\overline{\Sigma} = (c(\gamma_0), \dots, c(\gamma_{i_0-2}), c(\gamma_{i_0-1}), c(\gamma_{i_0+2}), \dots, c(\gamma_{2r}))$$

since $c(\gamma_{i_0+1})$ is reducible in Σ by the face relations (II) and (III); see Figure 3. Therefore, any digital representative of Γ is an S-loop of length r, while digital representatives of Σ have length $r - 1$.

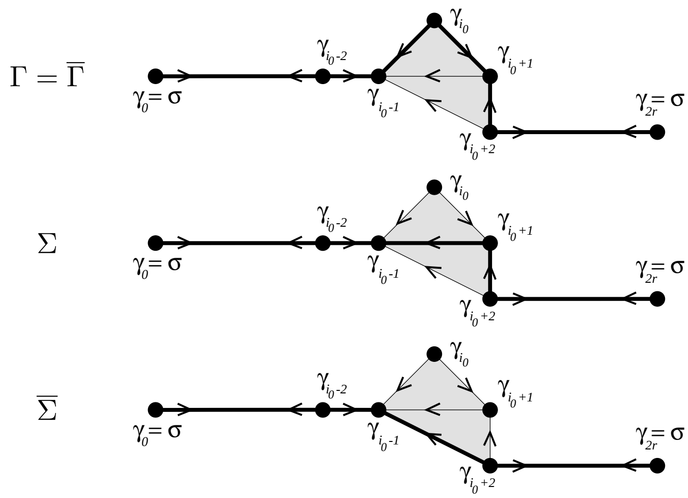

Fig. 3.

Notice that under a different set of face relations Γ and $\overline{\Gamma}$ may be distinct. In any case, the length of $\overline{\Gamma}$ is always greater than or equal to the length of $\overline{\Sigma}$, and the same happens for the digital representatives of Γ and Σ.

Step B. *Given a digital representative $\phi \in \mathcal{D}(\Gamma)$ of Γ, derive a digital representative $\psi \in \mathcal{D}(\Sigma)$ of Σ.*

Given $\phi = \phi_r \in \mathcal{D}(\Gamma)$, it is not difficult to check from Step A that the S-loop $\psi = \psi_{r-1}$, given by $\psi_{r-1}(j/2) = \phi_r(j/2)$, for $0 \leq j \leq i_0 - 1$, and $\psi_{r-1}(j/2) = \phi_r(j/2+1)$, for $i_0 \leq j \leq 2r-2$, is a digital representative of the edge–loop Σ.

Step C. *Obtain a new S-loop $\overline{\psi}$ d-homotopic to ψ and such that $\overline{\psi}$ and ϕ have the same length.*

By Definition 2.8, the S-loops $\psi = \psi_{r-1}$ and $\psi_{r-1} * \psi_1^\sigma$ are d-homotopic, where ψ_1^σ is the constant S-loop of length 1 at $\sigma = \psi_{r-1}(0) = \psi_{r-1}(r-1)$. Then, Proposition 2.9 yields the following d-homotopy

$$\psi_{r-1} * \psi_1^\sigma \simeq_d \psi_{\frac{i_0}{2}} * \psi_1^\tau * \psi_{r-1-\frac{i_0}{2}} = \overline{\psi}_r \ ,$$

where $\psi_{\frac{i_0}{2}}$ and $\psi_{r-1-\frac{i_0}{2}}$ are the S-walks in O given by $\psi_{\frac{i_0}{2}}(j/2) = \psi_{r-1}(j/2)$, for $0 \leq j \leq i_0$ and $\psi_{r-1-\frac{i_0}{2}}(j/2) = \psi_{r-1}((j+i_0)/2)$, for $0 \leq j \leq 2r - i_0 - 2$, respectively, and moreover ψ_1^τ is the constant S-loop of length 1 at $\tau = \psi_{r-1}(i_0/2)$.

In general, different constant S-loops may be required for other sets of face relations. Notice also that this step could be not necessary in case the original digital representatives ϕ and ψ have the same length.

Step D. *Describe a d-homotopy between ϕ and $\overline{\psi}$. As a consequence, the lemma follows.*

From the face relations (II) and (III) it is not difficult to show that the d-map given by

$$H(\frac{j}{2}, \frac{k}{2}) = \begin{cases} \phi_r(j/2) & \text{if } k = 0 \text{ and } 0 \leq j \leq 2r \\ \phi_r(j/2) & \text{if } k = 1 \text{ and } 0 \leq j \leq i_0 - 1 \text{ or } i_0 + 1 \leq j \leq 2r \\ \gamma_{i_0 - 1} & \text{if } k = 1 \text{ and } j = i_0 - 1 \\ \overline{\psi}_r(j/2) & \text{if } k = 2 \text{ and } 0 \leq j \leq 2r - 2 \end{cases}$$

is a d-homotopy between ϕ_r and the S-loop $\overline{\psi}_r \simeq_d \psi_{r-1}$.

Any other set of face relations leads to a possibly different d-homotopy, anyway of the same nature as H above. ■

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