Generating families of surface triangulations. The case of punctured surfaces with inner degree at least 4

María-José Chávez[*] Seiya Negami [†] Antonio Quintero [‡]
María Trinidad Villar[§]

Abstract

We present two versions of a method for generating all triangulations of any punctured surface in each of these two families: (1) triangulations with inner vertices of degree ≥ 4 and boundary vertices of degree ≥ 3 and (2) triangulations with all vertices of degree ≥ 4. The method is based on a series of reversible operations, termed reductions, which lead to a minimal set of triangulations in each family. Throughout the process the triangulations remain within the corresponding family. Moreover, for the family (1) these operations reduce to the well-known edge contractions and removals of octahedra. The main results are proved by an exhaustive analysis of all possible local configurations which admit a reduction.

Keywords: punctured surface, irreducible triangulation, edge contraction, vertex splitting, removal/addition of octahedra.

This work has been partially supported by PAI FQM-164; PAI FQM-189; MTM 2010-20445.

1 Introduction

By a *triangulation* of a surface F^2 we mean a simple graph G (i.e., a graph without loops and multiple edges) embedded in F^2 so that each face is bounded by a 3-cycle and any two faces share at most one edge. In other words, the vertices, edges and faces of G (the corresponding sets denoted by $V(G)$, $E(G)$ and $F(G)$, respectively) form a simplicial complex whose underlying space is F^2. Two triangulations G and G' of F^2 are *equivalent* if there is a homeomorphism $\varphi : F^2 \to F^2$ with $\varphi(G) = G'$. In this paper surfaces are supposed to be compact and connected and possibly with boundary. Surfaces without boundary will be termed *closed surfaces*. Here, we distinguish between triangulations only up to equivalence.

Generation from irreducible triangulations of a surface F^2 is a well-known procedure for obtaining all the triangulations of F^2. Recall that an edge of a triangulation G of F^2 is *contractible* if the vertices of the edge can be identified (multiple edges are removed, if they appear) and the result is still a triangulation of F^2 ([2]). A triangulation is said to

[*]Departamento de Matemática Aplicada I, Universidad de Sevilla, Spain, mjchavez@us.es
[†]Research Institute of Environment and Information Sciences, Yokohama National University, 79-2 Tokiwadai, Hodogaya-Ku, Yokohama 240-8501, Japan, negami@ynu.ac.jp
[‡]Departamento de Geometría y Topología Universidad de Sevilla, Spain, quintero@us.es
[§]Departamento de Geometría y Topología, Universidad de Sevilla, Spain, villar@us.es

be *irreducible* if it has no contractible edge. Irreducible triangulations form a generating set for all triangulations of the same surface in the sense that every triangulation of the surface can be obtained from some irreducible triangulation by a sequence of vertex splittings (the inverse of the edge contraction operation); see [2].

Barnette and Edelson [2] showed that every closed surface has finitely many irreducible triangulations. More recently, Boulch, Colin de Verdière, and Nakamoto [4] showed the same result for compact surfaces with a nonempty boundary. Notwithstanding, it is far from being trivial to enumerate the irreducible triangulations of a given surface. Complete lists of irreducible triangulations are available only for some low genus surfaces. See [20], [22], [23] for a comprehensive reference for the class of closed surfaces.

So far, the research on irreducible triangulations of closed surfaces has produced a considerable literature. This is not the case for surfaces with boundary, for which few references can be presently found; see [4], [8]. This paper is a contribution to the study of irreducible triangulations for *punctured surfaces* (i.e., surfaces with a hole produced by the deletion of the interior of a disk in closed surfaces).

It is well known that any irreducible triangulation G of an arbitrary non-spherical closed surface F^2 has minimum degree ≥ 4 [20]. This is no longer true if F^2 has non-empty boundary. However, if a boundary vertex v has degree 2 then v lies in exactly one face of G whose boundary edges are trivially contractible (unless G reduces to a triangle). Therefore we will deal exclusively with triangulations with minimum degree ≥ 3. Notice that this condition implies that no face shares more than one edge with the boundary of the surface. This way, all irreducible triangulations of a surface with boundary F^2 (other than the disk) are elements of the class $\mathcal{F}^2_\circ(4)$ consisting of all triangulations of F^2 with minimum degree ≥ 3 and $deg(v) \geq 4$ for all vertices v missing the boundary.

In this paper we give a generating theorem for such triangulations in terms of internal operations in the class $\mathcal{F}^2_\circ(4)$; that is, we show that all triangulations of a punctured surface other than the disk can be reduced to an irreducible triangulation by performing such operations (Theorem 18). The particular case of the disk is also treated (Theorem 19).

Similarly, we introduce a set of internal operations in the subfamily $\mathcal{F}^2(4) \subseteq \mathcal{F}^2_\circ(4)$ consisting of all triangulations with minimum degree ≥ 4. In contrast with the case of closed surfaces, for a surface with non empty boundary, the minimal triangulations obtained by the use of such operations may contain contractible edges whose contraction produce 3-valent vertices. We prove that such contractible edges are necessarily located in two particular configurations given in Definitions 4 and 34, see Theorem 38.

The main results collected in this work can be regarded as extensions to punctured surfaces of the main theorems by Nakamoto and Negami for closed surfaces in [17].

2 Notation and preliminaries

If G is a triangulation of the surface F^2, let $\partial G \subset G$ denote the subgraph triangulating the boundary ∂F^2. The vertices and edges of ∂G will be called *boundary vertices* and *boundary edges* of G, respectively. The vertices and edges of $G - \partial G$ will be called *inner vertices* and *inner edges* of G, respectively. Let us now recall that the link of a vertex $x \in G$, denoted $link(x)$, is the set of edges in G which jointly with the vertex x form a triangle in G.

Let $e = v_1 v_2$ be an edge in G. Let us recall that the distance from e to ∂G, denoted

$d(e, \partial G)$, is defined to be the minimum number of edges needed to connect e and ∂G. The resulting graph obtained by contracting e in G is denoted by G/e. The contraction of a pair of disjoint edges in two adjacent faces in G is named *double contraction*. If $e = v_1 v_2$ is a contractible edge of G, then the new vertex $v = v_1 = v_2$ in G/e satisfies $deg(v) = deg(v_1) + deg(v_2) - 3$ when e is a boundary edge of G, and $deg(v) = deg(v_1) + deg(v_2) - 4$ otherwise. Here $deg(v)$ denotes the degree of the vertex v, if $deg(v) = k$ we say that v is a *k-valent* vertex. A 3-cycle in G is *critical* if it consists of three edges which do not bound a face of G. Besides, if xv_1v_2 is a face of G, then $deg(x)$ diminishes by one after the contraction of e.

When a lower bound k for the degree of the vertices is preserved after contractions we will use the term *k-contraction*. Namely, given a triangulation G with minimum degree $\geq k$, an edge e is said to be *k-contractible* (*kc-edge* for short) if the minimum degree of G/e is at least k. If an edge e is contractible but not k-contractible, we call e to be a *cnkc-edge*, for short. The inverse operation of the contraction of the edge $e = v_1 v_2$ is the *splitting* of $v_1 = v_2$. When $deg(v_i) \geq k$ for $i = 1, 2$, after the splitting, this will be called a *k-splitting*.

Remark 1. Notice that the contraction of an edge $e \in G$ belonging to a critical 3-cycle produces a double edge. On the other hand, if e belongs to no critical 3-cycle and at most one of its end vertices belongs to ∂G, then e is contractible. In other words, the impediments to the contractibility of e are the two following locations of e in G:

(1) e belongs to a critical cycle of G. This is the case if e lies on the boundary of a hole of length 3.

(2) e is an inner edge but its two vertices belong to ∂G.

Remark 2. Notice that a necessary condition for an interior edge $e = v_1 v_2$ to be 4-contractible is that e belongs to faces $v_1 v_2 v_3$ and $v_1 v_2 v_4$ so that $\deg(v_i) \geq 5$ for $i = 3, 4$. If e is a boundary edge lying in a face $v_1 v_2 v_3$, the necessary condition for e to be 4-contractible is that $\deg(v_3) \geq 5$.

The following definitions extend to surfaces with boundary the one given in [17] for closed surfaces.

Definition 3. Let G be a triangulation of a surface F^2 possibly with non-empty boundary. Let $v_1 v_2 v_3$ be a cycle of G such that $deg(v_i) = 4$ for $i =$ 1, 2, 3 and $\{a_1, a_2, a_3\}$ be the only three vertices such that a_i is adjacent to v_j and v_k for $\{i, j, k\} = \{1, 2, 3\}$. The subgraph $H \subset G$ induced by the vertex set $\{a_1, a_2, a_3, v_1, v_2, v_3\}$ is said to be an *octahedron component* centered at $v_1 v_2 v_3$ with remaining vertices a_1, a_2, a_3 if the cycle $a_1 a_2 a_3$ exists in G (and hence in H) and one of the following conditions holds:

1. $v_i \notin \partial G$ for all $i = 1, 2, 3$.
2. Only one v_i lies in ∂G and ∂G coincides with $v_i a_j a_k$.
3. Exactly $v_i, v_j \in \partial G$ and hence $\partial G = v_i v_j a_k$.
4. $v_i \in \partial G$ for all $i = 1, 2, 3$ and hence $\partial G = v_i v_j v_k$.

An octahedron component of G is said to be *external* if two edges $a_i a_j$, $a_j a_k$ lie in ∂G (in particular, $\delta(a_j) = 4$). Observe that this happens only under condition 1.

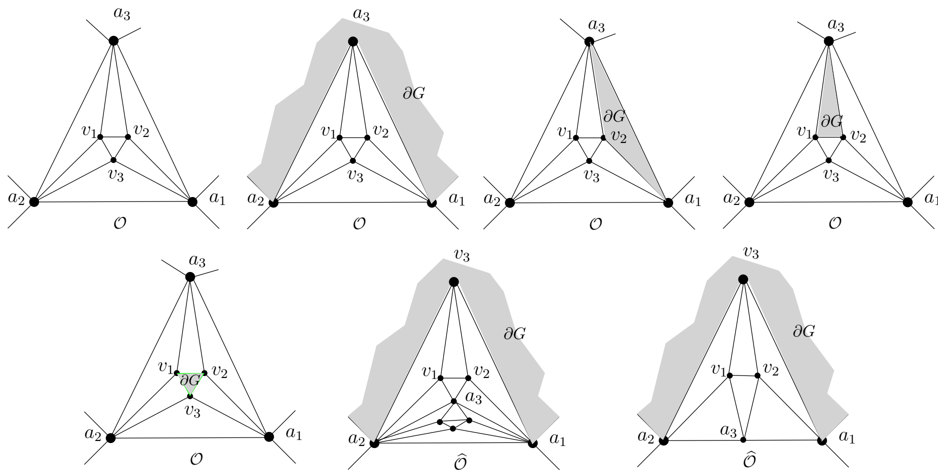

Figure 1: Octahedron and quasi-octahedron components in G.

Definition 4. The subgraph H in Definition 3 will be termed a *quasi-octahedron component* of G centered at $v_1v_2v_3$ and remaining vertices a_1, a_2, a_3 if one of the following conditions holds. (Figure 1).

1. The cycle $a_1a_2a_3$ exists but does not define a face in G and only one vertex v_i belongs to ∂G but (in contrast with 2 above) $a_ja_k \notin \partial G$.
2. Only the edge a_ia_j fails in closing the cycle $a_1a_2a_3$ in G, hence $v_k \in \partial G$ and all the 3-cycles of H are faces of G.

Remark 5. Notice that the possible occurrences of the subgraph H other than the ones considered in Definitions 3 and 4 appear when at least two edges a_ia_j do not exist in G (and then $v_k \in \partial G$). If no edge a_ia_j exists in G, then $H = G$ is a triangulation of the disk. On the other hand, if only one edge a_ia_j is in G, then a_k has degree 2 in ∂G.

Notation: Octahedron and quasi-octahedron components will be denoted $\mathcal{O}$ and $\widehat{\mathcal{O}}$, respectively.

Remark 6. Let us remark also that at most one 3-cycle of an octahedron component of G may not be a face of G. In such case, ∂G reduces to that 3-cycle.

Let us note that for a quasi-octahedron component of G the edge a_iv_3 is always a boundary edge of G, for $i = 1, 2$.

Notice that no quasi-octahedron component $\widehat{\mathcal{O}}$ can be extended to an octahedron component. Indeed, if $v_i \in \widehat{\mathcal{O}} \cap \partial G$, then the 3-cycle $a_1a_2a_3$ is not a face even though the edge a_ja_k opposite to v_i exists (and it is necessarily an inner edge).

3 Characterizing the triangulations of inner degree at least 4.

It is well known that any irreducible triangulation of a closed surface other than the sphere has minimum degree ≥ 4. Therefore irreducible triangulations of punctured surfaces F^2 (other than the disk) must have minimum inner degree ≥ 4 (that is, only boundary vertices are allowed to have degree 3); that is, they are in the class $\mathcal{F}^2_\circ(4)$ defined above.

In this section we give a method to construct all triangulations in $\mathcal{F}^2_\circ(4)$ from irreducible ones by operations which keeps all triangulations within this class (Theorem 18). The special case of the disk is also considered (Theorem 19). This way we generalize Theorems 1 and 2 in [17]. The method in [17] is based on the use of 4-splitting and adding octahedra. The existence of 3-valent vertices in the boundary requires two further operations: adding flags and triode 3-splittings.

Throughout this section F^2 will denote a surface with connected (possibly empty) boundary. Recall that $G \in \mathcal{F}^2_\circ(4)$ denotes an arbitrary but fixed triangulation of F^2 with all its inner vertices of degree ≥ 4.

Let us start by fixing some notation.

Notation: If x is a vertex of G with $deg(x) = 4$ we fix notation by calling x_1, x_2, a, b its neighbours and for the sake of simplicity $link(x)$ is written $link(x) = x_1abx_2x_2x_1$ if $x \notin \partial G$ or $link(x) = x_1abx_2$ if $x \in \partial G$. This notation will be used throughout this paper without any further comment.

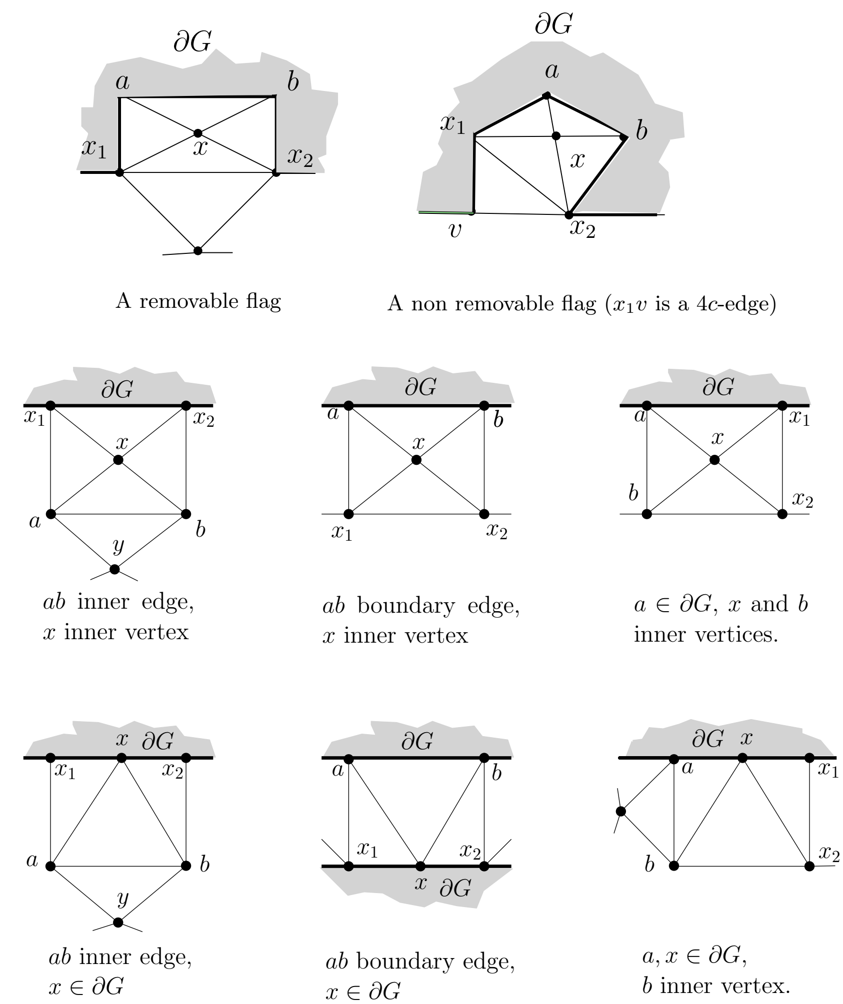

Figure 2: Different configurations for $link(x)$, with $deg(x) = 4$ and distance ≤ 1 from ∂G.

The vertices of degree 3 in ∂G play a crucial role in the family $\mathcal{F}_\circ^2(4)$. We will give them a special name.

Definition 7. Given $G \in \mathcal{F}_\circ^2(4)$, a boundary vertex of degree 3 is called a *triode* of G. A contractible edge of G is said to be a *triode detecting edge* if the vertices of degree 3 produced by its contraction are triodes.

Remark 8. For every face abx such that ab is a contractible boundary edge, x lies in the boundary and $degh(x) = 4$ it readily follows from Definition 7 that ab is a triode detecting edge.

On the other hand, the contraction of any inner triode detecting edge produces at most three triodes since any contraction modifies the degree of at most three vertices.

It is also readily checked that two adjacent triodes define a contractible edge in ∂G, say ab, unless G is isomorphic to the complete graph K_4 (and so G triangulates the disk). If, in addition, ab shares a face with a 4-valent inner vertex, the contraction of any edge incident at x or in $link(x)$ is allowed in $\mathcal{F}_\circ^2(4)$, but ab is a $cn4c-$edge. To get rid of this obstacle, we define the following configuration termed flag. Recall that a vertex v is said to be independent of degree k if all neighbors of v have degree $\neq k$.

Definition 9. Given $G \in \mathcal{F}_\circ^2(4)$, let x be an independent inner vertex of degree 4 such that $link(x) = x_1abx_2x_1$ verifies $\{x_1a, ab, bx_2\} \subset \partial G$, $x_1x_2 \cap \partial G = \{x_1, x_2\}$, and $deg(a) = deg(b) = 3$. The subgraph induced by $\{x, x_1, x_2, a, b\}$ is called a *flag centered at* x. If the graph $G' = G - \{a, b, x\}$ remains in $\mathcal{F}_\circ^2(4)$, the flag is said to be *removable* (see Figure 2). Conversely, we say that G is obtained from G' by *adding a flag* along a boundary edge of G'.

Remark 10. Observe that any flag is removable unless $deg(x_1) = 4$ (or $deg(x_2) = 4$) and this is the only impediment for a flag to being removable. If a flag is non-removable then either x_1v or x_2v is a boundary $4c$-edge.

The next lemma follows immediately from definitions and it will be used in the proof of Theorem 18 below.

Lemma 11. *Let ab be a contractible inner edge of $G \in \mathcal{F}_\circ^2(4)$, let x and y be vertices so that x is a boundary vertex with $deg(x) = 4$ and abx and aby define two faces of G. Then ab is a triode detecting edge whenever y is a boundary vertex of $deg(y) \geq 4$ or else y is an inner vertex of degree $deg(y) \geq 5$.*

Next definitions introduce the family of removable octahedron components, which added to 3-contractions and 4-contractions of edges lead to a minimal class of irreducible triangulations for any punctured surface in the spirit of Nakamoto and Negami's theorem in [17]. Recall the notation in Definition 3.

Definition 12. We will say that an octahedron component $\mathcal{O}$ in a triangulation $G \in \mathcal{F}_\circ^2(4)$ is *removable* in $\mathcal{F}_\circ^2(4)$ if the graph $G' = G - \{v_1, v_2, v_3\}$ remains in $\mathcal{F}_\circ^2(4)$. We also say that G' is obtained by *removing* the octahedron $\mathcal{O}$ from G. Conversely, G is obtained from G' by *adding an octahedron.*

Remark 13. If $G \in \mathcal{F}_\circ^2(4)$ has an octahedron component $\mathcal{O}$, then no edge of $\mathcal{O}$ is 4-contractible (see Remark 2). However, removing the inner set of vertices $\{v_1, v_2, v_3\}$ is equivalent to three consecutive edge 3-contractions (v_1a_2, v_2a_3 and v_3a_1, for instance). Therefore, we can regard this set of 3-contractions as a single operation within the class $\mathcal{F}^2_\circ(4)$ except in case that $\mathcal{O}$ is external.

From Definition 12, the following result gives us sufficient conditions for an octahedron being removable.

Remark 14. An octahedron component $\mathcal{O}$ is removable in $G \in \mathcal{F}^2_\circ(4)$ when any of the following cases holds:

- All vertices a_1, a_2, a_3, have degree ≥ 6.
- At least one of the vertices $\{a_1, a_2, a_3\}$ lies in ∂G and its degree is equal to 5. Observe that at least one boundary vertex of degree 3 appears after the removal of $\mathcal{O}$.

On the other hand, if $\mathcal{O}$ does not hit ∂G and $deg(a_i) = 5$ for some $i \in \{1, 2, 3\}$, then $\mathcal{O}$ is not removable but the edge $a_i v$ is 4c-edge where v is the only neighbour of a_i outside $\mathcal{O}$.

Observe that external octahedra are configurations with many vertices and edges which are not relevant from the topological point of view. We let them to be deleted according to the following definition.

Definition 15. Let $\mathcal{O}$ be an external octahedron component in a triangulation $G \in \mathcal{F}^2_\circ(4)$. If the graph $G' = G - \{a_j, v_i, i = 1, 2, 3\}$ remains in $\mathcal{F}^2_\circ(4)$, $\mathcal{O}$ is said to be *redundant* (see Figure 3). Conversely, we say that G is obtained from G' by *adding an octahedron* along a boundary edge of G'.

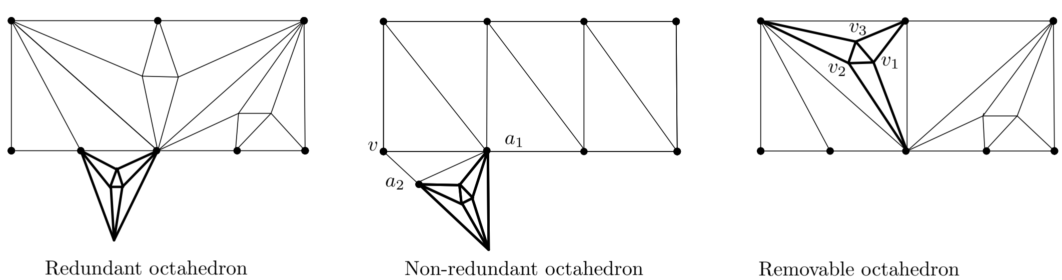

Figure 3: Different types of octahedra in triangulations of the Möbius strip. Here the surface is represented by a rectangular unfolding with the opposite vertical sides identified in the usual way.

Notice that after deleting any redundant octahedron component, the new triangulation; that is $G' = G - \{a_j, v_i, i = 1, 2, 3\}$, remains in $\mathcal{F}^2_\circ(4)$. Alternatively one can regard the deletion of a redundant octahedron component $\mathcal{O}$ as the composite of the folding of $\mathcal{O}$ into a face (as in Definition 22 below) and the removal of the folded octahedron component according Definition 12

Remark 16. Any addition of an octahedron in Definitions 12 and 15 is equivalent to apply three consecutive splittings in an appropriate set of vertices.

Thus, if G is a triangulation of any surface F^2 containing an octahedron component $\mathcal{O}$, then G is reducible. It suffices to check that the interior edges v_iv_j and a_iv_j of $\mathcal{O}$ in Figure 1 are contractible.

Notice that by contracting the three edges $v_i v_j$ of the triangulation on the right-hand side of Figure 3, we obtain another triangulation of the Möbius strip, depicted with thin lines.

Remark 17. An external octahedron $\mathcal{O}$ is not redundant whenever $deg(a_j) = 5$ for some $j \in \{1, 2, 3\}$ and this is the only impediment to being redundant. This provides a triangle $va_j a_k$ with a boundary $4c$-edge va_j (see Figure 3). After contracting va_j two possible situations appear:

(i) $\mathcal{O}$ becomes redundant.

(ii) $\mathcal{O}$ remains non-redundant.

In case *(ii)*, a new boundary $4c$-edge $v'a_j$ appears whose contraction leads us again to case *(i)* or *(ii)*. By iterating this procedure, $\mathcal{O}$ reaches situation (i) in finitely many steps. Otherwise G reduces to $\mathcal{O}$, and so G triangulates the disk.

Next we present the main theorems of this section.

Theorem 18. *Every triangulation $G \in \mathcal{F}^2_\circ(4)$, of a punctured surface F^2, except the disk, can be obtained from an irreducible triangulation of F^2 by a sequence of 3-splitting triodes, additions of flags, 4-splittings and additions of octahedra.*

Theorem 19. *Every triangulation $G \in \mathcal{F}^2_\circ(4)$ of the disk can be obtained from a flag or an octahedron component by a sequence of 3-splitting triodes, additions of flags, 4-splittings and additions of octahedra.*

The proofs of these results are consequence of the following technical lemma which deals with the possible configurations near the boundary. This is the crucial difference with the ordinary case of closed surfaces studied in [17].

Lemma 20. *Assume that ab is a $cn4c$-edge in G, that is, there is a face abx in G with $deg(x) \leq 4$.*

1. *Let $G \in \mathcal{F}^2_\circ(4)$ be a triangulation of a punctured surface different from the disk. If $d(ab, \partial G) \leq 1$, then either a $4c$-edge or a subgraph $H \subseteq G$ in the family*

$$\mathcal{A} = \{\textit{octahedron component},\ \textit{triode detecting edge},\ \textit{flag}\}$$

   can be found at distance at most 1 from ab.

2. *If G triangulates the disk, then the subgraph G' may reduce to a flag or an octahedron.*

For the sake of simplicity we will give the proof of Lemma 20 in the final appendix.

Remark 21. It is straightforwardly checked that an octahedron component $\mathcal{O}$ is not contractible to a flag within the family $\mathcal{F}^2_\circ(4)$; that is, any contraction of any inner edge in $\mathcal{O}$ produces an inner 3-valent vertex. Hence, flags and octahedra are needed for generating all triangulations with minimum inner degree 4.

Proof of Theorems 18 and 19. We will show that for any reducible triangulation $G \in \mathcal{F}^2_\circ(4)$, every $cn4c$-edge which is not a triode detecting edge lies in a removable octahedron or in a removable flag. This way, an irreducible triangulation G' can be obtained recursively from G. Conversely, G is constructed from G' by a sequence of 3-splitting triodes, additions of flags, 4-splittings and additions of octahedra.

Assume G contains neither $4c$-edges nor triode detecting edge. Since G is reducible, let ab be a $cn4c$-edge in G and therefore, a vertex x with $deg(x) \leq 4$ defines a face abx of G and $x_1abx_2 \subseteq link(x)$ (the edge x_1x_2 may exist or not). Let us remark that the case $d(ab, \partial G) \geq 2$ admits the same kind of arguments given in Lemma 1 of [17] for closed surfaces to find either an octahedron or a $4c$-edge. Hence, we focuss on the case $d(ab, \partial G) \leq 1$. In that case Lemma 20 leads us to one of the following cases:

(a) There exists a flag X such that $d(ab, X) \leq 1$, therefore X is removable (otherwise a $4c$-edge exists according to Remark 10).

(b) There exists an octahedron component $\mathcal{O}$ such that $d(ab, \mathcal{O}) \leq 1$, thus $\mathcal{O}$ is removable or redundant in $\mathcal{F}^2_\circ(4)$ (see Remarks 14 and 17). This finishes the proof.

□

4 On reductions of triangulations of degree at least 4.

Henceforth, unless otherwise is stated, by F^2 we mean any punctured surface.

Recall that $\mathcal{F}^2(4)$ denotes the set of triangulations of the surface F^2 with all its vertices of degree ≥ 4. In this section we give a series of reductions involving exclusively triangulations in $\mathcal{F}^2(4)$. The two operations introduced by Nakamoto and Negami in [17] are among such reductions and they are the only ones which are defined in absence of boundary. In particular, the triangulations of closed surfaces which are minimal for such reductions coincides with the irreducible triangulations in [17]. In sharp contrast with the class of closed surfaces, for a surface with non empty boundary, the minimal triangulations obtained by such reductions may contain contractible edges whose contraction produce 3-valent vertices. For this case, we prove in Theorem 38 that those possible contractible edges are located in two particular configurations given in Definitions 23 and 34 below.

Notice that Definitions 12 and 15 restrict to the family $\mathcal{F}^2(4)$ in the obvious way so that *removable* and *redundant* octahedra as well as *removing* and *addition* of such configurations are defined in $\mathcal{F}^2(4)$. Besides these operations, we introduce new ones in Definitions 22, 23, 25 and 28 below.

Definition 22. Let $G \in \mathcal{F}^2(4)$ be a triangulation of the surface F^2. Let $\mathcal{O}$ be an external octahedron of G so that $deg(a_3) = 4$ and $deg(a_i) = 6$ for $i = 1$ or 2. Let v be a vertex of G such that a_1a_2v is a face of G. By *folding the octahedron $\mathcal{O}$ onto the face a_1a_2v* we mean the removal of $\mathcal{O}$ followed by the addition of an octahedron to the face a_1a_2v (Figure 4). The inverse operation is called *unfolding an octahedron* with respect to the boundary of G.

Definition 23. Let $G \in \mathcal{F}^2(4)$ be a triangulation of F^2. A *quasi-octahedron component* of G, $\widehat{\mathcal{O}}$, is said to be *removable in* $\mathcal{F}^2(4)$ (or *4-removable*, for short) if one of the following conditions holds:

1. The graph $G' = G - \{v_1, v_2, v_3\}$ yields a triangulation of F^2 in $\mathcal{F}^2(4)$.

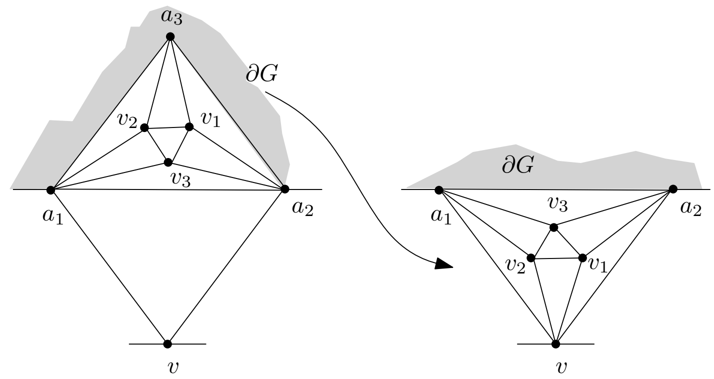

Figure 4: Folding the octahedron $\mathcal{O}$ onto the face a_1a_2v.

2. The graph $G' = (G - \{v_i, v_j\})/a_iv_k$, with $v_k \in \partial G$, yields a triangulation of F^2 in $\mathcal{F}^2(4)$.

In both cases, we will simply say that G' is obtained by *removing a quasi-octahedron* from G. Conversely, if case (1) happens, we say that G is obtained from G' by *adding a quasi-octahedron* along two boundary edges a_ia_3 (for $i = 1, 2$) of G'. In case (2) we say that G is obtained from G' by *embedding a quasi-octahedron* in a boundary face $a_1a_2a_3$ of G'.

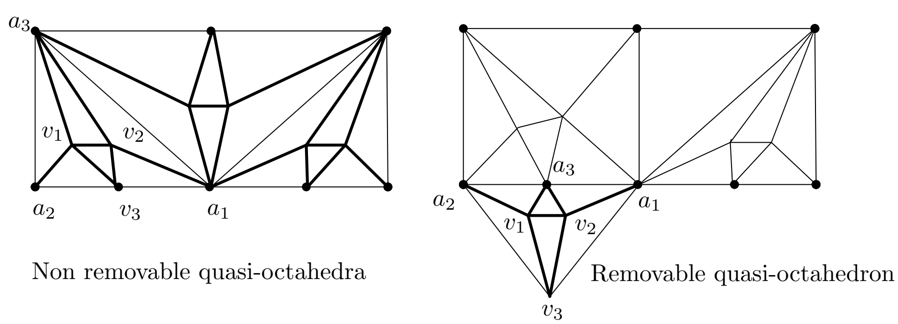

Figure 5: Triangulations for the Möbius strip with some quasi-octahedra components.

Remark 24. Adding a quasi-octahedron is equivalent to apply three consecutive splittings starting in a boundary vertex.

As a consequence, if G is a triangulation of a surface F^2 containing a quasi-octahedron component $\widehat{\mathcal{O}}$, then G is reducible. Indeed, similarly to Remark 16, the interior edges v_iv_j and a_iv_j of $\widehat{\mathcal{O}}$ in Definition 23 are readily checked to be *cn*4*c*-edges.

Definition 25. Let $G \in \mathcal{F}^2(4)$ be a triangulation of the surface F^2. Let $\mathcal{O}$ be an octahedron component of G so that only the edge a_1a_2 lies in the boundary and $deg(a_2) = 5$. Let v be the only neighbor of a_2 outside $\mathcal{O}$. The *replacement of the boundary octahedron* $\mathcal{O}$ by a quasi-octahedron $\widehat{\mathcal{O}}$ is defined to be the removal of the edge a_1a_2 followed by the contraction of the edge a_2v in G. (Figure 6). The inverse operation is called the *replacement of the quasi-octahedron* $\widehat{\mathcal{O}}$ by a boundary octahedron $\mathcal{O}$.

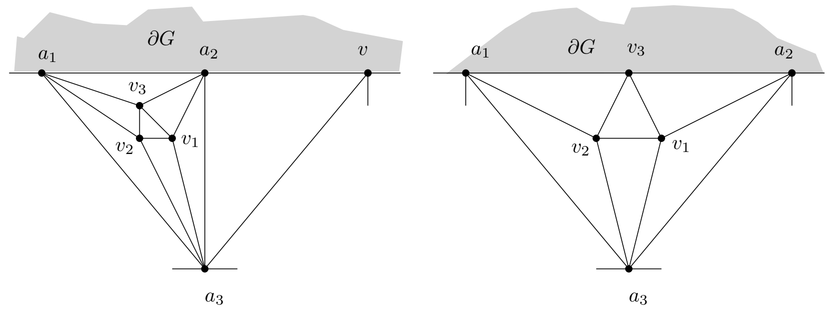

Figure 6: A replacement of a boundary octahedron by a quasi-octahedron.

Remark 26. Let us observe that the operation in Definition 25 is well defined, that is the edge a_2v always becomes 4-contractible after the removal of a_1a_2. Indeed, otherwise, there would be a critical 3-cycle a_1a_2v but no such a cycle exists when the edge a_1a_2 is removed. Notice also that the replacement of a boundary octahedron is needed only in case that the edge a_2v is not contractible (or, equivalently, 4-contractible since $deg(a_3) \geq 5$). Moreover, if $a_1v \in \partial G$ ($a_1v \notin \partial G$, respectively), then an octahedron (quasi-octahedron, respectively) component would be obtained after applying this reduction. In case $a_1v \notin \partial G$, a quasi-octahedron would be obtained.

Observe that with the previous operations, triangulations with arbitrarily many vertices may appear by repeating the pattern shown in Figure 8 and they could not be simplified (or reduced). Next we define a new operation in order to avoid such an undesirable repetitive construction.

Definition 27. Let $G \in \mathcal{F}^2(4)$ be a triangulation of the surface F^2. An *N-component*, of G consists of a subgraph $\mathcal{N}$ of G determined by two faces sharing an edge, where their two non-incident edges are contractible and at least one of them lies in ∂G (Figure 7).

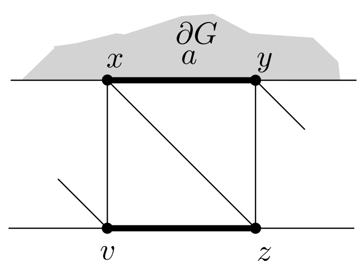

Figure 7: $N-$component: two faces share the edge xz the non-incident edges xy and vz must be contractible and, at least one of them must lie in ∂G.

Definition 28. Let $G \in \mathcal{F}^2(4)$ be a triangulation of the surface F^2. An N-component $\mathcal{N} \subset G$ is termed *contractible* if both contractible edges lie in the boundary or else some inner vertex in $\mathcal{N}$ has degree ≥ 5. In such cases the simultaneous contraction of the two non-incident contractible edges in $\mathcal{N}$ can be carried out within the family $\mathcal{F}^2(4)$. This reduction will be called a *double contraction*.

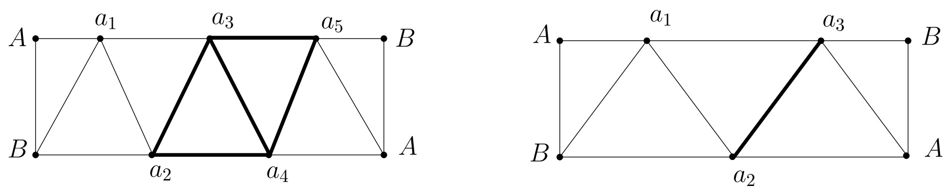

Figure 8: The double contraction of the edges a_2a_4, a_3a_5 in a triangulation of the Möbius strip. In this example, both contractible edges lie in ∂G.

Remark 29. Observe that any double contraction performed in an N-component with its two inner vertices of degree 4 expelled the triangulation from $\mathcal{F}^2(4)$. In particular, no double contraction is allowed in any of the two N-components sharing an edge in any quasi-octahedron component.

The operations defined above and their inverses are summarized in Table 1 where they are classified into two classes: reductions and expansions.

	reductions		**expansions**	**Figure**
R_1	edge contraction	E_1	vertex splitting	
R_2	octahedron removal	E_2	octahedron addition	Figure 3
R_3	folding an octahedron	E_3	unfolding an octahedron	Figure 4
R_4	quasi-octahedron removal	E_4	quasi-octahedron addition	Figure 5
R_5	boundary octahedron replacement	E_5	quasi-octahedron replacement	Figure 6
R_6	double contraction in a N-component	E_6	double splitting of vertices	Figure 7

Table 1:

By a 4-reduction (4-expansion, respectively) we mean any reduction (expansion, respectively) in Table 1 which provides a new triangulation in $\mathcal{F}^2(4)$. Notice that reductions and expansions R_i, E_i for $i \geq 3$ are always 4-reductions and 4-expansions.

The operations R_1 and E_1 preserving $\mathcal{F}^2(4)$ are the usual 4-contractions and 4-splitting of Nakamoto and Negami in [17].

By the use of these operations we eventually get a class which is minimal in $\mathcal{F}^2(4)$ in the sense of the following definition.

Definition 30. A triangulation $G \in \mathcal{F}^2(4)$ of the surface F^2 is said to be *minimal in* $\mathcal{F}^2(4)$ (or 4-*minimal*[1], for short) if G does not admit any further 4-reduction.

In particular, the 4-minimal triangulations in $\mathcal{F}^2(4)$ of any non-spherical closed surface coincide with the usual irreducible ones since operations R_i and E_i, for $i \geq 3$ in Table 1 make sense only for boundary surfaces triangulations. This way, Theorem 1 in [17] can be restated as follows.

Theorem 31. *Any triangulation of a closed surface in* $\mathcal{F}^2(4)$ *can be obtained from a 4-minimal triangulation by a sequence of 4-expansions.*

[1]This term appears in [16] with a different meaning.

Next we focuss our interest on punctured surfaces. Let us start by considering the following examples.

Example 32. a) Let G be the triangulation of the punctured connected sum of three projective planes constructed by identifying the edges with equal labels. See Figure 9. Notice that G is 4-minimal, it contains a non-removable quasi-octahedron component and a $cn4c$-edge, namely ab, outside the quasi-octahedron component.

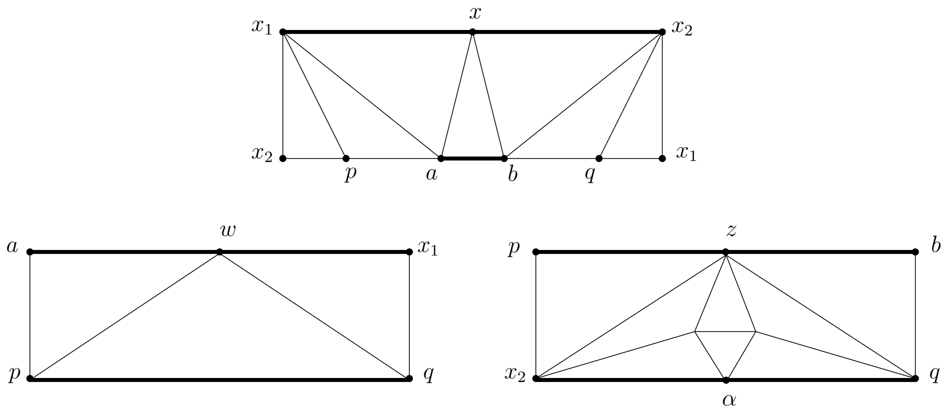

Figure 9: Observe that $\delta(a) = \delta(b) = 5$, $\delta(x_1) = \delta(x_2) = 6$ and ∂G consists of bold edges.

b) Let G be the triangulation of the punctured connected sum of three projective planes with a torus, depicted in Figure 10. The connected sum is done by firstly identifying the edges with equal labels (namely x_2p, pa, bq) and then by attaching two copies of M_3 along the boundaries of the polygons labeled M_3 in the strip on the right-top. Notice that G is 4-minimal and ab is its only $cn4c$-edge. Here, M_3 denotes the so-called irreducible triangulation of the Möbius strip collected in [8].

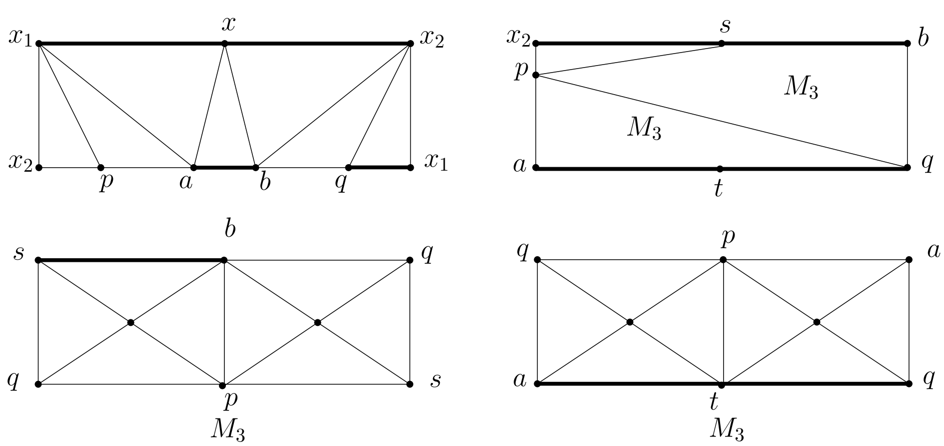

Figure 10: Observe that $\delta(a) \geq 6$, $\delta(b) \geq 6$, $\delta(x_2) \geq 6$, $\delta(x_1) = 5$. Here $\partial G = absx_2xx_1qta$.

c) Let G be the triangulation of the punctured torus depicted in Figure 11. Notice that G is 4-minimal and it contains a unique non-removable quasi-octahedron component.

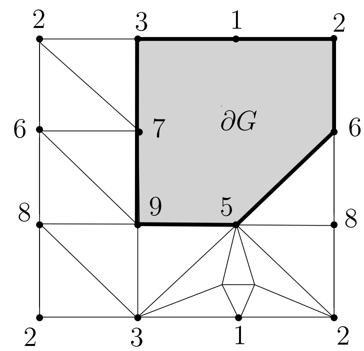

Figure 11: A 4-minimal triangulation of the punctured torus with precisely one non-removable quasi-octahedron component. Here, the triangulation is obtained by removing vertex 4 in T^{13} triangulation of the torus given in [13] and $\partial G = 31265973$.

Remark 33. Examples 32 a) and b) illustrate a procedure to construct 4-minimal triangulations by leaving M fixed and by choosing as M_3 any irreducible triangulation of a punctured surface with appropriate boundary length.

Example 32 depicts 4-minimal triangulations with contractible edges. In fact, as we prove in Theorem 38 bellow, the configurations of contractible edges in any 4-minimal triangulations are exactly two: the non-removable quasi-octahedron and the one termed M-configuration in the following definition.

Definition 34. Let $G \in \mathcal{F}^2(4)$ be a triangulation of the surface F^2. Let abx be a face with x a 4-valent vertex and ab a boudary *cn*4*c*-edge. An *M-component* centered at abx in G consists of a subgraph $\mathcal{M} \subseteq G$ determined by three faces $\{xab, xax_1, xbx_2\}$ such that xx_1, xx_2 lie in the boundary and x_1x_2 is an inner edge. Notice that xx_1x_2x is a critical 3-cycle. (Figure 12).

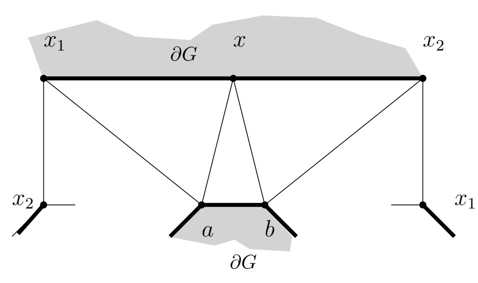

Figure 12: M-component centered at abx.

Remark 35. In Figure 9 above both a non-removable quasi-octahedron and an M-component appear. On the other hand, each Figure 10, 13, 14 only contains one M-component. Finally, Figure 11 contains exactly a non-removable quasi-octahedron.

Remark 36. In any M-component centered at abx in a triangulation $G \in \mathcal{F}^2(4)$, the boundary edge ab is triode detecting and $deg(x_i) \geq 5$ for $i = 1, 2$. Hence there are vertices p, q so that apx_1 and bqx_2 are faces in G. This way, $\{q, x_2, p, a, x\} \subseteq V(link(x_1))$ and $\{p, x_1, q, b, x\} \subseteq V(link(x_2))$. Moreover,

1. If $deg(a) = 4$ (or $deg(b) = 4$), then, ap (bq respectively) is a boundary 4c-edge. Möbius
2. Otherwise, there are faces apw, bqr. If $deg(a) = 5$ (or $deg(b) = 5$) then the edge $aw \subset \partial G$ and it is not a triode detecting edge although aw may be contractible.

Let us observe that in case that all 3-cycles x_1x_2p, x_1ap, x_1x_2q and x_2bq are faces of the triangulation, the M-component centered at abx coincides with the triangulation obtained by the splitting of a 5-valent boundary vertex in the irreducible triangulation of the strip M_2 collected in [8].

On the other hand, it is not difficult to see that the set of vertices $\{a, b, x_1, x_2, x\}$ in the M-component are principal vertices of a subdivision of the complete graph K_5 in G. Hence, M is not present in any triangulation of the disk.

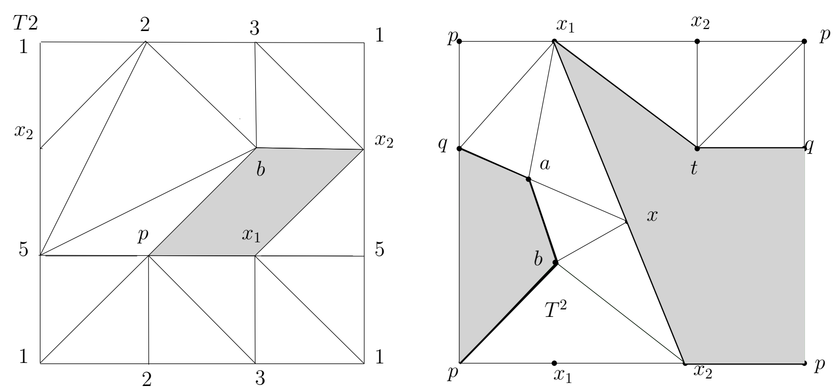

Figure 13: A triangulation of the punctured double torus with precisely one M-component centered at abx. Here $\partial G = x_1xx_2pbaqtx_1$. The connected sum is done by attaching the punctured torus T^2 along the cycle bpx_1x_2b on the right side torus.

The interest of the M-component is pointed out by the following proposition.

Proposition 37. *Let $G \in \mathcal{F}^2(4)$ be a triangulation of the surface F^2, different from the disk. Any M-component $\mathcal{M} \subset G$ remains unaltered after performing any reduction R_i ($i = 1, \ldots, 6$).*

Proof. Let $\mathcal{M}$ be an M-component centered at abx. The edge ab is the only contractible one in $\mathcal{M}$ (in fact, it is a $cn4c$-edge), hence no reduction R_1 can be applied to $\mathcal{M}$.

Furthermore, the only possible octahedron or quasi-octahedron components containing ab must be centered at abx (since $deg(x_i) \geq 5$ for $i = 1, 2$, by Remark 36). However, such a quasi-octahedron component can not exist since $a, b \in \partial G$. Similarly no octahedron components exists since otherwise ∂G reduces to abx. Therefore, no reduction R_2 to R_5 may be performed to $\mathcal{M}$.

Finally, the existence of an N-component containing ab is ruled out by existence of the edge x_1x_2. Then, reduction R_6 does not affect $\mathcal{M}$. This finishes the proof. $\square$

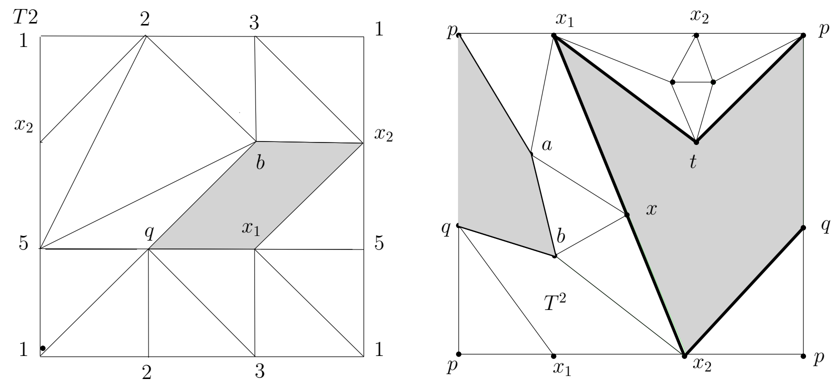

Figure 14: A triangulation of the punctured double torus with precisely one M-component centered at abx and one non-removable quasi-octahedron. Here $\partial G = x_1 x x_2 q b a p t x_1$. The construction is done in a similar way as in Figure 13.

We are now ready to establish and prove the main result of this section. Namely,

Theorem 38. *A triangulation $G \in \mathcal{F}^2(4)$ of the surface F^2, different from the disk, is 4-minimal if and only if each contractible edge (if any) of G is located in either a non-removable quasi-octahedron component or an M-component.*

Theorem 39. *The only 4-minimal triangulation of the disk is the octahedron.*

In order to prove Theorems 38 and 39 we will need the following technical lemmas.

Lemma 40. *Let $G \in \mathcal{F}^2(4)$ be a triangulation of the surface F^2 different from the disk. Assume in addition that G contains a non-4-removable octahedron component, $\mathcal{O}$, then exactly one of the following statements hold:*

1. *There is at least one 4c-edge $a_i t$ with $a_i \in \mathcal{O}$, $t \notin \mathcal{O}$ and $\mathcal{O}$ turns to be 4-removable after contracting $a_i t$.*
2. *$\partial G \cap V(\mathcal{O}) \subseteq \{a_1, a_2, a_3\}$ and this intersection contains at least two vertices a_i, a_j.*

Proof. Since $\mathcal{O}$ is non-removable, $\mathcal{O}$ has an external vertex, say a_1, such that $deg(a_1) = 5$. Let us suppose that $\mathcal{O}$ does not intersect the boundary, then $a_1 \notin \partial G$, $deg(a_2), deg(a_3) \geq 6$. Hence there is an edge $a_1 t$ in $G - E(\mathcal{O})$ existing the two faces $t a_1 a_2$ and $t a_1 a_3$ in G. This way, $a_1 t$ is a 4c-edge. After contracting it, $deg(a_1)$ increases and the octahedron turns to be 4-removable.

Moreover, if the intersection $V(\mathcal{O}) \cap \partial G$ reduces to a single vertex, let us suppose that this intersection is precisely the vertex a_2. Since no edge $a_i a_j$ lie in ∂G then $deg(a_2) \geq 6$ holds and since $\mathcal{O}$ is non-removable, $deg(a_i) = 5$ for $i = 1$ or $i = 3$. Let us suppose $deg(a_1) = 5$ and let t be the boundary vertex adjacent to a_2 and a_1. Therefore $t a_1 a_2$ and $t a_1 a_3$ are faces of G, and since a_1 is not a boundary vertex, it readily follows that $a_1 t$ is a 4c-edge of G. Again, after contracting it, the octahedron become 4-removable. Therefore, $\partial G \cap V(\mathcal{O})$ contains at least two vertices a_i, a_j. □

In the following lemma we will use the operations R_i in Table 1.

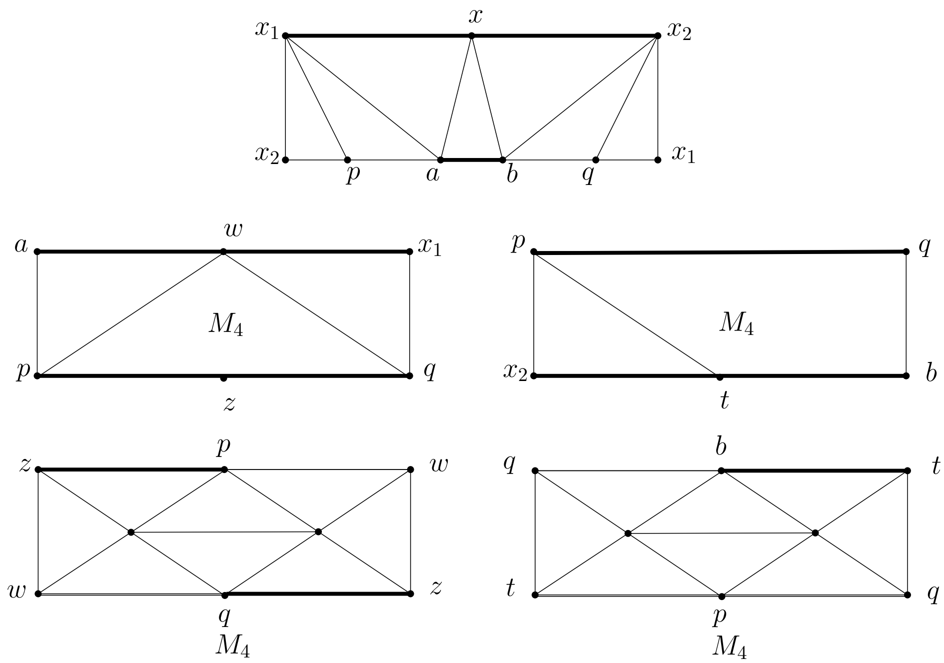

Figure 15: Observe that $\delta(a) = 5$, $\delta(b) \geq 6$, $\delta(x_2) \geq 6$, $\delta(x_1) \geq 6$.

Lemma 41. *Let $G \in \mathcal{F}^2(4)$ be a triangulation of the surface F^2, and let $\mathcal{O}$ be a non-removable octahedron component of G with at least two vertices a_i, a_j in ∂G. Then*

1. *If $E(\mathcal{O}) \cap \partial G = \{a_1 a_2\}$ and $deg(a_2) = 5$, then, there is precisely one vertex $v \in link(a_2) - V(\mathcal{O})$ and by applying an operation R_5 to G, the new triangulation G' belongs to $\mathcal{F}^2(4)$.*
2. *If $E(\mathcal{O}) \cap \partial G = \{a_2 a_3, a_1 a_3\}$ and $\delta(a_1) = 6$ or $\delta(a_2) = 6$, then by applying an operation R_3 to G, the new triangulation G' belongs to $\mathcal{F}^2(4)$.*
3. *If $E(\mathcal{O}) \cap \partial G = \emptyset$, then there exists precisely one vertex a_j such that $deg(a_j) = 5$ and there is a 4c-edge incident with a_j.*

Proof. It is straightforwardly deduced from the definitions involved in the statements and Lemma 40 (2). Observe that $V(\mathcal{O}) \cap \partial G = \{a_1, a_2, a_3\}$ implies that $\mathcal{O}$ is 4-removable. □

As a consequence of Lemmas 40 and 41 we get

Corollary 42. *Any octahedron component of a triangulation in $\mathcal{F}^2(4)$ of the surface F can be deleted by one of the reductions R_1, R_2, R_3 or R_5 of Table 1.*

The following is the corresponding analogue of Lemma 20 for the class $\mathcal{F}^2(4)$.

Lemma 43. *Let $G \in \mathcal{F}^2(4)$ be a triangulation of the surface F^2. If ab is a cn4c-edge in G so that $d(ab, \partial G) \leq 1$, then one of the following configurations can be found at distance at most 1 from ab:*

1. *A* $4c$*-edge*
2. *A subgraph in the family*
   $\mathcal{B} = \{$*octahedron component, quasi-octahedron component,* $N-$*component*$\}$
3. *An* M*-component centered at* abx*.*

The proof of Lemma 43 is a specialization of the proof of Lemma 20 and it will be postponed to the final appendix.

Proof of Theorems 38 and 39: Let ab a contractible edge in G. As G is 4-minimal, ab is a $cn4c$-edge. Moreover, if $d(ab, \partial G)) \geq 2$ then the same arguments given in Lemma 1 of [17] for closed surfaces allows us to find a $4c$-edge or an octahedron component at distance ≤ 1 from ab. This contradicts the 4-minimality of G. Thus, necessarily, $d(ab, \partial G) \leq 1$ and Lemma 43, Corollary 42 and, again, the 4-minimality of G yield that ab lies in a non-removable quasi-octahedron component or an M-component.

Conversely, if the contractible edge ab belongs to a non-removable quasi-octahedron component $\widehat{\mathcal{O}}$ then it is not a $4c$-edge since $\widehat{\mathcal{O}}$ does not contain such edges by Remark 13. Moreover, $\widehat{\mathcal{O}}$ cannot be extended to an octahedron in G by Remark 5. Finally, no N-component contained in $\widehat{\mathcal{O}}$ can be reduced by Remark 29. Hence, no reduction R_i can be applied to remove ab.

On the other hand, if ab belongs to an M-component $\mathcal{M} \subset G$, we know by Proposition 37 that $\mathcal{M}$ is stable under reductions R_i $(i = 1, \ldots, 6)$. This finishes the proof of Theorem 38.

Let us consider the case of the triangulated disk. From Remark 36 no M-component may appear in a triangulation of the disk. Besides, a quasi-octahedron component $\widehat{\mathcal{O}}$ will be always removable according to Definition 23. In fact, it is clear that the degree ≥ 4 condition expels the quasi-octahedron from the set of disk triangulations. Moreover, according to Definition 4, vertex a_3 must have degree ≥ 5. Let a_3t be an edge with t outside $\widehat{\mathcal{O}}$. Observe that $deg(a_3) = 5$ leads to the contractibility of at, which contradicts the minimality of G, hence $deg(a_3) \geq 6$. Besides, $deg(a_i) \geq 5$ for $i = 1, 2$ since otherwise a 4-contractible edge incident at a_i appears, which is impossible. Therefore, $\widehat{\mathcal{O}}$ can be removed by applying Definition 23 (1) if $deg(a_i) \geq 6$ for $i = 1, 2$ and $a_3 \in \partial G$ or Definition 23 (2) otherwise. This finishes the proof of Theorem 39.

□

Corollary 44. *Let* G *be a triangulation of a punctured surface different from the disk such that* G *is* 4*-minimal. Then* G *is irreducible if and only if* G *contains neither quasi-octahedron component nor* M*-component.*

Theorem 38 shows that 4-reductions do not suffice to get all irreducible triangulations within the class $\mathcal{F}^2(4)$. If, similarly as in [11] for closed surfaces, we allow diagonal flips then we get the following theorem.

Theorem 45. *If diagonal flips are added to 4-reductions as admissible operations in the family* $\mathcal{F}^2(4)$ *of triangulations of any punctured surface* F^2*, then the 4-minimal triangulations reduce to the irreducible triangulations in* $\mathcal{F}^2(4)$*.*

Proof. The diagonal flip operation is a way of getting rid of quasi-octahedra and M-components in 4-minimal triangulations. For instance, if we flip the edge xa in an M-configuration when $\deg(a) \geq 5$ (similarly, flip xb when $\deg(b) \geq 5$) we still have a triangulation in $\mathcal{F}^2(4)$ but now the edge ab is 4-contractible. Notice that $deg(x_1) \geq 5$ by definition of an M-configuration and, moreover, that some 4-contractible edge is detected whenever $\deg(a) = 4$ ($deg(b) = 4$, respectively); see Remark 36(1).

On the other hand, by flipping an edge a_ia_3 of a quasi-octahedron component, new 4-contractible edges are available to perform further 4-reductions and dismantle the original quasi-octahedron component.

This way, any 4-minimal triangulation turns to be irreducible within the class $\mathcal{F}^2(4)$. □

Appendix: Proofs of Lemmas 20 and 43.

In order to prove Lemma 20, let $G \in \mathcal{F}^2_\circ(4)$ be a fixed triangulation of the punctured surface F^2. Assume that ab is a $cn4c$-edge in G, that is, there is a face abx in G with $deg(x) \leq 4$. We start with the following technical lemmas which detect possible contractible edges and $4c$-edges around ab.

Lemma 46. *Let ab be a $cn4c$-edge of $G \in \mathcal{F}^2_\circ(4)$ so that $deg(a) \geq 4$, $deg(b) \geq 4$, and let x be a vertex of degree 4 so that abx defines a face of G and $V(link(x)) = \{x_1, a, b, x_2\}$. Then, the following statements hold:*

1. *ax_2 and bx_1 are not edges of G.*
2. *Whenever ab and x do not intersect ∂G simultaneously, ax and bx are contractible edges of G.*
3. *Whenever xx_1x_2 is not a critical 3-cycle, both xx_1 and xx_2 are contractible edges of G.*
4. *If x is an inner vertex of G and the two vertices $\{a, x_2\}$ ($\{b, x_1\}$, respectively) have degree ≥ 5, then the edges xv with $v \in \{b, x_1\}$ ($\{a, x_2\}$, respectively), are 4-contractible.*

Proof. Suppose ax_2 (or bx_1) is an edge of G, since $deg(b) \geq 4$, the 3-cycle ax_2ba (ax_1ba) does not define a face of G, and hence ab lies in a critical 3-cycle, contradicting the hypothesis.

As a consequence, if ab and x do not intersect ∂G simultaneously, then the edge ax is contractible. Otherwise, a critical 3-cycle contains ax, namely $axta$ with $t \in \{b, x_1, x_2\}$, but it is easily deduced that $t = x_2$ is the only possibility, reaching a contradiction. A similar argument works for proving the contractibility of bx. Similarly, xx_1 and xx_2 are contractible whenever xx_1x_2 is not a critical 3-cycle. □

Lemma 47. *If $deg(x) = 3$ then there exists a triode detecting edge, a flag or a $4c$-edge meeting $link(x)$.*

Proof. Observe that if $deg(x) = 3$, then $x \in \partial G$, $link(x) = x_1ab$ and a or b is an inner vertex, hence $deg(a) \geq 4$ or $deg(b) \geq 4$. Let us consider a to be the inner vertex, whence $x_1, b \in \partial G$.

Assume $deg(a) = 4$. Then a flag centered at a appears whenever $deg(x_1) = 3$ or $deg(b) = 3$. In case $deg(x_1) = 4$ or $deg(b) = 4$, the edge xa is a triode detecting. This edge turns to be a $4c$-edge when $deg(x_1), deg(b) \geq 5$.

On the other hand, if $deg(a) \geq 5$, by Lemma 46 the edge of vertices x_1 and b does not exist.

□

In order to simplify the notation, let $V(link(x)) = \{x_1, a, b, x_2\}$ be the vertex set of the link of the vertex x fixed in Lemma 46. Then the degree of vertices defines a map $\delta : V(link(x)) \to \{n \in \mathbb{N}; n \geq 3\}$ by $\delta(v) = deg(v)$.

Henceforth, let $m = min(\delta)$ denote the minimum of this map, and $\sharp Min$ be the cardinal of the set $Min = \delta^{-1}(m)$.

Proof of Lemma 20

Since ab is a $cn4c$-edge in G, there is a face abx in G with $deg(x) \leq 4$. The case $deg(x) = 3$ is studied in Lemma 47. For $deg(x) = 4$, recall that we denote $link(x) = x_1abx_2x_1$ if $link(x) = \{x_1a, ab, bx_2, x_2x_1\}$ and $link(x) = x_1abx_2$ if $link(x) = \{x_1a, ab, bx_2\}$. (See Figure 2).

Case 0. Notice that if $x \in \partial G$ and $ab \subset \partial G$, then it is clear that ab is a triode detecting edge.

According to Case 0, we will consider hereafter that x and ab do not lie simultaneously in ∂G.

Case 1: m=3. Since three or more vertices of degree 3 lead to the triangulated disk, G is isomorphic to the wheel graph of 4 radii, i.e. a flag, and we reach statement 2 in the Lemma.

Next we deal with the case $\sharp Min = 2$. If Min consists of two adjacent vertices then one readily finds a flag centered at x. Otherwise if two vertices in Min are not adjacent, the only possibility is that $x \in \partial G$. Let $v \in Min$ and let us consider $link(v) = \{x, t_1, t_2\}$ with $vt_2 \in \partial G$. If $t_1 \in G - \partial G$ and $deg(t_1) \geq 4$, then vt_1 is a triode detecting edge. If $t_1 \in \partial G$, then, by hypothesis, $deg(t_1) \geq 4$ and xv is contractible and so a $4c$-edge if $deg(t_1) \geq 5$ or a triode detecting edge if $deg(t_1) = 4$.

Finally, assume $Min = \{v\}$ reduces to a single vertex and $V(link(v)) = \{x, t_1, t_2\}$. We consider the two following cases.

Case 1.1: x is independent of degree 4.

Since there are three vertices $u \in V(link(x))$ verifying $deg(u) \geq 5$, Lemma 46 shows that xv is a $4c$-edge unless x and ab lie in the boundary of G simultaneously and x_1x_2 is an inner edge of G (otherwise ab is a triode detecting edge, as pointed out in Case 0).

Case 1.2: x is adjacent to some vertex of degree 4.

1.2.1 If $x \in G - \partial G$, then $vt_i \in \partial G$ for $i = 1, 2$ and xv is a contractible edge. Moreover, xv is a $4c$-edge whenever $deg(t_i) \geq 5$ and it is a triode detecting otherwise.

1.2.2 If $x \in \partial G$ there are two possibilities:

a) $v \in \{a, b\}$. Let us suppose $v = a$ (analogously for $v = b$). In this case $vt_2 = at_2 \in \partial G$ and $b \in G - \partial G$. Now, ab is triode detecting edge if $deg(t_2) \geq 4$ and xa is a $4c$-edge if $deg(t_2) = 3$ (since $t_2 \neq x_1$ and $deg(t_2) = 3$ implies $deg(b) \geq 5$).

b) $v \in \{x_1, x_2\}$. If $v = x_1$ (analogously for $v = x_2$) then $vt_2 = x_1t_2 \in \partial G$. In this case, ax_1 is triode detecting edge if $deg(t_2) \geq 4$ and xx_1 is a $4c$-edge if $deg(t_2) = 3$ (since $t_2 \neq x_2$ and $deg(t_2) = 3$ implies $deg(a) \geq 5$).

Case 2: m $\geq$ 4. If $m \geq 5$, Lemma 46.(4) and the assumption after Case 0 yield that a $4c$-edge incident in x must appear. The same situation occurs when $m = 4$ and $\sharp Min = 1$.

Thus Case 2 reduces to $m = 4$ and $\sharp Min \geq 2$. Let $u, v \in Min$ two distinct vertices. We will study the following possibilities according to the positions of the vertex x and the edge ab with respect to ∂G:

2.1 **$x \in \partial G$ and a and b are inner vertices** (and hence, ab is an inner edge). Observe that $x_i \in \partial G$ for $i = 1, 2$ since $deg(x) = 4$. Let aby be the other face sharing ab with abx.

If y is a boundary vertex or else is an inner vertex of degree at least 5, then ab is a triode detecting edge, by definition. Let us study the case when y is an inner vertex of degree 4.

2.1.1 Suppose that uv is an edge. Then xuv is a 3-cycle and, moreover, by Lemma 46 $uv \neq x_2a$, x_1b for the vertices $u, v \in Min$ chosen above. If x_1x_2 is an edge, an octahedron centered at xuv is found. If x_1x_2 is not an edge the possibilities of the edge uv are:

1) $uv = ab$ then an octahedron centered at yab appears.

2) $uv = x_1a$ (analogously $uv = x_2b$). We can assume $b \notin Min$ ($a \notin Min$, respectively) since, otherwise, we are in, previous subcase 1). Then the edge xx_1 (xx_2, respectively) is 4-contractible.

2.1.2 Suppose that uv is not an edge (that is uxv is an arc). Notice that if $\sharp Min \geq 3$ then at least two vertices in Min form a face with x, and we are in case 2.1.1. Thus we can assume $Min = \{u, v\}$ and Lemma 46 (2) yields that either xb or xa is a $4c$-edge.

2.2 **x, a and b are inner vertices.**

2.2.1 Suppose that uv is an edge (or, equivalently, xuv is a face). If, in addition, u and v are inner vertices (in particular, if $uv = ab$) then an octahedron centered at xuv is found. Thus we can assume that $\{a, b\} \nsubseteq Min$ and $\{u, v\} \cap \partial G \neq \emptyset$.

1) If $uv = x_1a$, as $b \notin Min$ and $x_1 \in \partial G$, we easily check that xa is a triode detecting edge. Similarly, if $uv = x_2b$ ($x_2 \in \partial G$), xb turns to be a triode detecting edge.

2) If $uv = x_1x_2$, we can assume $Min = \{x_1, x_2\}$ (otherwise we are in one of the previous situations) and then xx_1 or xx_2 are triode detecting edges. Recall $\{x_1, x_2\} \cap \partial G \neq \emptyset$.

2.2.2 Suppose that uv is not an edge. The same arguments as in 2.1.2 reduces this case to 2.2.1 if $\sharp Min \geq 3$ or, otherwise, $Min = \{u, v\}$ and Lemma 46 (2) yields that either xb or xa is a $4c$-edge.

2.3 x **is an inner vertex and** ab **inner edge at distance 0 from** ∂G (that is, precisely a or b (but not both) lies in ∂G). Let us suppose $a \in \partial G$ and b an inner vertex (thus xb is an inner edge). From Lemma 46 (2), xb is a contractible edge. Moreover, if $\{a, x_2\} \cap Min = \emptyset$, then xb is a $4c$-edge. Otherwise, $\{a, x_2\} \cap Min \neq \emptyset$ and this case reduces to previous cases. Indeed, if $a \in Min$, we are in case 2.1. with a playing the role of x. Similarly, if $x_2 \in Min$, x_2 can play the role of x in case 2.1 and 2.2 when $x_2 \in \partial G$ and $x_2 \notin \partial G$, respectively.

2.4 $x \in \partial G$ **and** ab **is an inner edge at distance 0 from** ∂G (that is $\partial G \cap \{a, b\}$ reduces to a vertex). Assume $a \in \partial G$ (the case $b \in \partial G$ is analogous) and let aby be the other face containing ab. If ab is a triode detecting, we are done; otherwise (from definition of triode detecting edge) y must be an inner vertex with $deg(y) = 4$. This case was studied just in the case 2.3 by interchanging x and y.

2.5 x **is an inner vertex and** $ab \in \partial G$. From Lemma 46(2) xa and xb are contractible edges. Since $m = 4$, we are in case 2.3 when $x_i \in Min - \partial G$, for some $i = 1, 2$, with xa or xb playing the role of ab, for $i = 1, 2$, respectively. Otherwise, we are in case 2.4. and xa (analogously xb) playing the role of ab and the role of x is played by u where $u \in Min - \{a\}$.

□

The proof of Lemma 43 only deals with the occurrences of triode detecting edges in the proof of Lemma 20 as explained below. In Table 2 we summarize the conditions under which a triode detecting edges are located in case 2 of the proof of Lemma 20.

$m \geq 4$, $u, v \in V(link(x))$, with $u, v \in Min$ (see Case 2)

Case 2.1.	$x \in \partial G$ $a, b \in G - \partial G$	$y \in \partial G$
		$y \notin \partial G$ and $deg(y) \geq 5$
Case 2.2.	$x, a, b \in G - \partial G$	uv edge, $uv \cap \partial G \neq \emptyset$ and $a \notin Min$ or $b \notin Min$
Case 2.4.	$x, a \in \partial G$ $b \in G - \partial G$	$deg(y) \geq 5$ or $deg(y) = 4$ with $y \in \partial G$

Table 2: Occurrences of triode detecting edges in the proof of Case 2 of Lemma 20.

Proof of Lemma 43

Let us start by fixing a triangulation $G \in \mathcal{F}^2(4)$ of the surface F^2. As $\mathcal{F}^2(4) \subseteq \mathcal{F}^2_0(4)$, we can follow the pattern of the proof of Lemma 20 above. Since G does not contain flags, only the cases in the proof of Lemma 20 when a triode detecting edge appears require a deeper analysis (otherwise the same arguments as in the proof of Lemma 20 work). Recall that triode detecting edges appear only in Case 0 and Case 2 of that proof. Occurrences in Case 2 are described in Table 2.

Case 0 $x \in \partial G$ and $ab \subset \partial G$

Case 0.1 m=4 and the edge x_1x_2 does not exist. If $a \notin Min$, then the edge xx_1 turns to be a $4c$-edge. Otherwise, there is an N-component with parallel edges xx_1, ab.

Case 0.2 m=4 and the edge x_1x_2 exists (and it is necessarily an inner edge). Observe that $Min = \{a, b, x_1, x_2\}$ yields that G is necessarily the irreducible triangulation M_1 (K_6 minus a vertex) of the Möbius strip given in [8]. This contradicts that ab is contractible in G. Similarly, the contractibility of ab implies that the edges x_2a and x_1b do not exist in G (Lemma 46). Hence an M-component centered at abx is found and Remark 36 assures that $\{x_1, x_2\} \cap Min = \emptyset$

Case 0.3 m=5 and the edge x_1x_2 does not exist. Then xx_i is a $4c$-edge for $i = 1, 2$.

Case 0.4 m=5 and the edge x_1x_2 exists. In this case, an M-configuration centered at abx is found and the proof for the Case 0 is finished.

Case 2 m=4 See Table 2; recall that Min contains at least two vertices $u, v \in V(link(x))$.

Case 2.1 $x \in \partial G$, $a, b \in G - \partial G$. This implies $x_1, x_2 \in \partial G$. Let aby be the other face sharing ab with abx.

Case 2.1.a) $y \in \partial G$ and then ab is a triode detecting edge (see Definition 7).

1. Assume $deg(y) = 4$. If $a \in Min$, then $yx_1 \in \partial G$ and $deg(x_1) = 3$, which contradicts the hypothesis. (Analogously, $b \in Min$ leads to $deg(x_2) = 3$). Hence $\{u, v\} = \{x_1, x_2\}$. Notice that x_1x_2 is not a boundary edge since otherwise $\partial G = x_1xx_2$, which leads to a contradiction with $y \in \partial G$. Moreover, as $x_1, x_2 \in Min$, if x_1x_2 is an inner edge there exists the face x_1ax_2 which contradicts the contractibility of ab. Therefore, the edge x_1x_2 does not exist in G and hence xx_i is a $4c$-edge for $i = 1, 3$ by Lemma 46.

2. Assume $deg(y) \geq 5$. It is clear that in case that the edge x_1x_2 exists, it cannot be a boundary edge. If $\{u, v\} = \{a, b\}$, then a quasi-octahedron centered at abx and remaining vertices $\{x_1, x_2, y\}$ is found. Suppose now $a \notin Min$. If x_1x_2 is not an edge, then by Lemma 46 xx_1 is a $4c$-edge. If x_1x_2 is an edge, then there must be $x_2 \notin Min$ (otherwise bx_1 is an edge contradicting the hypothesis of ab contractible edge) and then by Lemma 46 xb is a $4c$-edge.

Case 2.1.b) $y \in G - \partial G$, $deg(y) \geq 5$ (recall that ab is a triode detecting edge).

Firstly, we consider $\{u, v\} = \{a, b\}$. If x_1x_2 is not an edge, clearly an N-configuration with parallel edges ab, xx_1 is detected. (Moreover, xab is the center of a quasi-octahedron component with remaining vertices $\{x_1, x_2, y\}$). If x_1x_2 is a boundary edge, then an octahedron centered at abx appears. If x_1x_2 is an inner edge, a quasi-octahedron component centered at abx and remaining vertices $\{x_1, x_2, y\}$ is found.

Secondly, we consider $\{u, v\} = \{x_1, x_2\}$. Notice that x_1x_2 cannot be an inner edge since otherwise, as $x_1 \in \partial G$ and $x_1 \in Min$, there should be a boundary vertex p defining a face x_1x_2p, and then x_2a should be an edge contradicting the fact that ab is a contractible edge. Moreover, if x_1x_2 is a boundary edge, then an octahedron centered at xx_1x_2 is found. It remains to consider that x_1x_2 is not an edge. Then there is an N-component with parallel edges ab, xx_1.

Next, we consider $\{u, v\} = \{a, x_2\}$. Here $b \notin Min$ and then xa is a $4c$-edge by Lemma 46. (Observe that $b \in Min$ corresponds to the first subcase studied above). An analogous argument works for $\{u, v\} = \{b, x_1\}$.

Finally, we consider $\{u, v\} = \{a, x_1\}$. Observe that in case the edge x_1x_2 exists, it can not be an inner edge since $x_1 \in Min$. If x_1x_2 is a boundary edge, then an octahedron centered at xx_1a is found. In case that x_1x_2 is not an edge, we get that xx_2 is a 4-contractible edge by Lemma 46. An analogous argument works for $\{u, v\} = \{b, x_2\}$.

Case 2.2 x, $a, b \in G - \partial G$.

According to Table 2 we have to study only the case when uv is an edge (hence xuv is a triangle) and $u \in \partial G$ or $v \in \partial G$, whence xv or xu is a triode detecting edge.

Moreover, if $\{u, v\} \cap \{x_1, x_2\} \neq \emptyset$ we distinguish two possibilities.

a) If $\{u, v\} \cap \{x_1, x_2\} = \{x_1\}$ (or x_2), then $v \in \{a, b\}$.

If $uv = x_1a$ (similarly $uv = x_2b$), then Lemma 46 yields that xa is contractible. Now it is not difficult to see that this case reduces to Case 2.1 above: xa plays the same role as ab, $x_1 \in \partial G$ plays the role of x and b plays the role of y.

b) If $uv = x_1x_2$, then x_1x_2 is necessarily a boundary edge and xu (and xv) is a $cn4c$-edge. This case reduces to Case 2.4 in Lemma 20: xu plays the same role as ab (v plays the role of x).

Case 2.4 $x, a \in \partial G$, $b \in G - \partial G$. It readily follows that in this case x_1x is a boundary edge (here we use $deg(x) = 4$) and x_1x_2 is an inner edge (otherwise $deg(x_1) = 2$).

1. First we analyze the case $uv = ab$. If ax_1 is an edge, then an octahedron centered at abx is located. If ax_1 is not an edge and $deg(x_2) \geq 5$, then an N-component with parallel edges ax and bx_2 is found. If ax_1 is not an edge and $deg(x_2) = 4$, then a quasi-octahedron component centered at abx_2 and remaining vertices x_1, a, y is detected (observe that y must be an inner vertex, otherwise a boundary vertex of degree 2 appears).

2. Notice that when $uv = x_1x_2$ we have $x_i \in \partial G$ and $x_2 \notin \partial G$ and x_1x_2 is a $cn4c$-edge. Then a similar argument as above works with x_1x_2 playing the role of ab to obtain either an N-component or a quasi-octahedron component.

   It remains to deal with the subcase when the sets $\{u, v\} \neq \{a, b\}$ share exactly one element.

3. If $u = a$ and $v = x_1$ and ax_1 is an edge, then an octahedron centered at axx_1 is found and $\partial G = axx_1$. If ax_1 is not an edge, as $deg(b) \geq 5$, then ax is a $4c$-edge.

4. If $u = a$, $v = x_2$ then $deg(x_1) \geq 5$ and $deg(b) \geq 5$ and we conclude that by Lemma 46 xx_2 is a $4c$-edge. Observe that x_2 is an inner vertex since $xx_1 \subset \partial G$, $deg(x_2) = 4$ and $b \notin \partial G$.

5. If $u = b$, $v = x_1$, necessarily $deg(x_2)$, $deg(a) \geq 5$ and so xb is a $4c$-edge by Lemma 46.

6. If $u = b$, $v = x_2$ then x_1x_2y is necessarily a face. Moreover, if ax_1 is a boundary edge, then an octahedron component centered at bxx_2 is found and $\partial G = axx_1$.

   On the other hand, if ax_1 is an inner edge then an octahedron component centered at byx_2 appears whenever ax_1y is a face (so that $deg(y) = 4$), if it is not a face then a quasi-octahedron component centered at bxx_2 and remaining vertices x_1, a, y is found.

Finally, the same quasi-octahedron component is located in G when ax_1 is not an edge. This finishes the proof of Lemma 43.

□

References

[1] D.W. Barnette, *Generating the triangulations of the projective plane*, J. Combin. Theory Ser. B 33, 222-230, (1982).

[2] D.W. Barnette, A. L. Edelson, *All* $2-$ *manifolds have finitely many minimal triangulations*, Israel J. Math. 67, 123-128, (1989).

[3] D.W. Barnette, A. L. Edelson, *All orientable* $2-$ *manifolds have finitely many minimal triangulations*, Israel J. Math. 62, 90-98, (1988).

[4] A. Boulch, É. Colin de Verdière, A. Nakamoto, *Irreducible triangulations of surfaces with boundary*, Graphs Comb., **29** No. 6:1675–1688, (2013).

[5] R. Bowen, S. Fisk, *Generation of triangulations of the sphere*, Math. Comput. 21, 250–252, (1967).

[6] M.J. Chávez, S. Lawrencenko, J.R. Portillo, M.T. Villar, *An algorithm that constructs irreducible triangulations of once-punctured surfaces* "Proceedings of the XV Spanish Meeting on Computational Geometry" (June 26-28, 2013, Sevilla, Spain) / J.M. Díaz-Báñez, D. Garijo, A. Márquez, and J. Urrutia (Eds). - Seville: Prensa Universidad de Sevilla, 2013, pp. 43-46. http://congreso.us.es/ecgeometry/proceedingsECG2013.pdf

[7] M.J. Chávez, S. Lawrencenko, J.R. Portillo, M.T. Villar, *An algorithm that constructs irreducible triangulations of once-punctured surfaces*, arXiv e-print service, Cornell University Library, Paper No. arXiv:math/0606690v1, 9 p., (2014).

[8] M.J. Chávez, S. Lawrencenko, A. Quintero, M. T. Villar, *Irreducible triangulations of the Möbius band*, Bul. Acad. Sţiinţe Repub. Mold. Mat., No. 2(75), 44–50, (2014).

[9] B. Chen, J.H. Kwak, S. Lawrencenko, *Weinberg bounds over nonspherical graphs*, Journal of Graph Theory, vol. 33, issue 4, 220-236, (2000).

[10] H.H. Glover, J.P. Huneke, C.S. Wang, *103 Graphs that are irreducible for the projective plane*, J. Combin. Theory Ser. B 27, 332-370, (1979).

[11] H. Komuro, A. Nakamoto, S. Negami, *Diagonal flips in triangulations on closed surfaces whith minimum degree at least 4*, J. Combin. Theory Ser. B 76, 68-92, (1999).

[12] S. Lawrencenko, *On the number of triangular embeddings of a labeled graph in the projective plane* [in Russian], Ukrainskii Geometricheskii Sbornik, No. 32, 71-84, (1989). [English translation: S. A. Lawrencenko, *Number of triangular packings of a marked graph on a projective plane*, Journal of Mathematical Sciences 59, no. 2, 741-749, (1992).]

[13] S. Lawrencenko, *The irreducible triangulations of the Torus*, (in Russian) Ukrain. Geom. Sb. 30 (1987), 52-62; Engl. transl. J. Soviet Math. 51, 2537-2543, (1990).

[14] S. Lawrencenko, *Tables of all essentially different triangulations of the projective plane with some interesting graphs* [in Russian], Preprint deposited at VINITI, No. 613-V89, Moscow, 23 p, (1989).

[15] S. Lawrencenko, S. Negami, *Irreducible triangulations of the Klein Bottle*, J. Combin. Theory Ser. B 70, 265-291, (1997).

[16] A. Malnič, R. Nedela, *K-Minimal triangulations of surfaces*, Acta Math. Univ. Comenianae 64, 1, pp. 5776, (1995).

[17] A. Nakamoto, S. Negami, *Generating triangulations on closed surfaces with minimum degree at least 4*, Discrete Math. 244, 345-349, (2002).

[18] S. Negami, *Diagonal flips in triangulations of surfaces*, Discrete Math. 135, 225-232, (1994).

[19] S. Negami, *Diagonal flips of triangulations on surfaces, a survey*, Yokohama Mathematical Journal, vol. 47, 1-40, (1999).

[20] S. Negami, *Triangulations*, Handbook of Graph Theory, Second Edition. J. L. Gross, J. Yellen and P. Zhang (Ed.) Chapman and Hall/CRC Press, 876-901, (2014).

[21] E. Steinitz, H. Rademacher, *Vorlesungen über die Theorie der Polyeder*, Springer, Berlin, (1934).

[22] T. Sulanke, Generating irreducible triangulations of surfaces, arXiv:math/0606687v1 [math.CO], (2006).

[23] T. Sulanke, Irreducible triangulations of low genus surfaces, arXiv:math/0606690v1 [math.CO], (2006).