# Semi-Galerkin Approximation and Strong Solutions to the Equations of the Nonhomogeneous Asymmetric Fluids

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**Abstract**

This paper analyzes an initial/boundary value problem for a system of equations modelling the nonstationary flow of a nonhomogeneous incompressible asymmetric (polar) fluid. Under conditions similar to those usually imposed to the nonhomogeneous 3D Navier-Stokes equations, by using a spectral semi-Galerkin method, we prove the existence of a local in time strong solution. We also prove the uniqueness of the strong solution and some global existence results. Several estimates for the solutions and their approximations are given. These can be used to find useful error bounds of the Galerkin approximations.

**Résumé**

Dans ce papier, on analyse un problème de valeurs initiales et valeurs aux limites pour un systeème d'équations aux dérivées partielles qui modélise le flux instationnaire d'un fluide asymmétrique incompressible non homogène. Sous des conditions similaires aux conditions usuellement imposées aux équations tridimensionelles de Navier-Stokes non homogènes, à l'aide d'une méthode de type semi-Galerkin, nous démontrons l'éxistence d'une solution forte locale en temps. On établit aussi l'unicité de solution forte et quelques résultats d'éxistence globale. Tous ces

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résultats reposent sur des estimations appropriées pour les solutions et leurs approximations qui permettent d'ailleurs déduire des estimations de l'erreur.

## 1 Introduction

In this paper we will study the equations for the motion of a nonhomogeneous viscous incompressible asymmetric fluid. These equations will be considered in a set of the form $\Omega \times (0,T)$, where $\Omega \subset \mathbf{R}^3$ is a bounded and regular domain with boundary $\partial\Omega$ and $T > 0$.

Thus, let us denote by $u$, $w$, $\rho$ and $p$ the velocity field, the angular velocity of rotation of the fluid particles, the mass density and the pressure distribution, respectively. The governing equations are the following:

$$\begin{cases} \rho(u_t + (u\cdot\nabla)u) - (\mu+\mu_r)\Delta u + \nabla p = 2\mu_r \operatorname{curl} w + \rho f, \\ \operatorname{div} u = 0, \\ \rho(w_t + \rho(u\cdot\nabla)w) - (c_a + c_d)\Delta w - (c_0 + c_d - c_a)\nabla \operatorname{div} w \\ \qquad + 4\mu_r w = 2\mu_r \operatorname{curl} u + \rho g, \\ \rho_t + u\cdot\nabla\rho = 0. \end{cases} \tag{1}$$

For simplicity, they will be completed with the following boundary and initial conditions

$$\begin{cases} u(x,t) = 0, \quad w(x,t) = 0 \quad \text{on} \quad \partial\Omega\times(0,T), \\ u(x,0) = u_0(x), \quad w(x,0) = w_0(x), \quad \rho(x,0) = \rho_0(x) \quad \text{in} \quad \Omega, \end{cases} \tag{2}$$

In (1), $f$ and $g$ are known density functions of external sources for the linear and the angular momentum of particles, respectively. The positive constants $\mu$, $\mu_r$, $c_0$, $c_a$ and $c_d$ characterize the physical properties of the fluid. Thus, $\mu$ is the usual *Newtonian viscosity;* $\mu_r$, $c_0$, $c_a$ and $c_d$ are additional *viscosities* related to the lack of symmetry of the stress tensor and, consequently, to the fact that the field of internal rotation $w$ does not vanish. These constants must satisfy the inequality

$$c_0 + c_d > c_a\,.$$

The symbols $\nabla$, $\Delta$, div and curl denote the *gradient*, *Laplacian*, *divergence* and *rotational* operators, respectively; $u_t$, $w_t$ and $\rho_t$ stand for the time derivatives of $u$, $w$ and $\rho$; the $i$-th components of $(u\cdot\nabla)u$ and $(u\cdot\nabla)w$

in cartesian coordinates are given by

$$[(u \cdot \nabla)u]_i = \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j}$$

and

$$[(u \cdot \nabla)w]_i = \sum_{j=1}^{n} u_j \frac{\partial w_i}{\partial x_j} .$$

We also have

$$u \cdot \nabla \rho = \sum_{j=1}^{n} u_j \frac{\partial \rho}{\partial x_j} .$$

For the derivation of equations (1) and a discussion on their physical meaning, see [5]. Observe that this system includes as a particular case the classical Navier-Stokes equations, which have been largely studied (see for instance the classical books by Ladyzhenskaya [10] and Temam [20] and the references therein). It also includes as a reduced model the nonhomogeneous Navier-Stokes system, that are less known (cf. [19],[9],[11],[18]).

Concerning the model considered in this paper, let us recall that, under certain assumptions, by using linearization and an almost fixed point Theorem, Lukaszewicz established in [17] the existence of weak solutions for short time. In this same paper, there are considerations on the possible proof of the existence of strong solutions (assuming that the initial density is strictly separated from zero) by using the techniques of [15] and [16] (linearization and fixed point Theorems; recall that in [15] and [16] the density is a positive constant).

In this paper we are also concerned with the existence of strong solutions of (1)–(2). However, since we are mainly motivated by techniques directly related to numerical applications, we have preferred an approach based on spectral semi-Galerkin methods. In this way, by assuming that the initial data are more regular than in [17] and the initial density is separated from zero, we will prove that more regular strong solutions exist. In this process, we will find appropriate estimates that become fundamental to derive error bounds for the Galerkin approximations, as was already explained in the previous paper [2]. Actually, these estimates can be viewed as one of the main objectives in this paper.

The rest of the paper is organized as follows. In Section 2, we will explain what is a strong solution $(u, w, \rho)$ of (1)–(2) and we will present our main results, Theorems 1 and 2: the existence and uniqueness of a local in time strong solution. We will also introduce in this Section the main tool in this paper, namely a spectral semi-Galerkin approximation scheme for (1)–(2).

In Section 3, we will deduce the *a priori* estimates needed to ensure the regularity of $(u, w, \rho)$. To this end, we will combine arguments and techniques that have been used in other similar contexts by several authors, in particular by Heywood [6],[7], Kim [9] and Boldrini and Rojas-Medar [3].

In Section 4, we will use these *a priori* estimates to extract a convergent subsequence and, then, to pass to the limit in the equations. In this way, we will prove that a strong solution exists in a (possibly small) maximal time interval $[0, T_0)$, with $T_0 \leq T$.

Section 5 deals with uniqueness. There, we will prove that any strong solution must coincide with the solution furnished by Theorem 1.

In Section 6, we prove the existence of a (regular) pressure. In particular, we see that the triplet $(u, w, \rho)$ provided by Theorem 1 is such that, for some $p$, the equations (1) are satisfied a.e. in $\Omega \times (0, T_0)$. Finally, Section 7 is concerned with the existence of global strong solutions for small regular data.

## 2 Preliminaries and Main Results

In the sequel we will assume that $\Omega$ is a bounded domain in $\mathbf{R}^3$, with regular boundary $\partial\Omega$. We will consider the usual Sobolev spaces

$$W^{m,q}(\Omega) = \{\, f \in L^q(D) : \|\partial^\kappa f\|_{L^q(D)} < +\infty \text{ for } |\kappa| \leq m \,\}$$

for $m \geq 1$ and $1 \leq q \leq \infty$ with the usual norms $\|\cdot\|_{W^{m,q}}$. When $q = 2$, we will set $H^m(\Omega) = W^{m,2}(\Omega)$. As usual, $H_0^m(\Omega)$ will stand for the closure of $C_0^\infty(\Omega)$ in $H^m(\Omega)$. For simplicity, if $B$ is a Banach space with norm $\|\cdot\|_B$, the natural product norm in $B^m$ will be also denoted by $\|\cdot\|_B$.

We will set

$$\begin{aligned} &\mathcal{V}(\Omega) = \{\, v \in (C_0^\infty(\Omega))^3 : \operatorname{div} v = 0 \text{ in } \Omega \,\}, \\ &H = \text{ the closure of } \mathcal{V}(\Omega) \text{ in } (L^2(\Omega))^3, \\ &V = \text{ the closure of } \mathcal{V}(\Omega) \text{ in } (H_0^1(\Omega))^3. \end{aligned}$$

For any Banach-space $B$ and any $T > 0$, we will denote by $L^r(0, T; B)$ the Banach space of the $B$-valued (classes of) functions defined a.e. in $[0, T]$ that are $L^r$-integrable in the sense of Bochner. Frequently, we will consider Banach spaces $L^r(0, T; B)$ with $B = W^{m,q}(\Omega)$. In such cases, for any $v \in L^r(0, T; W^{m,q}(\Omega))$, $v(t)$ stands for the function $v(\cdot, t)$.

Let $P$ be the orthogonal projection of $(L^2(\Omega))^3$ onto $H$ induced by the usual *Helmholtz decomposition* of $(L^2(\Omega))^3$. By definition, the *Stokes operator* is the unbounded linear mapping $A : D(A) \subset H \mapsto H$, with domain $D(A) = V \cap (H^2(\Omega))^3$, given by

$$Av = P(-\Delta v) \qquad \forall v \in D(A).$$

It is well known that $A$ is a positive self-adjoint operator. It is characterized by the equalities

$$(Aw, v) = (\nabla w, \nabla v) \qquad \forall w \in D(A), \quad \forall v \in V.$$

Here and in the sequel, $(\cdot\,,\cdot)$ stands for the usual scalar $(L^2(\Omega))^3$-product. The associated norm will be denoted by $\|\cdot\|$.

We will also consider in the sequel the strongly uniformly elliptic operators $L_0$ and $L$, with $D(L_0) = D(L) = (H_0^1(\Omega))^3 \cap (H^2(\Omega))^3$,

$$L_0 z = -(c_a + c_d)\Delta z - (c_0 + c_d - c_a)\nabla \mathrm{div}\, z \qquad \forall z \in D(L)$$

and

$$Lz = L_0 z + 4\mu_r z \qquad \forall z \in D(L).$$

Due to the assumption $c_0 + c_d > c_a$ , $L$ is indeed a positive operator.

The following assumptions on the initial velocity, angular velocity and density will be imposed throughouth this paper:

$$u_0 \in D(A), \tag{3}$$

$$w_0 \in D(L), \tag{4}$$

$$\rho_0 \in C^1(\overline{\Omega}), \quad 0 < \alpha \le \rho_0(x) \le \beta \text{ a.e. in } \Omega. \tag{5}$$

We will also assume that

$$f, g \in L^2(0, T; (H^1(\Omega))^3), \qquad f_t\,, g_t \in L^2(0, T; (L^2(\Omega))^3). \tag{6}$$

Using the orthogonal projector $P$ and the operators $A$ and $L$, we can give a rigorous formulation of problem $(1)-(2)$: Find a time $T_0 \in (0, T]$ and functions

$$\begin{aligned}
&u \in C^0([0, T_0); D(A)) \cap C^1([0, T_0); H),\\
&w \in C^0([0, T_0); D(L)) \cap C^1([0, T_0); (L^2(\Omega))^3) \quad \text{and}\\
&\rho \in C^1(\overline{\Omega} \times [0, T_0)),
\end{aligned}$$

such that

$$
\begin{cases}
(\rho u_t, v) + (\rho(u\cdot\nabla)u, v) + (\mu+\mu_r)(Au, v) \\
\quad = 2\mu_r(\operatorname{curl} w, v) + (\rho f, v) \ \text{ for } \ 0 < t < T_0\,, \quad \forall\, v \in V, \\
\\
(\rho w_t, \psi) + (\rho(u\cdot\nabla)w, \psi) + (Lw, \psi) \\
\quad = 2\mu_r(\operatorname{curl} u, \psi) + (\rho g, \psi) \ \text{ for } \ 0 < t < T_0\,, \quad \forall \psi \in (H_0^1(\Omega))^3, \\
\\
\rho_t + u\cdot\nabla\rho = 0 \ \text{ for } \ (x,t) \in \overline{\Omega}\times[0,T_0), \\
\\
u(x,0) = u_0(x), \quad w(x,0) = w_0(x), \quad \rho(x,0) = \rho_0(x) \ \text{ for } \ x \in \Omega.
\end{cases}
\tag{7}
$$

By definition, a triplet $(u, w, \rho)$ with these properties is a *strong solution* of (1)–(2) in $[0, T_0)$. We will prove below that, under assumptions (3)–(6), the initial/boundary value problem (1)–(2) possesses exactly one strong solution in a maximal time interval. Actually, we will see that this strong solution is still more regular than stated above.

To this end, let us first recall some properties of the Stokes operator $A$. If $\Omega$ is bounded and $\partial\Omega$ is of class $C^{1,1}$, the mapping $A : D(A) \mapsto H$ is one-to-one and onto (see for instance [1]). The inverse operator $A^{-1}$ is completely continuous as a mapping $A^{-1} : H \mapsto H$. Also, $A$ is symmetric and, therefore, so is its inverse. Consequently, $A^{-1}$ possesses an orthogonal sequence of eigenfunctions $\{\varphi^k\}$ which is complete in $H$, $V$ and $D(A)$. We will denote by $\lambda_k$ the $k$-th associated eigenvalue (that is, $A\varphi^k = \lambda_k\varphi^k$ for all $k$). It will be assumed that $\{\varphi^k\}$ is orthonormal in $H$. Accordingly, the eigenfunctions $\{\lambda_k^{-1/2}\varphi^k\}$ and $\{\lambda_k\varphi^k\}$ are complete and orthonormal respectively in $V$ (endowed with the scalar product $(\nabla u, \nabla v)$ and $D(A)$ (endowed with the scalar product $(Au, Av)$).

Notice also that, if $\partial\Omega$ is a $C^{m+1,1}$ manifold, then the eigenfunctions $\varphi^k$ belong to $(H^{m+2}(\Omega))^3$.

On the other hand, we will use the notation $\{\psi^k\}$ and $\{\gamma_k\}$ for the eigenfunctions and eigenvalues of $L$. Again, it is assumed that $\{\psi^k\}$ is orthonormal for the $L^2$-norm. The system $\{\psi^k\}$ is complete in $(L^2(\Omega))^3$, $(H_0^1(\Omega))^3$ and $D(L)$ and we have again regularity results for the eigenfunctions $\psi^k$ when $\partial\Omega$ is a $C^{m+1,1}$ manifold.

Let $P_k$ the orthogonal projection of $H$ onto the space $V_k$ spanned by the $k$ first eigenfunctions $\varphi^1, \dots, \varphi^k$ of $A$. Similarly, let $R_k$ be the orthogonal projection of $L^2(\Omega)$ onto the space $W_k$ spanned by the $k$ first eigenfunctions $\psi^1, \dots, \psi^k$ of $L$. Then the solutions of (7) can be obtained by using a semi-Galerkin method determined by the spaces $V_k$ and $W_k$ and the operators $P_k$ and $R_k$.

More precisely, for each fixed $k$, we consider the following finite dimensional problem: Find $T_k \in (0,T]$, $u^k \in C^1([0,T_k);V_k)$, $w^k \in C^1([0,T_k);W_k)$ and $\rho^k \in C^1(\overline{\Omega} \times [0,T_k))$ such that

$$
\left\{
\begin{array}{l}
P_k(\rho^k u^k_t + \rho^k(u^k \cdot \nabla)u^k - 2\mu_r \operatorname{curl} w^k - \rho^k f) + (\mu+\mu_r)Au^k = 0, \\
R_k(\rho^k w^k_t + \rho^k(u^k \cdot \nabla)w^k - 2\mu_r \operatorname{curl} u^k - \rho^k g + Lw^k = 0 \quad \text{and} \\
\rho^k_t + u^k \cdot \nabla \rho^k = 0 \\
\qquad \text{for} \quad 0 < t < T_k \,, \\
\\
u^k(0) = P_k u_0 \,, \quad w^k(0) = R_k w_0 \,, \quad \rho^k(0) = \rho_0 \,.
\end{array}
\right.
\tag{8}
$$

In (8), we have an initial value problem for a system of ordinary differential equations coupled to a transport equation. By using the characteristics method, it is not difficult to prove that (8) possesses exactly one solution $(u^k, w^k, \rho^k)$ defined in a time interval $[0,T_k)$. The *a priori* estimates established below prove that we can take in fact $T_k = T$ for all $k \geq 1$.

The $k$-th approximated problem (8) can also be written in the form

$$
\left\{
\begin{array}{l}
(\rho^k u^k_t + \rho^k(u^k \cdot \nabla)u^k - 2\mu_r \operatorname{curl} w^k - \rho^k f, v) + (\mu+\mu_r)(Au^k, v) = 0, \\
(\rho^k w^k_t + \rho^k(u^k \cdot \nabla)w^k - 2\mu_r \operatorname{curl} u^k - \rho^k g, \psi) + (Lw^k, \psi) = 0, \\
\qquad \text{for} \quad 0 < t < T_k \,, \ \forall v \in V_k \,, \quad \forall \psi \in W_k \,, \\
\\
\rho^k_t + u^k \cdot \nabla \rho^k = 0 \quad \text{in} \quad \Omega \times (0,T_k), \\
\\
u^k(0) = P_k u_0 \,, \quad w^k(0) = R_k w_0 \,, \quad \rho^k(0) = \rho_0 \,.
\end{array}
\right.
\tag{9}
$$

The first main result in this paper is the following:

**Theorem 1** *Assume that the initial data $u_0$, $w_0$ and $\rho_0$ satisfy* (3)–(5) *and the external fields $f$ and $g$ satisfy* (6)*. Then* (1)–(2) *possesses exactly one strong solution $(u,w,\rho)$ defined on a (possibly small) maximal time interval $[0,T_0)$, where $T_0 \leq T$. The functions $u$, $w$ and $\rho$ satisfy*

$$
P(\rho u_t + \rho(u \cdot \nabla)u - 2\mu_r \operatorname{curl} w - \rho f) + Au = 0, \tag{10}
$$

$$
\rho w_t + \rho(u \cdot \nabla)w - 2\mu_r \operatorname{curl} u + Lw = \rho g \tag{11}
$$

*and*

$$
\rho_t + u \cdot \nabla \rho = 0
$$

*a.e. in $\Omega \times (0,T_0)$.*

**Remark 1** As mentioned above, the functions $u$, $w$ and $\rho$ satisfy additional regularity properties. More precisely, the following will be proved for any small positive $\delta$ and $\gamma$:

$$u \in L^2(0,T_0;V)\cap L^\infty(0,T_0;H)\cap C^0([0,T_0);D(A))$$
$$\cap L^2(0,T_0-\gamma;(H^3(\Omega))^3)\cap L^\infty(\delta,T_0-\gamma;(H^3(\Omega))^3),$$

$$u_t \in L^{3/2}(0,T_0;V')\cap C^0([0,T_0);H)\cap L^2(0,T_0-\gamma;V)$$
$$\cap L^2(\delta,T_0-\gamma;D(A))\cap L^\infty(\delta,T_0-\gamma;V),$$

$$u_{tt}\in L^2(\delta,T_0-\gamma;H).$$

Similar regularity properties will also be established for the angular velocity $w$. ■

**Remark 2** As in the case of the classical Navier-Stokes equations, it can be proved that the strong solution furnished by Theorem 1 is *global in time*, i.e. it is defined for any $t\in[0,T)$ if the data $u_0$ , $w_0$ , $\rho_0$ , $f$ and $g$ are small enough. This situation will be analyzed in Section 7. ■

The proof of Theorem 1 relies on appropriate estimates for the approximations $(u^k,w^k,\rho^k)$. For future reference, let us gather them in the following:

**Proposition 1** *Let* $(u^k,w^k,\rho^k)$ *be the solution of* (8). *There exists* $T_0>0$ *(independent of* $k$*) such that the following estimates hold for all* $t\in[0,T_0)$*:*

$$\|\nabla u^k(t)\|^2+\|\nabla w^k(t)\|^2\le F_1(t),$$
$$\|u_t^k(t)\|^2+\|w_t^k(t)\|^2+\int_0^t\{\|\nabla u_t^k(s)\|^2+\|\nabla w_t^k(s)\|^2\}\,ds\le F_2(t),$$
$$\|Au^k(t)\|^2+\|\Delta w^k(t)\|^2\le F_3(t),$$
$$\alpha_0\le\rho^k(x,t)\le\beta_0\,,\quad(\alpha_0=\inf_\Omega\rho_0\,,\quad\beta_0=\sup_\Omega\rho_0)$$
$$\|\nabla\rho^k(t)\|_{L^\infty}^2\le F_4(t),\quad\|\rho_t^k(t)\|_{L^\infty}^2\le F_5(t),$$
$$\int_0^t\{\|u^k(s)\|_{H^3}^2+\|w^k(s)\|_{H^3}^2\}\,ds\le F_6(t),$$
$$\int_0^t\sigma(s)\{\|u_{tt}^k(s)\|^2+\|w_{tt}^k(s)\|^2\}\,ds\le F_7(t),$$
$$\sigma(t)\{\|\nabla u_t^k(t)\|^2+\|\nabla w_t^k(t)\|\}\le F_8(t),$$

$$\sigma(t)\{\|u^k(t)\|^2_{H^3} + \|w^k(t)\|^2_{H^3}\} \le F_9(t),$$
$$\int_0^t \sigma(s)\{\|Au^k_t(s)\|^2 + \|\Delta w^k_t(s)\|^2\}\, ds \le F_{11}(t).$$

*Here, the $F_i$ are nondecreasing continuous functions on $[0, T_0)$ and $\sigma(t) \equiv \min\{1, t\}$. Furthermore, the same estimates hold for the solution $(u, w, \rho)$ furnished by Theorem 1.*

**Remark 3** Proposition 1 implies the existence of a subsequence of approximations $(u^k, w^k, \rho^k)$ that converges to the solution $(u, w, \rho)$ in the following sense:

| | |
|---|---|
| (i) $u^k \to u, \quad w^k \to w$ | strongly in $L^p(\delta, T_0 - \gamma; (H^{3-\varepsilon}(\Omega))^3)$,<br>weakly in $L^2(0, T_0 - \gamma; (H^3(\Omega))^3)$<br>and weakly-$*$ in $L^\infty(\delta, T_0 - \gamma; (H^3(\Omega))^3)$, |
| (ii) $u^k_t \to u_t, \quad w^k_t \to w_t$ | strongly in $L^p(\delta, T_0 - \gamma; (H^{1-\varepsilon}(\Omega))^3)$,<br>weakly in $L^2(0, T_0 - \gamma; (H^1(\Omega))^3)$,<br>weakly-$*$ in $L^\infty(0, T_0 - \gamma; (L^2(\Omega))^3)$<br>and weakly in $L^2(\delta, T_0 - \gamma; (H^2(\Omega))^2)$, |
| (iii) $u^k_{tt} \to u_{tt}, \quad w^k_{tt} \to w_{tt}$ | weakly in $L^2(\delta, T_0 - \gamma; (L^2(\Omega))^3)$, |
| (iv) $\rho^k \to \rho$ | strongly in $L^p(0, T_0; C^{0,\beta}(\overline{\Omega}))$<br>$(0 \le \beta < 1)$<br>and uniformly in $\overline{\Omega} \times [0, T_0 - \gamma]$, |
| (v) $\nabla\rho^k \to \nabla\rho$ | weakly-$*$ in $L^\infty(\Omega \times (0, T_0 - \gamma))^3$, |
| (vi) $\rho^k_t \to \rho_t$ | weakly-$*$ in $L^\infty(\Omega \times (0, T_0 - \gamma))$. |

The above is true for all small $\delta$, $\gamma$ and $\varepsilon > 0$ and any $p \in (1, +\infty)$. This justifies the regularity properties of $u$ and $w$ mentioned in Remark 1. ■

**Remark 4** Actually, more information can be obtained for $\rho^k$ and $\rho$. Arguing as in [14], we deduce that the distribution function for the mass density, i.e. the function

$$\lambda \mapsto \text{meas}\,\{\, x \in \Omega : \rho(x, t) \le \lambda \,\}$$

is independent of $t$. This leads in particular to the inequalities $\alpha_0 \le \rho(x, t) \le \beta_0$. ■

Let us now refer to the uniqueness of the strong solution. Let $T' > 0$ be given, with $0 < T' \leq T_0$ and set

$$\begin{aligned}\mathcal{H}' = \{\, (v,\psi,\sigma) : v \in L^4(0,T';V), \quad & v_t \in L^2(0,T';(L^3(\Omega))^3), \\ \psi \in L^2(0,T';(H_0^1(\Omega))^3), \quad & \psi_t \in L^2(0,T';(L^3(\Omega))^3), \\ \sigma \in L^\infty(\Omega \times (0,T')), \quad & \nabla\sigma \in L^2(0,T';(L^\infty(\Omega))^3) \,\}.\end{aligned}$$

With this notation, we can state the following uniqueness result:

**Theorem 2** *Assume that $(v,z,\sigma)$ is a solution to (1)–(2) in $[0,T')$, with $(v,z,\sigma) \in \mathcal{H}'$. Then $(v,z,\sigma) = (u,w,\rho)$ in $\Omega \times (0,T')$, where $(u,w,\rho)$ is the strong solution of $(1)-(2)$ furnished by Theorem 1.*

In the following Sections, we will denote by $C$ a generic positive constant depending at most on $\Omega$, $T$ and the data of the problem (the parameters $\mu$, $\mu_r$, $c_a$, $c_d$, $c_0$, the initial conditions $u_0$, $w_0$ and $\rho_0$ and also $f$ and $g$). This will appear in most estimates below. When, for any reason, we want to emphasize the dependence of a $C$ on a given parameter or function, we will put a suitable subscript.

## 3 A Priori Estimates

We will prove in this Section the estimates stated in Proposition 1. This will be done in several steps, combining variants of arguments used by Heywood [6],[7], Kim [9] and Boldrini and Rojas-Medar [3].

In the sequel, we will find several real-valued functions $F_i$, $G_i$, $H_i$, all them defined, nondecreasing and continuous in a time interval of the form $[0,T_0)$. They all depend on $\Omega$, $T$ and the data of the problem.

**Lemma 1** *There exists $T_0$ with $0 < T_0 \leq T$ such that the approximations $(\rho^k,u^k,w^k)$ satisfy the following for all $t \in [0,T_0)$:*

$$\alpha_0 \leq \rho^k(x,t) \leq \beta_0\,, \tag{12}$$

$$\|u^k(t)\|^2 + \|w^k(t)\|^2 + \int_0^t \{\|\nabla u^k(s)\|^2 + \|\nabla w^k(s)\|^2\}\,ds \leq C, \tag{13}$$

$$\|\nabla u^k(t)\|^2 + \|\nabla w^k(t)\|^2 \leq F_1(t), \tag{14}$$

$$\int_0^t \{\|Au^k(s)\|^2 + \|\Delta w^k(s)\|^2\}\,ds \leq H_1(t), \tag{15}$$

$$\int_0^t \{\|u_t^k(s)\|^2 + \|w_t^k(s)\|^2\}\,ds \leq H_2(t). \tag{16}$$

**Proof:** From the method of characteristics applied to the transport equation

$$\rho_t^k + u^k \cdot \nabla \rho^k = 0,$$

it follows immediately that, whenever $\rho^k$ exists, it satisfies (12).

Now, taking $v = u^k$ and $\psi = w^k$ in (9) and working as in [12] and [13], we easily obtain:

$$\frac{1}{2}\frac{d}{dt}\|(\rho^k)^{\frac{1}{2}} u^k\|^2 + (\mu + \mu_r)\|\nabla u^k\|^2 = (\rho^k f, u^k) + 2\mu_r(\operatorname{curl} w^k, u^k),$$
$$\frac{1}{2}\frac{d}{dt}\|(\rho^k)^{\frac{1}{2}} w^k\|^2 + (c_a + c_d)\|\nabla w^k\|^2 + (c_0 + c_d - c_a)\|\operatorname{div} w^k\|^2$$
$$+4\mu_r\|w^k\|^2 = 2\mu_r(\operatorname{curl} u^k, w^k) + (\rho^k g, w^k).$$

By adding these two equations, manipulating the terms in the right-hand side in a standard way, using (12) and integrating with respect to time in $[0, t]$, we find:

$$\begin{cases} \|u^k(t)\|^2 + \|w^k(t)\|^2 + \displaystyle\int_0^t \{\|\nabla u^k(s)\|^2 + \|\nabla w^k(s)\|^2\}\, ds \\ \qquad \le C + C \displaystyle\int_0^t \left\{\|f(s)\|^2 + \|g(s)\|^2\right\} ds \\ \qquad +C \displaystyle\int_0^t \left\{\|u^k(s)\|^2 + \|w^k(s)\|^2\right\} ds \end{cases} \tag{17}$$

(recall that $C$ is a generic constant which depends on the data of the problem but not on $k$). Thus, from Gronwall's Lemma, we obtain (13). This proves in particular that the approximate solutions $(u^k, w^k, \rho^k)$ are defined in the whole interval $[0, T)$.

Now, let us take $v = u_t^k$ and then $v = -\varepsilon A u^k$ in the $u$-equation in (9). By adding the resulting equations and working as in [9], we easily find:

$$\begin{cases} \|u_t^k\|^2 + (\mu + \mu_r)\dfrac{d}{dt}\|\nabla u^k\|^2 + \|Au^k\|^2 \\ \qquad \le C\left(\|f\|^2 + \|\nabla w^k\|^2 + \|(u^k \cdot \nabla)u^k\|^2\right). \end{cases} \tag{18}$$

We can find a similar differential inequality for $w^k$. Indeed, if we take $\psi = w_t^k$ in the $w$-equation in (9), we immediately see that

$$\frac{\alpha}{2}\|w_t^k\|^2$$
$$+\frac{1}{2}\frac{d}{dt}\left((c_a + c_d)\|\nabla w^k\|^2 + (c_0 + c_d - c_a)\|\operatorname{div} w^k\|^2 + 2\mu_r\|w^k\|^2\right)$$
$$\le C\left(\|g\|^2 + \|\nabla u^k\|^2 + \|(u^k \cdot \nabla)w^k\|^2\right).$$

Taking $\psi = -\Delta w^k$ in the same equation, we also have

$$\|Lw^k\|^2 \leq C\left(\|g\|^2 + \|\nabla u^k\|^2 + \|(u^k \cdot \nabla)w^k\|^2\right) + \frac{\beta_0^2}{2}\|w_t^k\|^2.$$

Consequently,

$$\left\{\begin{array}{l} \dfrac{\alpha}{2}\|w_t^k\|^2 + \|Lw^k\|^2 \\ \quad +\dfrac{1}{2}\dfrac{d}{dt}\left((c_a+c_d)\|\nabla w^k\|^2 + (c_0+c_d-c_a)\|\operatorname{div} w^k\|^2 + 2\mu_r\|w^k\|^2\right) \\ \quad \leq C\left(\|g\|^2 + \|\nabla u^k\|^2 + \|(u^k \cdot \nabla)w^k\|^2\right). \end{array}\right. \tag{19}$$

Observe that, by standard interpolation and Sobolev inequalities, one has

$$\|(u^k \cdot \nabla)u^k\|^2 \leq C\|\nabla u^k\|^3\|Au^k\| \leq \varepsilon\|Au^k\|^2 + C_\varepsilon\|\nabla u^k\|^6$$

for any $\varepsilon > 0$ and suitable $C_\varepsilon > 0$.

On the other hand, since $L$ is strongly elliptic, we also have

$$\|(u^k \cdot \nabla)w^k\|^2 \leq \varepsilon\|Lw^k\|^2 + C_\varepsilon\|\nabla u^k\|^4\|\nabla w^k\|^2.$$

Adding (18) and (19) and using these inequalities with suitable small $\varepsilon$, we conclude that

$$\dot{\theta}_k(t) + \psi_k(t) \leq \phi(t) + C\theta_k(t)^3 \tag{20}$$

for all $t$, where

$$\begin{array}{rcl} \theta_k(t) & = & (\mu+\mu_r)\|\nabla u^k(t)\|^2 + (c_a+c_d)\|\nabla w^k\|^2 \\ & & +(c_0+c_d-c_a)\|\operatorname{div} w^k(t)\|^2 + 2\mu_r\|w^k(t)\|^2, \\ \psi_k(t) & = & \|u_t^k(t)\|^2 + \|Au^k(t)\|^2 + \|Lw^k(t)\|^2, \\ \phi(t) & = & C(\|f(t)\|^2 + \|g(t)\|^2). \end{array}$$

Notice that $\theta_k(0)$ is bounded independently of $k$, in view of the facts that $u^k(0) = P_k u_0$, $w^k(0) = R_k w_0$, $u_0 \in D(A)$ and $w_0 \in D(L)$. Making use of Lemma 3 in [8], p. 656, we conclude that there exists $T_0$ with $0 < T_0 \leq T$ such that (14)–(16) hold in $[0, T_0)$ with suitable functions $F_1$, $H_1$ and $H_2$. ■

**Lemma 2** *For all $t \in [0, T_0)$, the approximations $u^k$ and $w^k$ satisfy*

$$\|u_t^k(t)\|^2 + \|w_t^k(t)\|^2 + \int_0^t \{\|\nabla u_t^k(s)\|^2 + \nabla w_t^k(s)\|^2\}\,ds \leq F_2(t), \tag{21}$$

$$\|Au^k(t)\|^2 + \|\Delta w^k(t)\|^2 \leq F_3(t). \tag{22}$$

**Proof:** By differentiating the first two equations in (9) with respect to $t$, setting $v = u_t^k$ and $\psi = w_t^k$, using the fact that $\rho_t^k = -\mathrm{div}\,(\rho^k u^k)$ and arguing as in [3], we obtain

$$\begin{aligned}
&\frac{d}{dt}\|(\rho^k)^{1/2}u_t^k\|^2 + (\mu+\mu_r)\|\nabla u_t^k\|^2 \\
&\qquad \le C_\varepsilon \|u_t^k\|^2\left(1+\|\nabla u^k\|^4 + \|\nabla u^k\|\|Au^k\|\right) \\
&\qquad +C\|\nabla u^k\|^4\|Au^k\|^2 \\
&\qquad +\varepsilon\|\nabla w_t^k\|^2 + \varepsilon\|f_t\|^2 + C\|f\|_{H^1}^2\|u^k\|\,\|\nabla u^k\|
\end{aligned}$$

and

$$\begin{aligned}
&\frac{d}{dt}\|(\rho^k)^{1/2}w_t^k\|^2 + (c_a+c_d)\|\nabla w_t^k\|^2 + (c_0+c_d-c_a)\|\mathrm{div}\, w_t^k\|^2 \\
&\qquad +4\mu_r\|w_t^k\|^2 \\
&\le C_\varepsilon\|w_t^k\|^2\left(1+\|\Delta w^k\|^2\right) + C\|\nabla u^k\|^4\|\Delta w^k\|^2 \\
&+\varepsilon\|\nabla u_t^k\|^2 + \varepsilon\|u_t^k\|^2 \\
&+\varepsilon\|g_t\|^2 + \|g\|_{H^1}^2\|u^k\|\|\nabla u^k\|
\end{aligned}$$

for any small $\varepsilon > 0$.

Adding and integrating in time these two inequalities, we obtain the following:

$$\left\{\begin{aligned}
&\|u_t^k(t)\|^2 + \|w_t^k(t)\|^2 \\
&\qquad + \int_0^t \left\{\|\nabla u_t^k(s)\|^2 + \|\nabla w_t^k(s)\|^2\right\} ds \\
&\qquad \le M_k(t) + C\int_0^t N_k(s)\left\{\|u_t^k(s)\|^2 + \|w_t^k(s)\|^2\right\} ds,
\end{aligned}\right. \tag{23}$$

where

$$\left\{\begin{aligned}
M_k(t) &= C\int_0^t (\|g_t(s)\|^2 + \|f_t(s)\|^2)\, ds \\
&+C\int_0^t (\|f(s)\|_{H^1}^2 + \|g(s)\|_{H^1}^2)\|u^k(s)\|\|\nabla u^k(s)\|\, ds \\
&+C\int_0^t \|\nabla u^k(s)\|^4(\|Au^k(s)\|^2 + \|\Delta w^k(s)\|^2)\, ds \\
&+C(\|u_t^k(0)\|^2 + \|w_t^k(0)\|^2)
\end{aligned}\right. \tag{24}$$

and

$$N_k(t) = 1 + \|\nabla u^k(t)\|^4 + \|Au^k(t)\|^2 + \|\Delta w^k(t)\|^2 \tag{25}$$

for all $t$. Notice that Gronwall's Lemma applied to (23) yields

$$\|u_t^k(t)\|^2 + \|w_t^k(t)\|^2 \leq M_k(t) + \int_0^t M_k(s) N_k(s) e^{\int_s^t N_k(\sigma)\, d\sigma}\, ds \tag{26}$$

for all $t \in [0, T_0)$.

Using (6) and the estimates in Lemma 1, we see that

$$M_k(t) \leq G_1(t) + C\left(\|u_t^k(0)\|^2 + \|w_t^k(0)\|^2\right) \quad \forall t \in [0, T_0), \tag{27}$$

for some function $G_1$. On the other hand, taking $t = 0$ and $v = u_t^k(0)$ in the first equation in (9), we deduce that

$$\left\{\begin{aligned} &\|u_t^k(0)\|^2 \\ &\quad \leq C\left(\|(u^k(0)\cdot\nabla)u^k(0)\|^2 + \|f(0)\|^2 + \|\nabla w^k(0)\|^2 + \|Au^k(0)\|^2\right) \\ &\quad \leq C\left(\|u_0\|^2\|\nabla u_0\|^2 + \|f(0)\|^2 + \|\nabla w_0\|^2 + \|Au_0\|^2\right), \end{aligned}\right.$$

since $u_0 \in D(A)$. In a similar way, we have

$$\|w_t^k(0)\|^2 \leq C\left(\|u_0\|^2\|\nabla w_0\|^2 + \|g(0)\|^2 + \|Lw_0\|^2\right).$$

Therefore, $C(\|u_t^k(0)\|^2 + \|w_t^k(0)\|^2)$ is uniformly bounded and

$$M_k(t) \leq G_2(t) \qquad \forall t \in [0, T_0),$$

for some nondecreasing function $G_2$.

From the estimates in Lemma 1, we also see that

$$\int_0^t N_k(s)\, ds \leq G_3(t) \qquad \forall t \in [0, T_0).$$

Thus, in view of (26), we obtain

$$\left\{\begin{aligned} &\|u_t^k(t)\|^2 + \|w_t^k(t)\|^2 \\ &\qquad\qquad \leq G_2(t) + \int_0^t G_2(s) N_k(s) e^{\int_s^t N_k(\sigma)\, d\sigma}\, ds \\ &\qquad\qquad \leq G_2(t)\left(1 + G_3(t) e^{G_3(t)}\right) \\ &\qquad\qquad \leq G_4(t). \end{aligned}\right.$$

Using again (23), we are finally led to (21).

In order to prove (22), let us take $v = Au^k$ in the first equation in (9). We deduce that

$$\|Au^k\| \le C\left(\|u_t^k\| + \|\nabla w^k\| + \|f\| + \|(u^k\cdot\nabla)u^k\|\right). \tag{28}$$

Observe that

$$\|(u^k\cdot\nabla)u^k\| \le C\|\nabla u^k\|^{3/2}\|Au^k\|^{1/2} \le \varepsilon\|Au^k\| + C_\varepsilon\|\nabla u^k\|^3 \tag{29}$$

for any small $\varepsilon > 0$. Choosing $\varepsilon$ appropriately and combining (28) and (29), we deduce that

$$\|Au^k\| \le C\left(\|u_t^k\| + \|\nabla w^k\| + \|\nabla u^k\|^3 + \|f\|\right).$$

A similar analysis gives

$$\|\Delta w^k\| \le C\left(\|w_t^k\| + \|\nabla u^k\| + \|\nabla u^k\|^2\|\nabla w^k\| + \|g\|\right).$$

These inequalities, together with (14) and (21), prove (22). This completes the proof of Lemma 2. ∎

**Lemma 3** *The approximations $u^k$ and $\rho^k$ satisfy the following for all $t \in [0, T_0)$:*

$$\int_0^t \|u^k(s)\|_{W^{2,6}}^2\,ds \le \tilde F_1(t), \tag{30}$$

$$\int_0^t \|\nabla u^k(s)\|_{L^\infty}^2\,ds \le \tilde F_2(t), \tag{31}$$

$$\|\nabla\rho^k(t)\|_{L^\infty} \le F_4(t), \quad \|\rho_t^k(t)\|_{L^\infty} \le F_5(t). \tag{32}$$

**Proof:** Observe that

$$\begin{aligned}(\mu+\mu_r)&(\nabla u^k, \nabla\phi)\\ &= -(P_k(\rho^k u_t^k + \rho^k(u^k\cdot\nabla)u^k - 2\mu_r \operatorname{curl} w^k - \rho^k f), \phi)\\ &\equiv (\chi^k, \phi).\end{aligned} \tag{33}$$

for any $\phi \in \mathcal{V}(\Omega)$. It is clear from the previous estimates that $\chi^k$ is uniformly bounded in $L^2(0, T'; (L^6(\Omega))^3)$ for all $T' < T_0$. Hence, from the results by Amrouche and Girault for the Stokes operator (see [1]), we get (30). In particular, from the usual Sobolev embedding results, we also have (31).

We can now apply Lemma 1.3 of [11] to $\rho^k$. We conclude that (32) must hold for some functions $F_4$ and $F_5$. ∎

**Lemma 4** *The approximations $w^k$ satisfy the following estimates for any $t \in [0, T_0)$:*

$$\int_0^t \|w^k(s)\|^2_{W^{2,6}}\, ds \le \tilde{F}_3(t), \tag{34}$$

$$\int_0^t \|\nabla w^k(s)\|^2_{L^\infty}\, ds \le \tilde{F}_4(t). \tag{35}$$

**Proof:** For any $\psi \in C_0^\infty(\Omega)$, we have

$$\begin{aligned}(Lw^k, \psi)&\\ &= -(\rho^k w_t^k + \rho^k(u^k \cdot \nabla)w^k - 2\mu_r \operatorname{curl} u^k - \rho^k g, \psi)\\ &\equiv (\eta^k, \psi).\end{aligned} \tag{36}$$

As before, $\eta^k$ is uniformly bounded in $L^2(0, T'; (L^6(\Omega))^3)$ for all $T' < T_0$. Thus, $w^k$ is uniformly bounded in $L^2(0, T'; (W^{2,6}(\Omega))^3)$. From the Sobolev embeddings, $\nabla w^k$ is also uniformly bounded in $L^2(0, T'; (L^\infty(\Omega))^3)$. ■

**Lemma 5** *The approximations $u^k$ and $w^k$ satisfy the following for all $t \in [0, T_0)$:*

$$\int_0^t \{\|u^k(s)\|^2_{H^3} + \|w^k(s)\|^2_{H^3}\}\, ds \le F_6(t). \tag{37}$$

**Proof:** Again, we will take into account (33). From (30) and the estimates in Lemmas 1 and 2, it is not difficult to see that $\chi^k$ is uniformly bounded in $L^2(0, T'; (H^1(\Omega))^3)$ for all $T' < T_0$. Therefore, using Stokes regularity, we deduce that $u^k$ is also bounded in $L^2(0, T'; (H^3(\Omega))^3)$. This proves the estimate for $u^k$ in (37).

Arguing in a similar way for $w^k$ (starting from (36)), we finally deduce (37) for some $F_6$. ■

The following elementary remark will be useful for further estimates.

**Remark 5** Let $h : (a, b) \mapsto \mathbf{R}$ be a positive continuous function such that

$$\int_a^b h(s)\, ds < +\infty. \tag{38}$$

Then there exists a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to a^+$ such that $\varepsilon_n h(\varepsilon_n) \to 0$ as $n \to +\infty$. ■

**Lemma 6** *Under the assumptions in Theorem 1, we have the following for all* $t \in [0, T_0)$*:*

$$\int_0^t \sigma(s)\{\|u_{tt}^k(s)\|^2 + \|w_{tt}^k(s)\|^2\}\, ds \le F_7(t), \tag{39}$$

$$\sigma(t)(\|\nabla u_t^k(t)\|^2 + \|\nabla w_t^k(t)\|^2) \le F_8(t), \tag{40}$$

$$\sigma(t)\{\|u^k(t)\|_{H^3}^2 + \|w^k(t)\|_{H^3}^2\} \le F_9(t). \tag{41}$$

*Here, we have used the notation* $\sigma(t) = \min\{1,\, t\}$*.*

**Proof:** Differentiating the first equation in (9) with respect to $t$ and taking $v = u_{tt}^k$, in view of the estimates in the previous Lemmas, we get

$$\|u_{tt}^k\|^2 + \frac{d}{dt}\|\nabla u_t^k\|^2 \le G_5(t)\left(1 + \|\nabla u_t^k(t)\|^2\right). \tag{42}$$

Multiplying (42) by $\sigma(t)$ and integrating in $(\varepsilon, t)$, we obtain

$$\int_\varepsilon^t \sigma(s)\|u_{tt}^k(s)\|^2\, ds + \int_\varepsilon^t \sigma(s)\frac{d}{dt}\|\nabla u_t^k(s)\|^2\, ds \le \int_\varepsilon^t G_5(s)\sigma(s)\left(1 + \|\nabla u_t^k(s)\|^2\right) ds.$$

Observe that

$$\int_\varepsilon^t \sigma(s)\frac{d}{dt}\|\nabla u_t^k(s)\|^2\, ds = \sigma(t)\|\nabla u_t^k(t)\|^2 - \sigma(\varepsilon)\|\nabla u_t^k(\varepsilon)\|^2 - \int_\varepsilon^t \sigma'(s)\|\nabla u_t^k(s)\|^2\, ds. \tag{43}$$

In view of Remark 5, we can choose $\varepsilon = \varepsilon_n$ with $\varepsilon_n \to 0$ and

$$\sigma(\varepsilon_n)\|\nabla u_t^k(\varepsilon_n)\|^2 \to 0.$$

Then

$$\begin{aligned}\sigma(t)\|\nabla u_t^k(t)\|^2 &+ \int_0^t \sigma(s)\|\nabla u_{tt}^k(s)\|^2\, ds \\ &\le \int_0^t G_5(s)\sigma(s)\left(1 + \|\nabla u_t^k(s)\|^2\right) ds + \int_0^t \|\nabla u_t^k(s)\|^2\, ds \\ &\le G_6(t)\int_0^t \left(1 + \|\nabla u_t^k(s)\|^2\right) ds.\end{aligned} \tag{44}$$

In a similar way, we can deduce that

$$\begin{aligned}\sigma(t)\|\nabla w_t^k(t)\|^2 + \int_0^t \sigma(s)\|\nabla w_{tt}^k(s)\|^2\,ds \\ \leq G_7(t)\int_0^t \left(1+\|\nabla u_t^k(s)\|^2+\|\nabla w_t^k(s)\|^2\right)\,ds\end{aligned} \tag{45}$$

for all $t\in[0,T_0)$ for some $G_7$. Hence, we have (39) and (40).

The estimates (41) can be proved arguing as in the proof of Lemma 5 and using that (39) holds. This completes the proof. ■

With similar arguments, the following Lemma also holds:

**Lemma 7** *Under the hypotheses in Theorem 1, we have*

$$\int_0^t \sigma(s)\{\|Au_t^k(s)\|^2+\|\Delta w_t^k(s)\|^2\}\,ds \leq F_{10}(t) \tag{46}$$

*for all* $t\in[0,T_0)$. ■

# 4 Proof of Existence

In view of the estimates given in Proposition 1, we can find a triplet $(u,w,\rho)$ and a subsequence, again indexed by $k$, such that $(u^k,w^k,\rho^k)\to(u,w,\rho)$ in the sense indicated in Remark 3. We will now show that this suffices to pass to the limit in (9) and obtain (7).

Thus, let us first prove that

$$\int_0^{T_0}(\rho^k(t)u_t^k(t),v)\phi\,dt \to \int_0^{T_0}(\rho(t)u_t(t),v)\phi\,dt \tag{47}$$

and

$$\int_0^{T_0}(\rho^k(t)w_t^k(t),z)\zeta\,dt \to \int_0^{T_0}(\rho(t)w_t(t),z)\zeta\,dt \tag{48}$$

as $k\to\infty$, for any $v,z\in(C_0^\infty(\Omega))^3$ and any $\phi,\zeta\in\mathcal{D}(0,T)$.

We have

$$\begin{aligned}\left|\int_0^{T_0}(\rho^k u_t^k,v)\phi\,dt\right| \leq \left|\int_0^{T_0}((\rho^k-\rho)u_t^k,v)\phi\,dt\right| \\ +\left|\int_0^{T_0}(\rho^k(u_t^k-u_t),v)\phi\,dt\right|.\end{aligned} \tag{49}$$

Observe that

$$\left|\int_0^{T_0} ((\rho^k-\rho)u_t^k, v)\phi\,dt\right| \le C_{\phi,v}\int_0^{T'} \|\rho^k-\rho\|\;\|u_t^k\|\,dt$$

for some $T' < T_0$. Consequently, the first integral in the right hand side of (49) converges to zero.

On the other hand,

$$\left|\int_0^{T_0} (\rho(u_t^k-u_t), v)\phi\,dt\right| = \left|\int_0^{T'} (u_t^k-u_t, \rho v)\phi\,dt\right|$$

for some $T' < T_0$. Bearing in mind that $u_t^k \to u_t$ weakly-$*$ in $L^\infty(\Omega\times(0,T'))$ (for instance), we deduce that this integral also converges to zero. Thus, we have proved (47). The convergence of $\rho^k w^k$ in (48) can be proved similarly.

Next, let us show that

$$\int_0^{T_0} (\rho^k(u^k\cdot\nabla)u^k, v)\phi\,dt \to \int_0^{T_0} (\rho(u\cdot\nabla)u, v)\phi\,dt \tag{50}$$

and

$$\int_0^{T_0} (\rho^k(u^k\cdot\nabla)w^k, z)\zeta\,dt \to \int_0^{T_0} (\rho(u\cdot\nabla)w, z)\zeta\,dt \tag{51}$$

as $k\to\infty$ for any $v$, $z$, $\phi$ and $\zeta$ as above. We will only prove (50), since the proof of (51) is similar.

Notice that

$$\left\{\begin{aligned} &\int_0^{T_0} (\rho^k(u^k\cdot\nabla)u^k, v)\phi\,dt - \int_0^{T_0} (\rho(u\cdot\nabla)u, v)\phi\,dt \\ &\quad = \int_0^{T_0} ((\rho^k-\rho)(u^k\cdot\nabla)u^k, v)\phi\,dt \\ &\qquad + \int_0^{T_0} (\rho((u^k-u)\cdot\nabla)u^k, v)\phi\,dt \\ &\qquad + \int_0^{T_0} (\rho(u\cdot\nabla)(u^k-u), v)\phi\,dt. \end{aligned}\right. \tag{52}$$

Observe that the first integral in the right hand side of (52) converges to zero. Indeed, we have

$$\left|\int_0^{T_0} ((\rho^k-\rho)(u^k\cdot\nabla)u^k, v)\phi\,dt\right|$$

$$\leq C_{\phi,v}\|\rho^k - \rho\|_{L^2(\Omega\times(0,T'))} \left(\int_0^{T'}\int_\Omega |(u^k\cdot\nabla)u^k|^2\,dx\,dt\right)^{1/2}$$
$$\leq C_{\phi,v}\|\rho^k - \rho\|_{L^2(\Omega\times(0,T'))} \left(\int_0^{T'} \|Au^k\|\|\nabla u^k\|^3\,dt\right)^{1/2}$$

and this converges to zero, in view of the estimates (14) and (22).

The second integral in the right hand side of (52) is

$$\int_0^{T_0}\int_\Omega \rho((u^k - u)\cdot\nabla)u^k\cdot v\phi\,dx\,dt$$
$$\leq C_{\phi,v}\|\rho\|_{L^\infty(\Omega\times(0,T))}\int_0^{T'} \|u^k - u\|\|\nabla u^k\|\,dt$$

and also converges to zero. The third integral can be written in the form

$$\int_0^{T_0}\int_\Omega \rho(u\cdot\nabla)(u^k - u)\cdot v\phi\,dx\,dt.$$

Since $\nabla u^k \to \nabla u$ weakly in $L^2(\Omega\times(0,T))^3$, it also converges to zero as $k\to+\infty$.

By density, it is clear that (47), (48), (50) and (51) hold for any $v\in D(A)$, any $z\in D(L)$ and $\phi,\zeta\in C_0^\infty(0,T)$. We can now pass to the limit in (9). Indeed, for any $v\in\bigcup_{j\geq1}V_j$ and any $\phi\in\mathcal{D}(0,T)$, we have

$$\int_0^{T_0}(\rho^k u_t^k + \rho^k(u^k\cdot\nabla)u^k - 2\mu_r\ \mathrm{curl}\,w^k - \rho^k f - (\mu+\mu_r)\Delta u^k, v)\phi\,dt = 0$$

for all sufficiently large $k$. Letting $k\to+\infty$, we obtain

$$\langle\rho u_t + \rho(u\cdot\nabla)u - 2\mu_r\ \mathrm{curl}\,w - \rho f - (\mu+\mu_r)\Delta u,\ v\rangle = 0$$

a.e. in $(0,T_0)$, for every $v\in\bigcup_{j\geq1}V_j$. This gives

$$P(\rho u_t + \rho(u\cdot\nabla)u - 2\mu_r\ \mathrm{curl}\,w - \rho f - (\mu+\mu_r)\Delta u) = 0$$

a.e. in $\Omega\times(0,T_0)$.

The passage to the limit in the equation for $w^k$ is analogous. In order to deal with the equation for the density, let us simply observe that, for instance, $u^k\to u$ strongly in $L^2(\Omega\times(0,T'))$, $\rho_t^k\to\rho_t$ weakly in $L^2(\Omega\times(0,T'))$ and $\nabla\rho^k\to\nabla\rho$ weakly in $L^2(\Omega\times(0,T'))^3$ for all $T'<T_0$. This gives

$$\rho_t + u\cdot\nabla\rho = 0$$

in the distributional sense.

Let us now check that the initial conditions are satisfied. We will first prove the following result:

**Proposition 2** *Under the assumptions in Theorem 1, we have:*

$$\lim_{t\to 0^+} \|u(t)-u_0\|_{H_0^1} = 0 \tag{53}$$

*and*

$$\lim_{t\to 0^+} \|w(t)-w_0\|_{H_0^1} = 0, \tag{54}$$

*i.e. $u$ and $w$ assume the initial data continuously in the $H_0^1$-norm.*

**Proof:** We will only prove (53), since (54) can be proved similarly. For all $t$, we have

$$\nabla u^k(t) - \nabla u_0^k = \int_0^t \nabla u_t^k(s)\,ds \qquad \forall k\geq 1.$$

Therefore, in view of (21),

$$\|\nabla u^k(t)-\nabla u_0^k\| \leq \int_0^t \|\nabla u_t^k(s)\|\,ds \leq F_2(T_0/2)t \tag{55}$$

for any $t\in[0,T_0/2]$. Now, notice that $u^k$ is bounded in $L^\infty(0,T_0/2;D(A))$ and $u_t^k$ is bounded in $L^\infty(0,T_0/2;V)$. From Aubin-Lions' Lemma, we deduce that $u^k\to u$ strongly in $C^0([0,T_0/2];V)$. Thus, we can pass to the limit in (55), which gives

$$\|\nabla u(t)-\nabla u_0^k\| \leq F_2(T_0/2)t \qquad \forall t\in[0,T_0/2].$$

This proves (53). ■

**Proposition 3** *Under the assumtions in Theorem 1, we have:*

$$\lim_{t\to 0^+} \|Au(t)-Au_0\|_{L^2} = 0 \tag{56}$$

*and*

$$\lim_{t\to 0^+} \|u_t(t)-u_t(0)\|_{L^2} = 0. \tag{57}$$

**Proof:** To prove (56), it is sufficient to show that

$$\limsup_{t\to 0^+} \|Au(t)\| \le \|Au_0\|, \tag{58}$$

since we already know that $\|Au(t)\|$ is bounded near $t = 0$ and $u(t) \to u_0$ strongly in $V$. From the first equation in (9) with $v = Au^k$, after integration in time, we get

$$\|Au^k(t)\|^2 - \|Au_0^k\|^2 = -\int_0^t (m(s), Au_t^k(s))\, ds,$$

where

$$m(t) = \frac{1}{\mu+\mu_r}\,(\rho^k u_t^k + \rho^k (u^k\cdot\nabla)u^k - 2\mu_r \mathrm{curl}\, w^k - \rho^k f).$$

We have

$$\int_0^t (m(s, Au_t^k(s))\, ds = (m(t), Au^k(t)) - (m(0), Au_0) - \int_0^t (m'(s), Au^k(s))\, ds.$$

Therefore, in view of the estimates in Proposition 1, it is not difficult to prove that

$$\begin{aligned}
&\left|\int_0^t (m(s), Au_t^k(s))\, ds\right| \\
&\quad \le |(\rho^k(u^k\cdot\nabla)u^k - 2\mu_r \mathrm{curl}\, w^k - \rho^k f, Au^k)(t) \\
&\qquad -(\rho_0(u_0\cdot\nabla)u_0 - 2\mu_r \mathrm{curl}\, w_0 - \rho_0 f(0), Au_0)| \\
&\quad +G_8(T_0/2)\left(t + t^{1/4}\right)
\end{aligned}$$

for all $t\in[0, T_0/2]$, for some $G_8$. We deduce that

$$\begin{aligned}
\|Au^k(t)\|^2 &\le \|Au_0^k\|^2 + G_8(T_0/2)\left(t+t^{1/4}\right) \\
&\quad +|(\rho^k(u^k\cdot\nabla)u^k - 2\mu_r \mathrm{curl}\, w^k - \rho^k f, Au^k)(t) \\
&\qquad -(\rho_0(u_0\cdot\nabla)u_0 - 2\mu_r \mathrm{curl}\, w_0 - \rho_0 f(0), Au_0)|.
\end{aligned}$$

We know that, for each $t\in[0,T_0/2]$, $Au^k(t)\to Au(t)$ weakly in $L^2(\Omega)$ and $u^k(t)\to u(t)$ strongly in $V$. Thus,

$$\liminf_{k\to+\infty} \|Au^k(t)\|^2 \le \|Au(t)\|^2,$$

we have

$$\lim_{k\to+\infty}(\rho^k(u^k\cdot\nabla)u^k - 2\mu_r \operatorname{curl} w^k - \rho^k f, Au^k)(t)$$
$$= (\rho_0(u_0\cdot\nabla)u_0 - 2\mu_r \operatorname{curl} w_0 - \rho_0 f(0), Au_0)$$

and also

$$\|Au(t)\|^2 \le \|Au_0\|^2 + G_7(T_0/2)\left(t + t^{1/4}\right).$$

Obviously, this leads to (58). ∎

For the angular velocity, we have a similar result:

**Proposition 4** *Under the assumtions in Theorem 1, we have:*

$$\lim_{t\to 0^+}\|\Delta w(t) - \Delta w_0\|_{L^2} = 0 \tag{59}$$

*and*

$$\lim_{t\to 0^+}\|w_t(t) - w_t(0)\|_{L^2} = 0. \tag{60}$$

∎

**Remark 6** The argument used in the proofs of these two Propositions can also be made at any $t = t_0 \in (0, T_0)$ instead of $t = 0$. This implies continuity from the right of $u$, $u_t$ , $w$ and $w_t$ in the appropriate spaces. These arguments can also be adapted to prove continuity from the left at any $t_0 \in (0, T_0)$. In this way, we deduce the continuity properties indicated in Remark 1. ∎

## 5 Proof of Uniqueness

Let $(v, \psi, \sigma)$ be a solution to (1)–(2) in $[0, T')$ and assume that $(v, z, \sigma) \in \mathcal{H}'$. Let us introduce $(\eta, \xi, \pi)$, with $\eta = u - v$, $\xi = w - \psi$ and $\pi = \rho - \sigma$. Then these functions satisfy the following equations:

$$\begin{cases} P(\rho\eta_t + \sigma(v\cdot\nabla)\eta) + (\mu+\mu_r)A\eta \\ \qquad = P(2\mu_r \operatorname{curl}\xi + \pi f - \pi v_t - \pi(u\cdot\nabla)u - \sigma(\eta\cdot\nabla)u), \\ \rho\xi_t + \sigma(v\cdot\nabla)\xi + L\xi \\ \qquad = 2\mu_r \operatorname{curl}\eta + \pi g - \pi\psi_t - \pi(u\cdot\nabla)w - \sigma(\eta\cdot\nabla)w, \\ \pi_t + u\cdot\nabla\pi = -\eta\cdot\nabla\sigma. \end{cases} \tag{61}$$

Multiplying the first equation in (61) by $\eta$ and integrating over $\Omega$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\rho^{1/2}\eta\|^2 + (\mu+\mu_r)\|\nabla\eta\|^2$$
$$= (2\mu_r \operatorname{curl}\xi + \pi f - \pi v_t - \pi(u\cdot\nabla)u - \sigma(\eta\cdot\nabla)u, \eta)$$
$$+\frac{1}{2}(\rho_t\eta,\eta) - (\sigma(v\cdot\nabla)\eta,\eta).$$

Now, estimating the above terms in the usual way, we obtain the following integral inequality:

$$\|\eta(t)\|^2 + \int_0^t \|\nabla\eta(s)\|^2\,ds$$
$$\leq C\int_0^t \left(\|f(s)\|_{L^3}^2 + \|v_t(s)\|_{L^3}^2 + \|\nabla u(s)\|^2\|Au(s)\|^2\right)\|\pi(s)\|^2\,ds$$
$$+ C\int_0^t \|\xi(s)\|^2\,ds$$
$$+ C\int_0^t \left(\|\nabla v(s)\|^4 + \|\nabla u(s)\|^4 + \|\rho_t(s)\|_{L^\infty}\right)\|\eta(s)\|^2\,ds.$$

In a similar way, from the second equation in (61), we find

$$\|\xi(t)\|^2 + \int_0^t \|\nabla\xi(s)\|^2\,ds$$
$$\leq C\int_0^t \left(\|g(s)\|_{L^3}^2 + \|\psi_t(s)\|_{L^3}^2 + \|\nabla u(s)\|^2\|\Delta w(s)\|^2\right)\|\pi(s)\|^2\,ds$$
$$+ C\int_0^t \|\eta(s)\|^2\,ds + \frac{1}{2}\int_0^t \|\nabla\eta(s)\|^2\,ds$$
$$+ C\int_0^t \left(\|\nabla v(s)\|^2 + \|\Delta w(s)\|^2\|\rho_t(s)\|_{L^\infty}\right)\|\xi(s)\|^2\,ds.$$

On the other hand, multiplying the third equation in (61) by $\pi$ and integrating with respect to $x$ and $t$ in $\Omega\times(0,t)$, we obtain:

$$\|\pi(t)\|^2 \leq C\int_0^t \|\eta(s)\|\|\nabla\sigma(s)\|_{L^\infty}\|\pi(s)\|\,ds$$
$$\leq \int_0^t \|\eta(s)\|^2\,ds + C\int_0^t \|\nabla\sigma(s)\|_{L^\infty}^2\|\pi(s)\|^2\,ds.$$

From these estimates, we deduce that

$$\|\eta(s)\|^2 + \|\xi(s)\|^2 + \|\pi(s)\|^2 \leq \int_0^t h(s)(\|\eta(s)\|^2 + \|\xi(s)\|^2 + \|\pi(s)\|^2)\,ds$$

for all $t \in [0, T')$, where

$$h(t) = C(1 + \|f\|_{H^1}^2 + \|g\|_{H^1}^2 + \|v_t\|_{L^3}^2 + \|\psi_t\|_{L^3}^2 \\ + \|\nabla u\|^2(\|Au\|^2 + \|\Delta w\|^2) \\ + \|\nabla u\|^4 + \|\Delta w\|^2 + \|\rho_t\|_{L^\infty} + \|\nabla v\|^4 + \|\nabla \sigma\|_{L^\infty}^2).$$

Observe that $h$ is an integrable function, in view of the regularity of $(u, w, \rho)$ and $(v, \psi, \sigma)$. Consequently, we can apply Gronwall's Lemma, which gives

$$\|\xi(t)\|^2 + \|\eta(t)\|^2 + \|\pi(t)\|^2 \equiv 0,$$

i.e. $u = v$, $w = \psi$ and $\sigma = \rho$. This ends the proof of Theorem 2.

# 6 Some Additional Results Concerning the Pressure

We can now obtain some information on the pressure:

**Proposition 5** *Under the assumptions of Theorem 1, there exists a function $p \in C^0([0, T_0); H^1(\Omega))$ such that*

$$\rho u_t + \rho(u \cdot \nabla)u - (\mu + \mu_r)\Delta u + \nabla p = 2\mu_r \operatorname{curl} w + \rho f$$

*a.e. in $\Omega \times (0, T_0)$.*

**Proof:** Let $(u, w, \rho)$ be the strong solution furnished by Theorem 1 and let us set

$$j = \rho(f - u_t - (u \cdot \nabla)u) + 2\mu_r \operatorname{curl} w + (\mu + \mu_r)\Delta u.$$

Then, from the regularity of $(u, w, \rho)$ (see Remark 1), we easily deduce that

$$j \in C^0([0, T_0); L^2(\Omega)) \cap L^2(0, T_0 - \gamma; H^1(\Omega)) \qquad \forall \gamma > 0 \tag{62}$$

and

$$j_t \in L^2(\delta, T_0 - \gamma; L^2(\Omega)) \qquad \forall \delta, \gamma > 0. \tag{63}$$

From (10), we have

$$(j(t), v) = 0 \qquad \forall v \in \mathcal{V}(\Omega),$$

for $t$ a.e. in $[0, T_0)$. Consequently, we deduce from De Rham's Lemma that $j = \nabla p$ for some $p \in \mathcal{D}'(\Omega \times (0, T_0))$. Furthermore, since we have (62) and (62), we can choose $p$ satisfying

$$p \in C^0([0, T_0); H^1(\Omega)) \cap L^2(0, T_0 - \gamma; H^2(\Omega)) \qquad \forall \gamma > 0$$

and

$$j_t \in L^2(\delta, T_0 - \gamma; H^1(\Omega)) \qquad \forall \delta, \gamma > 0$$

(and depending continuously on $j$ in the corresponding norms). This ends the proof. ∎

**Remark 7** In order to obtain additional information on the pressure at time $t = 0$, appropriate compatibility conditions on the initial data have to be imposed. This can be done as in the case of the Navier-Stokes equations, see [8]. ∎

# 7 Global Existence for Small Data

We present in this Section two additional existence results concerning global in time strong solutions. The proofs can be easily obtained by combining the techniques in the previous Sections and the arguments of [3].

**Theorem 3** *Assume that the initial data $u_0$, $w_0$ and $\rho_0$ satisfy* (3)–(5)*. Also, assume that*

$$f, g \in L^\infty(0, +\infty; (H^1(\Omega))^3), \qquad f_t\,, g_t \in L^\infty(0, +\infty; (L^2(\Omega))^3). \tag{64}$$

*Then, if the norms*

$$\|u_0\|_{H_0^1}\,,\ \|w_0\|_{H_0^1}\,,\ \|f\|_{L^\infty([0,+\infty);L^2(\Omega))}\,,\ \|g\|_{L^\infty([0,+\infty);L^2(\Omega))} \tag{65}$$

*are sufficiently small, the strong solution $(\rho, u, w)$ of* (1)–(2) *exists globally in time and satisfies*

$$u \in C^0([0, +\infty); D(A)),\ w \in C^0([0, +\infty); D(L)),\ \rho \in C^1(\overline{\Omega} \times [0, +\infty)).$$

*Moreover, there exists $C$ such that*

$$\|u_t(t)\| + \|w_t(t)\| + \|Au(t)\| + \|\Delta w(t)\| \le C$$

*for all $t \ge 0$.*

**Proof:** We can repeat the arguments used in Lemma 1. Now, (17) indicates that the approximations $(u^k, w^k, \rho^k)$ are defined in the whole interval $[0, +\infty)$. From (20), which is also satisfied for all $t \ge 0$, we deduce that

$$\left\{ \begin{array}{l} \theta_k(t) \le C(\|\nabla u_0\|^2 + \|\nabla w_0\|^2) \\ \qquad + C \displaystyle\int_0^t (\|f(s)\|^2 + \|g(s)\|^2)\, ds \\ \qquad + C \displaystyle\int_0^t \theta_k(s)\, ds \end{array} \right.$$

for all $t \geq 0$. Consequently, if the norms in (65) are sufficiently small, we have the estimates (14)–(16) for all $t \geq 0$, i.e. Lemma 1 holds with $T_0 = +\infty$.

The arguments in the proofs of the other Lemmas presented in Section 3 can now be repeated with $T_0 = +\infty$. The same can be done in the proof of existence in Section 4. Accordingly, $u$, $w$ and $\rho$ are globally defined and the Theorem is proved. ∎

**Remark 8** Under the assumptions of Theorem 3, it can also be proved that, for each $\gamma > 0$, there exists a positive constant $C_\gamma$ such that

$$\sup_{t\geq 0} e^{-\gamma t} \int_0^t e^{\gamma s}\{\|\nabla u_t(s)\|^2 + \|\nabla w_t(s)\|^2\}\, ds \leq C_\gamma\,, \tag{66}$$

$$\sup_{t\geq 0} e^{-\gamma t} \int_0^t e^{\gamma s}\{\|u(s)\|^2_{W^{2,6}} + \|w(s)\|^2_{W^{2,6}}\}\, ds \leq C_\gamma\,. \tag{67}$$

Furthermore, the same estimates hold uniformly in $k$ for the semi-Galerkin approximations. ∎

With similar arguments, we can also prove the following result:

**Theorem 4** *Assume that we have* (3)–(5), (64) *and*

$$e^{\overline{\gamma} t}(f+g) \in L^\infty(0,+\infty; H^1(\Omega)^3), \quad e^{\overline{\gamma} t}(f_t+g_t) \in L^\infty(0,+\infty; L^2(\Omega)^3),$$

*for some* $\overline{\gamma} > 0$. *Then, if the norms*

$$\|u_0\|_{H_0^1}\,,\ \|w_0\|_{H_0^1}\,,\ \|e^{\overline{\gamma} t} f\|_{L^\infty([0,+\infty);L^2(\Omega)^3)}\,,\ \|e^{\overline{\gamma} t} g\|_{L^\infty([0,+\infty);L^2(\Omega)^3)}$$

*are sufficiently small, the strong solution* $(\rho, u, w)$ *of* (1)–(2) *exists globally in time.*

*There exists* $\gamma^* \in (0, \overline{\gamma})$ *such that*

$$\sup_{t\geq 0} e^{\gamma^* t}\left(\|\nabla u(t)\|^2 + \|\nabla w(t)\|^2\right) < +\infty.$$

*Furthermore, for any* $\theta \in [0, \overline{\gamma})$, *we have the following:*

$$\sup_{t\geq 0} e^{\theta t}\left(\|u_t(t)\|^2 + \|w_t(t)\|^2 + \|Au(t)\|^2 + \|Lw(t)\|^2\right) < +\infty,$$

$$\sup_{t\geq 0} \int_0^t e^{\theta s}\left(\|\nabla u_t(s)\|^2 + \|\nabla w_t(s)\|^2 + \|u(s)\|^2_{W^{2,6}} + \|w(s)\|^2_{W^{2,6}}\right) ds < +\infty,$$

$$\sup_{t\geq 0} (\|\nabla\rho(t)\|_{L^\infty} + \|\rho_t(t)\|_{L^\infty}) < +\infty,$$

$$\sup_{t\geq 0} \tilde{\sigma}(t) \left(\|\nabla u_t(t)\|^2 + \|\nabla w_t(t)\|^2\right) < +\infty,$$

$$\sup_{t\geq 0} \int_0^t \tilde{\sigma}(s) \left(\|u_{tt}(s)\|^2 + \|w_{tt}(s)\|^2 + |Au_t(s)\|^2 + \|Lw_t(s)\|^2\right) ds < +\infty.$$

*In the last three estimates, we have introduced $\tilde{\sigma}(t) = \min\{1, t\}e^{\theta t}$. Estimates of the same kind hold for the semi-Galerkin approximations.* ■

Notice that, under the assumptions in Theorem 3, the $L^\infty$-norms of $\nabla\rho$ and $\rho_t$ can blow up as $t \to +\infty$ (although these functions exist for all $t \geq 0$). Contrarily, the fourth estimate in Theorem 4 provides a uniform bound of these norms in $\overline{\Omega} \times [0, +\infty)$.

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