A logistic equation with degenerate diffusion and Robin boundary condition[1]

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Abstract

In this paper we study a model for a species confined in a bounded region. This species diffuses slowly, follows a logistic law in the habitat and there is a flux of population across the boundary of the habitat.

Basically, we give some theoretical results of the model depending on some parameters which appear in the model.

Key Words. Logistic equation, degenerate diffusion, Robin boundary condition, bifurcation methods.

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Proposed running head: Porous logistic equation and Robin condition

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1 Introduction

We consider some population inhabiting in a bounded region. We assume that this population grows following a logistic law. Moreover, we assume that this population diffuses in its habitat, that is the species is distributed in space and interacts with the physical environment.

This movement is not of random or linear type but the species moves from high density regions to low ones, and it is slower than in the linear case. Finally, unlike of previous works where only absorbing boundary conditions were considered, we assume that there is a flux of population across the boundary of the habitat.

So far, this model was only studied with linear diffusion and lethal boundary condition, that is, all the individuals who reach the boundary they cross it, die or simply leave and do not return. In this paper, we are interested in studying the effects of combining both aspects (nonlinear slow diffusion and flux across the boundary), showing that there is an important change of behavior on the model.

An outline of the paper is: in Section 2 we describe the model, present its novelty and state the main quantitative results. In Section 3 we present the formal model under a mathematical point of view. In Subsection 3.1 we employ the bifurcation method to show the existence of an unbounded continuum of positive solutions emanating from the trivial solution. In Subsection 3.2 we show non-existence results, in Subsection 3.3 we prove the main result of the paper and in the final section we give some conclusions and interpretations of our results.

2 Experimental motivation and main results

The main goal of this paper is a theoretical study of a system modelling some population inhabiting in a heterogeneous environment. The model is based on non-linear partial differential equations of reaction-diffusion type.

We assume mainly three assumptions in the model:

(A1) We assume that this population follows a logistic or Verhulst law; that is, the density of the population affects its growth rate. To be specific, when the population density increases the effects of crowding brings about that the birth rate decreases and the death rate increases.

(A2) The population diffuses, i.e., the population moves in its habitat. It is known that some species migrate to avoid crowding rather than random motion, that is the species moves from high density regions to low density ones. In this work, following the papers [9] and [12], we assume that this mobility depends upon their density. In fact, we consider the case where the diffusion is non-linear and degenerate, which provides a diffusion slower than the classical linear diffusion.

(A3) Finally, we assume that there is a flux of population across the boundary of the habitat. Specifically, we assume that this flux is proportional to the density of the species with constant proportionality $\gamma \in \mathbb{R}$. We consider here three possible cases: no-flux, positive and negative flux. The no-flux or reflecting boundary condition ($\gamma = 0$ or *Neumann* conditions) means that individuals encountering the boundary are always reflected back into the habitat so they do not leave. The case $\gamma \neq 0$ is called *Robin* conditions in the mathematical literature.

The combination of these assumptions gives us a more realistic model which describes better the previous models the reality, due to the previous models considered only some of these assumptions.

We state now some of the main results of the paper. For that, we say that the flux across the boundary is *positive* ($\gamma > 0$) when there is an influx of individuals from outside of the habitat to inside, and *negative* ($\gamma < 0$) in the opposite case. Moreover, we divide our study in *linear*, *slow* and *very slow* diffusion.

1. Fix a positive growth rate, that is assume that the species has a positive growth in its domain. In the case of linear diffusion, there exists a number $\gamma_0 < 0$ such that the species only persists if $\gamma > \gamma_0$. However, if the flux is very negative, $\gamma < \gamma_0$, then the species goes to the extinction. So, if the growth rate is positive the species can persist even if the flux is negative. On the other hand, if the diffusion is slow, then

the species persists for all the values of the flux. Finally, if the diffusion is very slow, then there exists a positive number γ_1 such that the species persists for $\gamma < \gamma_1$ and, however, the species grows in an uncontrolled way if $\gamma > \gamma_1$.

2. Assume now that the death rate is bigger than the birth rate, that is the species has a negative growth in its habitat. Now, in the linear and slow diffusion cases, there exists a number $\overline{\gamma}_0 > 0$ such that the species persists if $\gamma > \overline{\gamma}_0$ and goes to the extinction if $\gamma < \overline{\gamma}_0$. So, in this case in order that the species persists we need a positive flux at the boundary. On the other hand, if the diffusion is very slow, then the species persists for all the values of the flux.

A deeper biological interpretation of these results will be given in the final section.

3 The mathematical approach

We present now the formal model under the assumptions detailed in the above section. Denoting by Ω the habitat and by $w(x)$ the population density of the species in a point $x \in \Omega$, from [9] and [12] we know that, under assumptions $(A1) - (A2)$, w verifies the following logistic equation with degenerate diffusion

$$-\Delta(w^m) = \lambda w - w^2 \quad \text{in } \Omega.$$

The parameter $m > 1$ means that the diffusion is slower than in the linear case ($m = 1$). So, we are going to treat different cases: the case $1 < m < 2$ will be denoted by *slow diffusion*; $m > 2$ *very slow diffusion* and the special case $m = 2$ denoted by *self-diffusion* case, see [11]. Finally, the parameter λ represents the growth rate of the species in the habitat.

Under the change of variable $w^m = u$ we arrive at the equation

$$-\Delta u = \lambda u^{1/m} - u^{2/m} \quad \text{in } \Omega.$$

Now, we introduce assumption $(A3)$ related to the boundary condition, obtaining the model

$$\begin{cases} -\Delta u = \lambda u^{1/m} - u^{2/m} & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} + \alpha u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

being $\Omega \subset \mathbb{R}^N$, $N \geq 1$, a bounded and regular domain, $m \geq 1$, $\lambda, \alpha \in \mathbb{R}$ and n denotes the outward unit normal on $\partial\Omega$.

In Robin condition term, see for instance [2], α measures the rate of individuals which cross the boundary when they encounter it. When $\alpha = 0$ corresponds to the case where no individual crosses the boundary, mathematically called the Neumann case. When $\alpha \to +\infty$ all the individuals who encounter $\partial\Omega$ cross it, that is, $\partial\Omega$ absorbs all of them, this is also known as the lethal or Dirichlet condition. When $\alpha < 0$ the boundary in fact "produces" individuals, that is, there is an influx of individuals from outside of the habitat to inside.

Equation (3.1) has been studied under homogeneous Dirichlet boundary condition $u = 0$ on $\partial\Omega$ in [13], see also [4] and references therein. We summarize here the main results concerning to the Dirichlet case, including the linear diffusion case in order to compare the results.

1. Assume that $m = 1$. Then, there exists a positive solution if, and only if, $\lambda > \lambda_1$, where λ_1 denotes the principal eigenvalue of the laplacian under homogeneous Dirichlet boundary conditions. In case of existence, the positive solution is unique and stable.

2. Assume that $m > 1$. Then, there exists a positive solution if, and only if, $\lambda > 0$. In case of existence, the positive solution is unique and stable.

So, for Dirichelt case when the growth rate is large ($\lambda > \lambda_1$) the behaviour in both cases ($m = 1$ and $m > 1$) is similar. However, if the growth rate is positive but small ($0 < \lambda \leq \lambda_1$) when the diffusion is slow ($m > 1$) the species persists, however when the diffusion is linear the species moves faster and so the number of individuals who attain the lethal boundary is bigger, and the species tends to the extinction because there is an important loss of individuals at the boundary.

We use mainly the bifurcation and the sub-supersolution methods to obtain our results. They show that there are drastic changes on the set of positive solutions of (3.1) depending of the values of m and α. Before showing the result we need some notation. Denote by $\lambda_1(\alpha)$ the principal eigenvalue of the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} + \alpha u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

We will prove that for all the values of $m > 1$ there exists a bifurcation from the trivial solution at $\lambda = 0$ for (3.1). However, unlike that linear case where the bifurcation occurs at $\lambda = \lambda_1(\alpha)$ and it is always supercritical; the bifurcation direction (local and global) depends on the sign of α and the size of m. Specifically, we state now our main results (see Figures 1, 2 and 3):

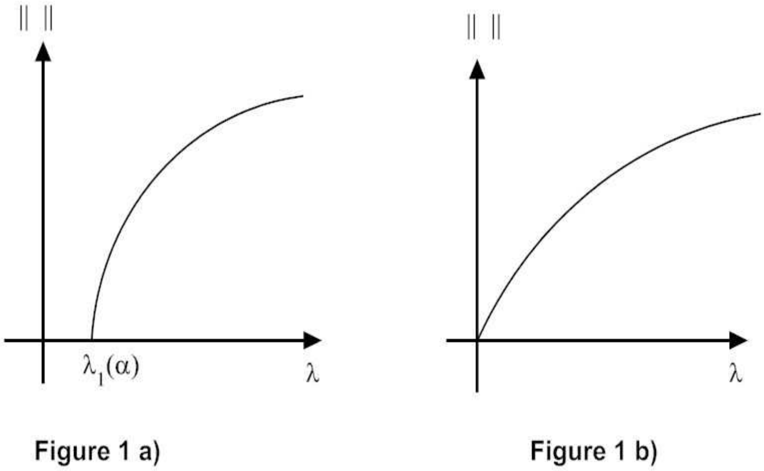

Figure 1: Bifurcation diagrams in the case $\alpha \geq 0$: Case a) $m = 1$; Case b) $m > 1$.

Theorem 3.1. *1. Assume that* $m = 1$. *Then, there exists a positive solution of (3.1) if, and only if,* $\lambda > \lambda_1(\alpha)$. *In case of existence, the solution is unique and stable.*

2. Assume $1 < m < 2$.

(a) If $\alpha \geq 0$. *Then, there exists a positive solution of (3.1) if, and only if,* $\lambda > 0$. *In case of existence, the solution is unique and stable.*

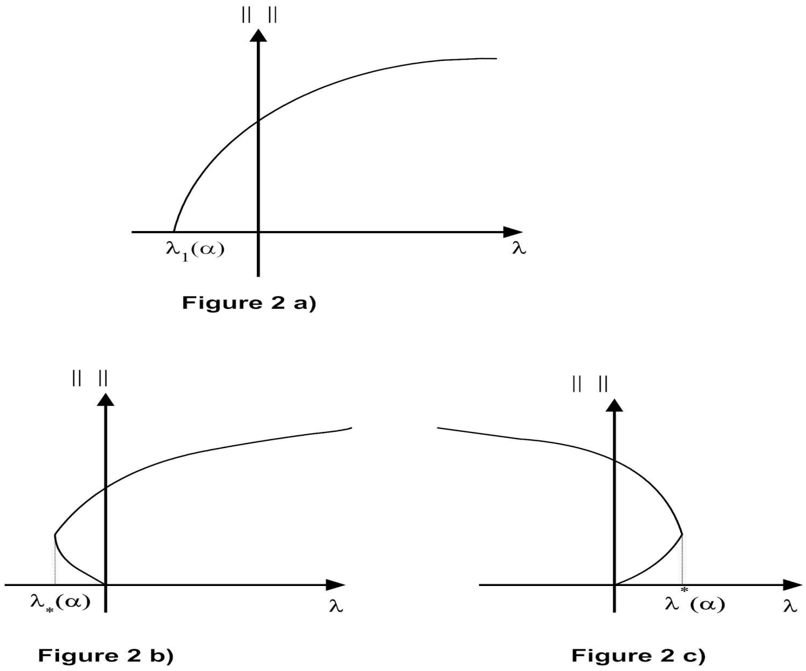

Figure 2: Bifurcation diagrams in the case $\alpha < 0$: Case a) $m = 1$; Case b) $1 < m < 2$; Case c) $m > 2$.

(b) *If* $\alpha < 0$. *There exists* $\lambda_*(\alpha) < 0$ *such that (3.1) possesses at least a positive solution if, and only if,* $\lambda \geq \lambda_*(\alpha)$. *Moreover, the map* $\alpha \in (-\infty, 0) \mapsto \lambda_*(\alpha)$ *is non-decreasing and*

$$\lim_{\alpha \to 0} \lambda_*(\alpha) = 0, \qquad \lim_{\alpha \to -\infty} \lambda_*(\alpha) = -\infty. \tag{3.3}$$

Finally, there exists $\lambda_{**}(\alpha) \in (\lambda_*(\alpha), 0)$ *such that for* $\lambda \in (\lambda_{**}(\alpha), 0)$ *equation (3.1) has at least two positive solutions.*

3. *Assume* $m = 2$.

(a) *If* $\lambda_1(\alpha) + 1 > 0$, *there exists positive solution of (3.1) if, and only if,* $\lambda > 0$. *In this case, the solution is unique and stable.*

(b) *If* $\lambda_1(\alpha) + 1 = 0$, *there exists positive solution of (3.1) if, and only if,* $\lambda = 0$.

(c) *If* $\lambda_1(\alpha) + 1 < 0$, *there exists positive solution of (3.1) if, and only if,* $\lambda < 0$.

4. *Assume* $m > 2$.

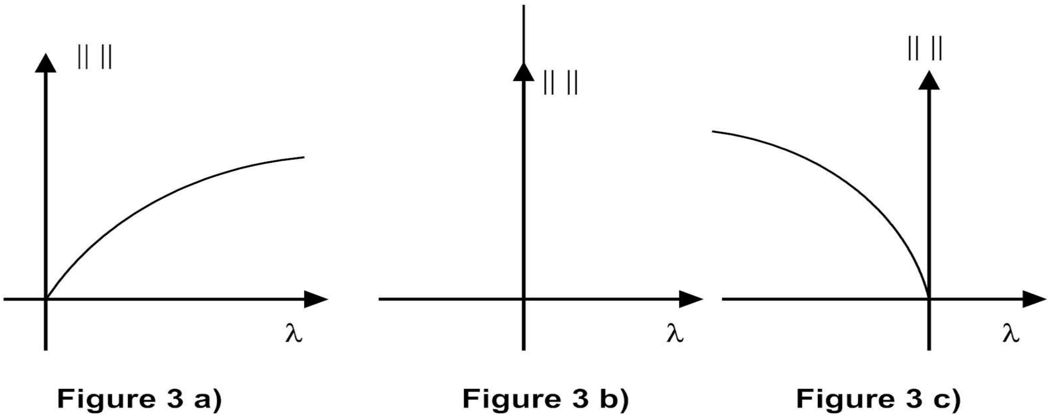

Figure 3: Bifurcation diagrams in the case $\alpha < 0$ and $m = 2$: Case a) $\lambda_1(\alpha) + 1 > 0$; Case b) $\lambda_1(\alpha) + 1 = 0$; Case c) $\lambda_1(\alpha) + 1 < 0$.

(a) *If* $\alpha \geq 0$*. Then, there exists a positive solution of (3.1) if, and only if,* $\lambda > 0$*. In case of existence, the solution is unique and stable.*

(b) *If* $\alpha < 0$*. There exists* $\lambda^*(\alpha) > 0$ *such that (3.1) possesses at least a positive solution if, and only if,* $\lambda \leq \lambda^*(\alpha)$*. Moreover, the map* $\alpha \in (-\infty, 0) \mapsto \lambda^*(\alpha)$ *is non-decreasing and*

$$\lim_{\alpha \to 0} \lambda^*(\alpha) = +\infty, \qquad \lim_{\alpha \to -\infty} \lambda^*(\alpha) = 0. \tag{3.4}$$

Finally, there exists $\lambda^{**}(\alpha) \in (0, \lambda^*(\alpha))$ *such that for* $\lambda \in (0, \lambda^{**}(\alpha))$ *equation (3.1) has at least two positive solutions.*

3.1 Existence results

We are interested in classical positive solutions of (3.1); positive means non-trivial and non-negative. Observe that, thanks to the strong maximum principle, when $\lambda > 0$ any positive solution is in fact strictly positive in $\overline{\Omega}$. However, when $\lambda \leq 0$ could occur that a positive solution vanishes in a part of Ω, appearing the called *dead cores*, but this topic is beyond the scope of this paper, see [3] for related results.

In this section we will show that for any value of the $m > 1$, a bifurcation from the trivial solution of (3.1) occurs at $\lambda = 0$. For that, we consider the Banach space

$$X := C(\overline{\Omega}).$$

Also, remember that the map $\alpha \mapsto \lambda_1(\alpha)$ is increasing and

$$\lim_{\alpha \to -\infty} \lambda_1(\alpha) = -\infty, \qquad \lim_{\alpha \to +\infty} \lambda_1(\alpha) = \lambda_1,$$

where λ_1 denotes the principal eigenvalue of $-\Delta$ under homogeneous Dirichlet boundary conditions. So, since $\lambda_1(0) = 0$ then $\lambda_1(\alpha) < 0$ if $\alpha < 0$ and $\lambda_1(\alpha) > 0$ if $\alpha > 0$.

Finally, we denote by φ_α a positive eigenfunction associated to $\lambda_1(\alpha)$. Remember that, by the strong maximum principle, $\varphi_\alpha > 0$ in $\overline{\Omega}$.

The main result of this section reads:

Theorem 3.2. *The value $\lambda = 0$ is the only bifurcation point from the trivial solutions for* (3.1)*. Moreover, there exists a continuum $\mathcal{C}_0$ of positive solutions of* (3.1) *unbounded in* $\mathbb{R} \times X$ *emanating from* $(0, 0)$*.*

The proof of this result is practically similar to Theorem 4.1 in [5], and so we only outline it. In fact, the main difference is writing our equation (3.1) equivalent to a fixed point equation. Since Ω is smooth, there exists (see [8] and Proposition 3.4 in [10]) a regular function $\psi > 0$ in $\overline{\Omega}$ such that $\partial\psi/\partial n \geq \rho_0 > 0$ in $\partial\Omega$. Consider the change of variable

$$u := e^{M\psi} v, \tag{3.5}$$

which transforms (3.1) into

$$\begin{cases} \mathcal{L}v = \lambda a(x) v^{1/m} - b(x) v^{2/m} & \text{in } \Omega, \\ \dfrac{\partial v}{\partial n} + c(x) v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

where

$$\mathcal{L}v := -\Delta v - 2M \nabla v \cdot \nabla \psi - (M^2 |\nabla \psi|^2 + M \Delta \psi) v,$$

and

$$a(x) := e^{M\psi(1/m-1)} > 0, \qquad b(x) := e^{M\psi(2/m-1)} > 0, \qquad c(x) := \alpha + M \frac{\partial \psi}{\partial n}.$$

Take M large such that $c(x) > 0$. Denote by $\lambda_1[\mathcal{L}; c(x)]$ the principal eigenvalue of the problem

$$\mathcal{L}w = \lambda w \quad \text{in } \Omega, \qquad \frac{\partial w}{\partial n} + c(x)w = 0 \quad \text{on } \partial\Omega.$$

Take $K > 0$ large enough such that $\lambda_1[\mathcal{L} + K; c(x)] > 0$. We extend the function $f(\lambda, x, s) := \lambda a(x)s^{1/m} - b(x)s^{2/m} + Ks$ by taking $f(\lambda, x, s) := 0$ if $s < 0$. Note that f can take negative values. Finally, we define the map

$$\mathcal{K}_\lambda : X \mapsto X; \qquad \mathcal{K}_\lambda(v) := v - (\mathcal{L} + K)^{-1}(f(\lambda, x, v))$$

where $(\mathcal{L} + K)^{-1}$ is the inverse of the operator $\mathcal{L} + K$ under boundary conditions $\frac{\partial v}{\partial n} + c(x)v = 0$, which is well-defined since $c > 0$ and $\lambda_1[\mathcal{L} + K; c(x)] > 0$. Now, we can prove that u is a nonnegative solution of (3.1) if, and only if, v is a zero of the map $\mathcal{K}_\lambda$.

In order to prove Theorem 3.2 we use the Leray-Schauder degree as in [5], see Lemmas 4.2 and 4.3 in [5]. We can get:

Lemma 3.3. *Denote by $i(\mathcal{K}_\lambda, v)$ the index of the isolated zero v of $\mathcal{K}_\lambda$.*

1. *If $\lambda < 0$, then $i(\mathcal{K}_\lambda, 0) = 1$.*
2. *If $\lambda > 0$, then $i(\mathcal{K}_\lambda, 0) = 0$.*

Idea of the proof of Theorem 3.2: The fact that $\lambda = 0$ is a bifurcation point follows by Lemma 3.3. That $\lambda = 0$ is the unique bifurcation point from the trivial solution follows again by [5]. Now, we can conclude the existence of an unbounded continuum of positive solutions of (3.1) emanating from $(\lambda, u) = (0, 0)$. □

3.2 Non-existence results

Concerning to the non-existence of positive solutions of (3.1) we have

Proposition 3.4. *Assume that there exists at least a positive solution u of (3.1) for λ.*

1. *If $\lambda_1(\alpha) \geq 0$, then $\lambda > 0$.*
2. *If $\lambda_1(\alpha) < 0$ and $1 < m < 2$, then*

$$\lambda \geq r(\alpha) := [-\lambda_1(\alpha)]^{1/(2-m)}(m-2)(m-1)^{(m-1)/(2-m)}.$$

3. If $\lambda_1(\alpha) < 0$ *and* $m > 2$*, then*

$$\lambda \leq R(\alpha) := [-\lambda_1(\alpha)]^{1/(2-m)}(m-2)(m-1)^{(m-1)/(2-m)}.$$

4. Assume $m = 2$*.*

(a) If $1 + \lambda_1(\alpha) > 0$*, then* $\lambda > 0$*.*

(b) If $1 + \lambda_1(\alpha) = 0$*, then* $\lambda = 0$*.*

(c) If $1 + \lambda_1(\alpha) < 0$*, then* $\lambda < 0$*.*

Proof. Multiplying (3.1) by a positive eigenfunction φ_α associated to $\lambda_1(\alpha)$ and integrating by parts, we obtain

$$0 = \int_\Omega \varphi_\alpha u^{1/m}(\lambda - u^{1/m} - \lambda_1(\alpha)u^{1-1/m}). \tag{3.7}$$

Now, paragraphs 1 and 4 follow easily from (3.7).

On the other hand, consider the function

$$h(r) := \lambda - r^{1/m} - \lambda_1(\alpha)r^{1-1/m}, \quad r \geq 0.$$

Assume $\lambda_1(\alpha) < 0$ and $1 < m < 2$. In this case, h attains a maximum at

$$r_M = [-\lambda_1(\alpha)(m-1)]^{m/(2-m)},$$

and $h(r_M) = \lambda - [-\lambda_1(\alpha)]^{1/(2-m)}(m-2)(m-1)^{(m-1)/(2-m)}$, whence the result follows. The case $m > 2$ can be reasoned in the same way. □

3.3 Proof of the main results

In the proof of Theorem 3.1 we employ the following result, proved in [7] under homogeneous Dirichlet boundary condition, but that it is also true for Robin boundary condition. Consider the general equation

$$\begin{cases} -\Delta u = f(\lambda, x, u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} + c(x)u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.8}$$

where c is a regular function on $\partial\Omega$ and f is a locally Lipschitz function.

Lemma 3.5. *Suppose that $I \subset \mathbb{R}$ is an interval and let $\Sigma \subset I \times C^2(\overline{\Omega})$ be a connected set of solutions of (3.8). Consider a continuous map $\overline{U} : I \mapsto C^2(\overline{\Omega})$ of supersolutions of (3.8) but not a solution. If for some $(\lambda_0, u_0) \in \Sigma$ we have that $u_0 < \overline{U}(\lambda_0)$, then $u < \overline{U}(\lambda)$ for all $(\lambda, u) \in \Sigma$.*

Now we are ready to prove the main result of the paper.

Proof of Theorem 3.1: The case $m = 1$ follows by Theorem 3.5 in [6]. In fact, there exists a positive solution if, and only if,

$$\lambda > \lambda_1(\alpha). \tag{3.9}$$

In this case the solution is unique and stable.

Assume now $m > 1$ and $\alpha \geq 0$, that is, $\lambda_1(\alpha) \geq 0$. By Proposition 3.4 we know that if there exists at least a positive solution then $\lambda > 0$. So, assume $\lambda > 0$. In this case, it is easily computable that the pair $(\underline{u}, \overline{u}) = (\varepsilon\varphi_\alpha, \lambda^m)$ is a pair of sub-supersolution of (3.1) where ε is a small constant. In fact, using the maximum principle we can show that for any positive solution u of (3.1) we have that

$$u \leq \lambda^m. \tag{3.10}$$

The uniqueness follows by [1], that is, from the fact that $u \mapsto (\lambda u^{1/m} - u^{2/m})/u$ is decreasing, for which we have to use (3.10).

For the stability, we need to show that the first eigenvalue of the linearized around a solution u_0 for $\lambda > 0$ is positive, i. e.,

$$\lambda_1(-\Delta - \frac{\lambda}{m} u_0^{1/m-1} + \frac{2}{m} u_0^{2/m-1}; \alpha) > 0.$$

For that, we use that u_0 is a positive supersolution of the above operator, that is

$$(-\Delta - \frac{\lambda}{m} u_0^{1/m-1} + \frac{2}{m} u_0^{2/m-1}) u_0 > 0 \quad \text{in } \Omega, \qquad \frac{\partial u_0}{\partial n} + \alpha u_0 = 0 \quad \text{on } \partial\Omega.$$

From now on we assume that $m > 1$ and $\alpha < 0$. We get by Theorem 3.2 the existence of an unbounded continuum $\mathcal{C}_0$ of positive solutions to (3.1).

Assume now that $1 < m < 2$. We show now that the bifurcation direction is subcritical. Assume on the contrary that there exists a sequence of positive solutions (λ_n, u_n) of (3.1) with $\lambda_n \geq 0$ and $\|u_n\|_\infty \to 0$. Take a positive constant $M > 0$, then for some $n_0 \in \mathbb{N}$ we have that

$$u_n^{1/m} \geq M u_n \quad \text{for } n \geq n_0.$$

Then, $-\Delta u_n \geq \lambda_n M u_n - u_n^{2/m}$ in Ω and so

$$\lambda_1(-\Delta + u_n^{2/m-1} - \lambda_n M; \alpha) \geq 0,$$

which yields to an absurdum as $\lambda_n \to 0$ because $m < 2$ and $\lambda_1(-\Delta; \alpha) = \lambda_1(\alpha) < 0$.

Hence, we know that the unbounded continuum $\mathcal{C}_0$ goes "to the left" near of $\lambda = 0$, and by Proposition 3.4 we also know that (3.1) does not possess positive solutions for $\lambda \leq r(\alpha)$. Now, we are going to show that $\mathcal{C}_0$ is unbounded because its projection over the λ-axis, called $Proj_{\mathbf{R}}(\mathcal{C}_0)$, is unbounded. First, we recall that, since $m < 2$, the equation

$$\begin{cases} -\Delta z = -z^{2/m} & \text{in } \Omega, \\ \dfrac{\partial z}{\partial n} + \alpha z = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.11}$$

is in the general setting studied in [6], and so (3.11) has positive solution if, and only if (see (3.9)), $0 > \lambda_1(\alpha)$, or equivalently $\alpha < 0$. Moreover, the solution is unique, we denote it by z.

Hence, if u is a positive solution of (3.1) with $\lambda \leq 0$, then u is a subsolution of (3.11) and then

$$u \leq z. \tag{3.12}$$

On the other hand, it can be proved that, since $m < 2$, $\overline{U}(\lambda) := K(\lambda)\varphi_\alpha$ is a supersolution of (3.1) for $K(\lambda)$ verifying

$$K^{1/m}\varphi_\alpha^{1/m} + \lambda_1(\alpha) K^{1-1/m} \varphi_\alpha^{1-1/m} \geq \lambda \quad \text{in } \Omega.$$

If $\lambda \geq 0$ the reaction term of (3.1) is locally Lipschitz and so we can apply Lemma 3.5 for $I := [0, \Gamma_0]$, for any $\Gamma_0 > 0$ and $\lambda_0 = 0$, and conclude that for all $(\lambda, u) \in \mathcal{C}_0$ we have that $u \leq \overline{U}(\lambda)$. Hence, we have proved that $Proj_{\mathbf{R}}(\mathcal{C}_0)$ is unbounded, in fact that $[0, +\infty) \subset Proj_{\mathbf{R}}(\mathcal{C}_0)$.

We can define

$$\lambda_*(\alpha) = \inf\{\lambda : (3.1) \text{ has at least a positive solution.}\}$$

We know that $-\infty < \lambda_*(\alpha) < 0$ and it can be shown, using (3.12) and a standard compactness argument, that there exists at least a positive solution for $\lambda = \lambda_*(\alpha)$. We show now that in fact there exists a positive solution for all $\lambda \geq \lambda_*(\alpha)$. Indeed, take $\lambda > \lambda_*(\alpha)$. Then, the pair $(u_*, K\varphi_\alpha)$ is a sub-supersolution of (3.1) for K large and being u_* a positive solution of (3.1) for $\lambda = \lambda_*(\alpha)$. Finally, the existence of $\lambda_{**}(\alpha)$ verifying the theorem follows by the connectedness of $\mathcal{C}_0$ and the subcritical direction of the bifurcation.

Now we prove (3.3). First, observe that $\lambda_*(\alpha)$ is non-decreasing. Indeed, take $\alpha_1 \leq \alpha_2$ and assume that for some λ there exists a positive solution of (3.1) for $\alpha = \alpha_2$. It can be shown that $(u_{\alpha_2}, K\varphi_{\alpha_1})$ is a sub-supersolution of (3.1) for $\alpha = \alpha_1$ where K is a large constant and u_{α_2} is a positive solution of (3.1) for $\alpha = \alpha_2$. So, there exists positive solution for λ of (3.1) for $\alpha = \alpha_1$ and then $\lambda_*(\alpha_1) \leq \lambda_*(\alpha_2)$.

On the other hand, we know that $r(\alpha) \leq \lambda_*(\alpha) < 0$ and so it is clear that $\lambda_*(\alpha) \to 0$ as $\alpha \to 0$ because $r(\alpha) \to 0$. Moreover, there exists the limit

$$\lim_{\alpha \to -\infty} \lambda_*(\alpha) := \underline{\lambda} < 0.$$

Assume that $-\infty < \underline{\lambda}$. Take $\lambda_0 < \underline{\lambda}$, we are going to show that there exists at least a positive solution for α very negative of (3.1) with $\lambda = \lambda_0$. For that, consider the equation

$$\begin{cases} -\Delta w = \lambda_0 w^{1/m} - w^{2/m} & \text{in } \Omega, \\ w = r & \text{on } \partial\Omega, \end{cases} \tag{3.13}$$

for $r > 0$. We claim that there exists a unique positive solution of (3.13), denoted by w_r. Indeed, the pair $(\underline{w}, \overline{w}) = (0, r)$ is a sub-supersolution of (3.13). Moreover, since $\lambda_0 < 0$, the reaction term is decreasing, and so the uniqueness follows.

Now, it can be shown that the pair

$$(\underline{u}, \overline{u}) := (w_r, Mz)$$

is a sub-supersolution of (3.1) with $\lambda = \lambda_0$, where z is the unique positive solution of (3.11), M is a positive constant large enough, and

$$\alpha \leq -\frac{\partial w_r/\partial n}{r}.$$

This shows the existence of positive solution of (3.1) with $\lambda = \lambda_0$.

Now consider the case $\alpha < 0$ and $m > 2$. Assume now that there exists a sequence of positive solutions (λ_n, u_n) of (3.1) such that $\lambda_n \leq 0$ and $\|u_n\|_\infty \to 0$. Take M large enough such that

$$\lambda_1(-\Delta + M; \alpha) > 0. \tag{3.14}$$

For such M, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ we have that $u_n^{2/m} \geq M u_n$. Then,

$$-\Delta u_n = \lambda_n u_n^{1/m} - u_n^{2/m} \leq -M u_n \quad \text{in } \Omega.$$

Then, multiplying by ψ_α, a positive eigenfunction associated to $\lambda_1(-\Delta + M; \alpha)$ and integrating by parts we obtain

$$\lambda_1(-\Delta + M; \alpha) \int_\Omega u_n \psi_\alpha \leq 0,$$

an absurdum with (3.14).

Now, it is clear that the unbounded continuum $\mathcal{C}_0$ goes "to the right" near of $\lambda = 0$ and, by Proposition 3.4, (3.1) does not possess positive solution for $\lambda \geq R(\alpha)$. We prove now that $Proj_{\mathbf{R}}(\mathcal{C}_0)$ is unbounded. Indeed, assume that there exists a value λ_∞ such that for a sequence (λ_n, u_n) of positive solutions of (3.1) we have that $\lambda_n \to \lambda_\infty$ and $\|u_n\|_\infty \to \infty$. Then, define

$$U_n := \frac{u_n}{\|u_n\|_\infty}.$$

It is clear that $U_n \to U$ in $C^2(\overline{\Omega})$ for some U non-negative and non-trivial and such that

$$-\Delta U = 0 \quad \text{in } \Omega, \qquad \frac{\partial U}{\partial n} + \alpha U = 0 \quad \text{on } \partial\Omega.$$

Again multiplying by φ_α in the above equation, an absurdum follows.

We can define

$$\lambda^*(\alpha) = \sup\{\lambda : \text{(3.1) has at least a positive solution.}\}$$

Again, we can prove that $0 < \lambda^*(\alpha) < \infty$ and that there exists at least a positive solution of (3.1) if $\lambda \le \lambda^*(\alpha)$. Indeed, we know that for all $\lambda \le 0$ there exists at least a positive solution of (3.13) because $(-\infty, 0] \subset Proj_{\mathbf{R}}(\mathcal{C}_0)$. Take $\lambda \in (0, \lambda^*(\alpha))$. Then the pair $(\varepsilon\varphi_\alpha, u^*)$ is a sub-supersolution of (3.13) with ε small and being u^* a positive solution of (3.13) with $\lambda = \lambda^*(\alpha)$.

Finally, we prove (3.4). For that, first we can see that $\lambda^*(\alpha)$ is non-decreasing in α. Consider $\alpha_1 \le \alpha_2$ and assume that for $\lambda > 0$ there exists a positive solution of (3.1) with $\alpha = \alpha_1$. It is clear that u_{α_1} is supersolution of (3.1) for $\alpha = \alpha_2$ and that $\varepsilon\varphi_{\alpha_2}$ is subsolution for ε small. Then, there exists at least a positive solution for λ of (3.1) for $\alpha = \alpha_2$, and so $\lambda^*(\alpha_1) \le \lambda^*(\alpha_2)$. On the other hand, since $\lambda^*(\alpha) \le R(\alpha)$ it is clear that $\lambda^*(\alpha) \to 0$ as $\alpha \to -\infty$ because $R(\alpha) \to 0$. We show now that $\lambda^*(\alpha) \to +\infty$ as $\alpha \to 0$.

Assume that

$$\lim_{\alpha \to 0} \lambda^*(\alpha) = \overline{\lambda} < \infty.$$

Consider $\lambda_0 > \overline{\lambda} > 0$, we will prove that there exists a positive solution of (3.1) for $\lambda = \lambda_0$. Indeed, it can be shown that the pair

$$(\underline{u}, \overline{u}) = (\varepsilon\varphi_\alpha, M\varphi_\alpha)$$

is a sub-supersolution of (3.1) with $\lambda = \lambda_0$ for ε small and M large enough. That $\underline{u}$ is a subsolution follows easily. On the other hand, $\overline{u}$ is supersolution provided of

$$F(M) := M^{1/m}\underline{\varphi}_\alpha^{1/m} + M^{1-1/m}\overline{\varphi}_\alpha^{1-1/m}\lambda_1(\alpha) \ge \lambda_0,$$

being

$$\underline{\varphi}_\alpha := \min_{x\in\overline{\Omega}} \varphi_\alpha(x), \qquad \overline{\varphi}_\alpha := \max_{x\in\overline{\Omega}} \varphi_\alpha(x).$$

The function F attains a maximum at

$$M_0 = \left(\frac{\underline{\varphi}_\alpha^{1/m}}{\overline{\varphi}_\alpha^{1-1/m}\lambda_1(\alpha)(1-m)}\right)^{m/(m-2)},$$

and its value is

$$F(M_0) = \left(\frac{m-2}{m-1}\right)\left(\frac{1}{\lambda_1(\alpha)(1-m)}\right)^{1/(m-2)}\left(\frac{\underline{\varphi}_\alpha}{\overline{\varphi}_\alpha}\right)^{(m-1)/(m(m-2))}.$$

Then, taking φ_α such that $\|\varphi_\alpha\|_\infty = \overline{\varphi}_\alpha = 1$ and using that $\lambda_1(\alpha) \to 0$ and $\underline{\varphi}_\alpha \geq \delta_0 > 0$ as $\alpha \to 0$, for some $\delta_0 > 0$, it follows that $\overline{u}$ is supersolution for α small.

Finally, the case $m = 2$ can be proved in a similar way. This completes the proof.

□

4 Conclusion

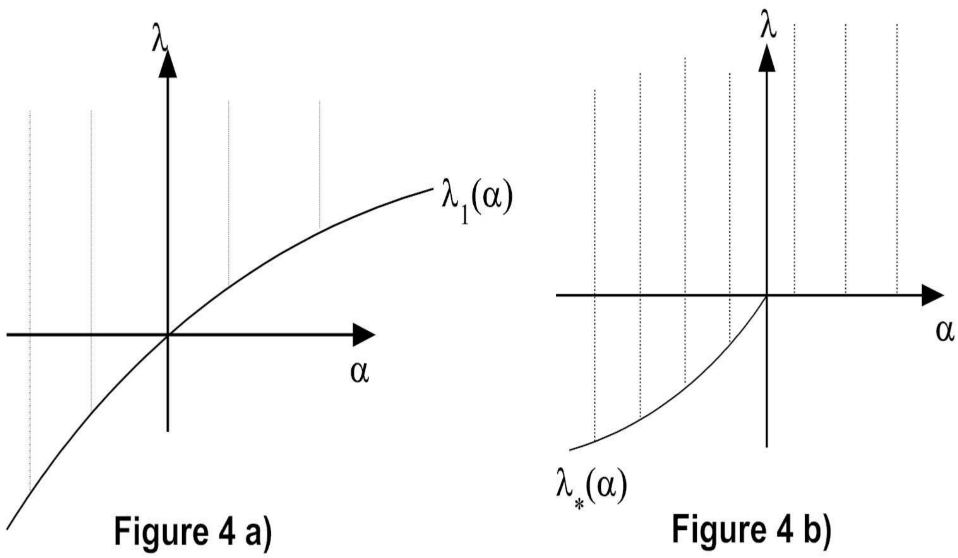

Figure 4: Region of existence of positive solution in the plane (α, λ) in the case $m = 1$, Figure 4 a) and $1 < m < 2$ Figure 4 b).

We have studied a logistic equation with degenerate (slow) diffusion and Robin boundary conditions. The results obtained depend on the values of m and α. We would like to show the different behaviours of the set of positive solutions fixing the value of λ and varying α. For that, according to Theorem 3.1, we have drawn in the (α, λ)-plane the region (lined region) where there exists at least a positive solution of (3.1), we recommend see the Figure 4.

In order to explain the results, we need some notation. Since $\lim_{\alpha \to -\infty} \lambda_1(\alpha) = -\infty$ and $\lim_{\alpha \to +\infty} \lambda_1(\alpha) = \lambda_1$, for all $\lambda < \lambda_1$ there exists a unique value $\overline{\alpha}$ such that $\lambda_1(\overline{\alpha}) = \lambda$. In a similar way, for all $\lambda < 0$ (resp. $\lambda > 0$) there exists a unique value α_* (resp. α^*) such

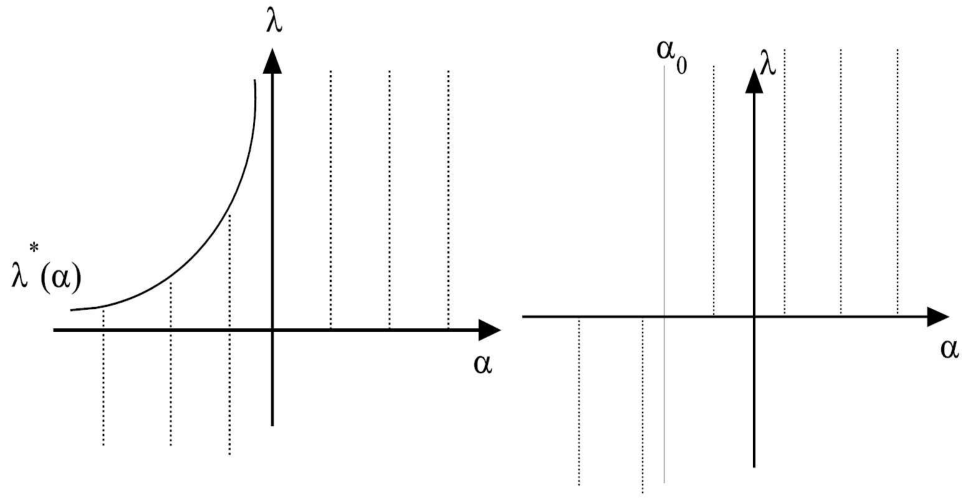

Figure 4 c)

Figure 4 d)

Figure 5: Region of existence of positive solution in the plane (α, λ) in the case $m > 2$, Figure 4 c) and $m = 2$ Figure 4 d).

that $\lambda_*(\alpha_*) = \lambda$ (resp. $\lambda^*(\alpha^*) = \lambda$). Finally, we denote by $\alpha_0 < 0$ the unique value such that

$$\lambda_1(\alpha_0) = -1.$$

Of course, $\overline{\alpha}, \alpha^*$ and α_* depends on λ. In fact, the maps $\lambda \mapsto \overline{\alpha}, \alpha^*, \alpha_*$ are non-decreasing.

Remember that $\alpha > 0$ means that there is a loss of population at the boundary, and that for $\alpha < 0$ there is an increasing of the population.

Now, we differentiate several cases:

Case $\lambda \geq \lambda_1$: In this case for $1 \leq m < 2$ there exists a positive solution for all $\alpha \in \mathbb{R}$. However, if $m = 2$ there exists positive solution if $\alpha > \alpha_0$ and for $\alpha > \alpha^*$ in the case $m > 2$. So, if the growth rate of the species is large, the species survives independently of the value of α in the case of linear or slow diffusion, but for the very slow diffusion and self-diffusion cases it is necessary that α is not very negative. That is, when the species moves very slowly, only some of the individuals attain the boundary, and so in order to avoid that the population grows in a uncontrolled way, it is necessary that the loss of individuals for the boundary is large, that is α positive and large.

Case $\lambda \in (0, \lambda_1)$: In the linear diffusion case, there exists positive solution if $\alpha < \overline{\alpha}$; for $m < 2$ for all $\alpha \in \mathbb{R}$ and if $\alpha > \alpha_0$ if $m = 2$ and for $\alpha > \alpha^*$ in the case $m > 2$. In this cases the possibilities are different. In the liner case, the species moves quickly and so it attains often the boundary, now it is necessary that the loss on the boundary is not so large, $\alpha < \overline{\alpha}$. However if the diffusion is slow then the species does not attain so often the boundary and the loss of population is small.

Case $\lambda = 0$: In this case, there exists a positive solution for all $\alpha < 0$ if $1 \leq m \neq 2$ and for $\alpha = \alpha_0$ if $m = 2$. Here the interpretation is clear: if there is not growth rate ($\lambda = 0$) then we need that the influx to inside occurs ($\alpha < 0$).

Case $\lambda < 0$: In this case, there exists a positive solution for all $\alpha < \overline{\alpha}$ if $m = 1$, for $\alpha < \alpha_*$ if $m < 2$, for $\alpha < \alpha_0$ if $m = 2$ and for all $\alpha < 0$ if $m > 2$. So, if the growth rate is negative, then we need to introduce populations across the boundary; of course we need to introduce less population if the diffusion is slower.

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