Bifurcation from zero of a complete trajectory for non-autonomous logistic PDEs

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Abstract

In this paper we extend the well-known bifurcation theory for autonomous logistic equations to the non-autonomous equation

$$u_t - \Delta u = \lambda u - b(t)u^2 \quad \text{with} \quad b(t) \in [b_0, B_0],$$

$0 < b_0 < B_0 < 2b_0$. In particular, we prove the existence of a unique uniformly bounded trajectory that bifurcates from zero as λ passes through the first eigenvalue of the Laplacian, which attracts all other trajectories. Although it is this relatively simple equation that we analyse in detail, other more involved models can be treated using similar techniques.

Keywords: Non-autonomous differential equations; bifurcation from zero; comparison techniques; complete trajectories

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1 Introduction

The bifurcation behaviour of positive solutions of the autonomous logistic PDE

$$u_t - \Delta u = \lambda u - b(x)u^2 \qquad x \in \Omega, \quad \text{with} \quad u = 0 \text{ on } \partial\Omega$$

under various conditions on the coefficient $b(x)$ is well-known (see for example Smoller (1983) and references therein). The simplest such result, when $b(x) \equiv b > 0$, guarantees that while $\lambda < \lambda_1$ (the first eigenvalue of the Laplacian on

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Ω) all such solutions tend to zero, while for $\lambda > \lambda_1$ there is a unique positive solution of the equation

$$-\Delta u = \lambda u - bu^2$$

which attracts all solutions as $t \to +\infty$ (see also Section 2, below).

In this paper we extend this result to the non-autonomous equation

$$\begin{cases} u_t - \Delta u = \lambda u - b(t)u^2 & \text{in } \Omega \times (s, \infty), \\ u = 0 & \partial\Omega, \\ u(s) = u_0 & \text{in } \Omega, \end{cases} \tag{1}$$

where Ω is a bounded and regular domain of $\mathbb{R}^N$, λ is a real parameter, $u_0 \in C(\overline{\Omega})$ with $u_0 > 0$ and $b \in C(\mathbb{R})$ satisfies

$$0 < b_0 \leq b(t) \leq B_0, \quad \text{for all } t \in \mathbb{R},$$

with $B_0 < 2b_0$.

We note here that Hess (1991, Theorem 28.1) proved a similar result (without the condition relating b_0 and B_0) when $b(t)$ is a periodic function. Two previous papers by the current authors (Langa & Suárez, 2000; Langa et al., 2003) treated related examples in which information could only be obtained 'in the pullback sense' (see Section 3); although here we still use a construction based on pullback ideas we focus on attraction forwards in time: to our knowledge this is the first result proving the existence of trajectories of (1) that are attracting in the conventional sense (as $t \to +\infty$) that allows for a more general non-autonomous term.

Whether the condition $B_0 < 2b_0$ is in fact necessary in general, or merely a technical artefact of our proof is unclear. From one point of view this condition is natural, in that it constrains $b(t)$ to be 'close to autonomous' in that it cannot fluctuate too wildly. On the other hand the non-autonomous logistic ODE

$$\dot{x} = \lambda x - b(t)x^2 \qquad \text{with} \qquad 0 < b_0 \leq b(t) \leq B_0$$

has a positive attracting trajectory when $\lambda > 0$ for any choice of b_0 and B_0 (see also Section 3).

2 Notation and preliminaries

2.1 Results for autonomous equations

Before studying equation (1), we first recall various results for the autonomous equation

$$u_t - \Delta u = \lambda u - Au^2 \qquad x \in \Omega, \quad \text{with} \quad u = 0 \text{ on } \partial\Omega. \tag{2}$$

Given an initial condition $u(s) = u_0$ this equation has a unique positive solution $u_A(t, s; u_0)$ for all $t \geq s$.

Denote by λ_1 the first eigenvalue of the Laplacian on Ω with Dirichlet boundary conditions. Then if $\lambda < \lambda_1$ we have

$$\lim_{t\to+\infty} u_A(t,s;u_0) = \lim_{s\to-\infty} u_A(t,s;u_0) = 0,$$

while if $\lambda > \lambda_1$ then

$$\lim_{t\to+\infty} u_A(t,s;u_0) = \lim_{s\to-\infty} u_A(t,s;u_0) = \theta_{[\lambda,A]},$$

where $\theta_{[\lambda,A]}$ is the unique positive solution of

$$\begin{cases} -\Delta u = \lambda u - Au^2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

We will need a condition to guarantee that solutions of the linear equation

$$\begin{cases} w_t - \Delta w = a(x)w & \text{in } \Omega \times (s,\infty), \\ w = 0 & \partial\Omega, \\ w(s) = u_0 & \text{in } \Omega. \end{cases} \tag{4}$$

(with $a \in L^\infty(\Omega)$) tend to zero. To this end, given $q \in L^\infty(\Omega)$ we will denote by $\lambda_1(q)$ the principal eigenvalue of the problem

$$\begin{cases} -\Delta w + q(x)w = \sigma w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

(note that $\lambda_1 = \lambda_1(0)$). If $\lambda_1(-a) < 0$ then $w \to \infty$ as $t \to +\infty$, while if $\lambda_1(-a) > 0$ then $w \to 0$ as $t \to +\infty$.

It is shown by López-Gómez (1996, Theorem 2.5) that if there exists a function ψ that is positive within the interior of Ω and satisfies

$$-\Delta\psi + q\psi > 0 \tag{5}$$

then $\lambda_1(q) > 0$.

2.2 Existence, uniqueness, and regularity properties

Following arguments due to Mora (1983) it is possible to show that solutions of (1) with continuous initial conditions exist and are unique, and enjoy parabolic smoothing. We denote by X the space $C_0^0(\overline{\Omega})$ of all continuous functions on $\overline{\Omega}$ that are zero on $\partial\Omega$, equipped with its natural norm $\|\cdot\|_\infty$.

Theorem 2.1 *Given an initial condition $u(s) = u_s \in X$, then there exists a unique solution $u(t,s;u_s)$ of (1), which satisfies*

$$u(t,s;u_s) \in C^0([s,T];X)$$

for each $T > s$, and depends continuously on u_s. Furthermore for each $t > s$ and $k > 0$ the solution operator $S(t,s)$ is a bounded map from X into $C^k(\overline{\Omega})$.

Note that in particular the solution operator $S(t,s)$ defined by

$$S(t,s)u_s = u(t,s;u_s)$$

is a continuous and compact operator from X into itself.

2.3 Order-preserving properties

Many of our arguments rely on the order-preserving nature of equation (1), and the resulting possibility of comparing solutions to those of suitable autonomous equations.

The following comparison principle, which in particular shows that solutions of (1) preserve the order of their initial conditions, is key to all that follows. We say that $u \geq v$ if $u(x) \geq v(x)$ everywhere in Ω.

Lemma 2.2 *Denote by $u(t,s;u_0)$ the solution of*

$$u_t - \Delta u = \lambda u - b(t)u^2 \qquad u(s) = u_0$$

and by $v(t,s;v_0)$ the solution of

$$v_t - \Delta v = \lambda v - \beta(t)v^2 \qquad v(s) = v_0.$$

Then

$$u_0 \geq v_0 \quad \text{and} \quad b(t) \leq \beta(t) \quad \Rightarrow \quad u(t,s;u_0) \geq v(t,s;v_0) \quad \textit{for all} \quad t \geq s.$$

In particular if $u(t,s;u_s)$ denotes the solution of (1) with $u(s) = u_s$ then

(i) if $u_s \geq 0$ then $u(t,s;u_s) \geq 0$ for all $t \geq s$, and

(ii) when $b_0 \leq b(t) \leq B_0$ we have

$$u_{B_0}(t,s;u_0) \leq u(t,s;u_0) \leq u_{b_0}(t,s;u_0), \tag{6}$$

where u_β is the solution of the autonomous equation that has $b(t) \equiv \beta$.

We will deduce this as a corollary of a more general nonlinear comparison principle.

Proposition 2.3 *(Comparison principle) Suppose that u and v are C^2 in space and C^1 in time,*

(i) $f(u,t) \geq g(u,t)$,

(ii) $f(u+s,t) - f(u,t) \leq Ls$ for all $0 < s < \delta$ for some $\delta > 0$,

(ii) $u|_{\partial\Omega} \geq v|_{\partial\Omega}$,

(iii) $u_0 \geq v_0$*, and*

(iv) $u_t - \Delta u - f(u,t) \geq v_t - \Delta v - g(v,t)$.

The $u(x,t) \geq v(x,t)$ *for all* $t \geq 0$.

(Proposition 2.3) Following Walter (2002) we first prove *Nagumo's Lemma*, in which most of the inequalities are strict: suppose that

(i) $f(u,t) \geq g(u,t)$,

(ii) $u|_{\partial\Omega} > v|_{\partial\Omega}$,

(iii) $u_0 > v_0$, and

(iv) $u_t - \Delta u - f(u,t) > v_t - \Delta v - g(v,t)$.

Then $u(x,t) > v(x,t)$ for all $t \geq 0$. Indeed, consider the difference $w(x,t) = u(x,t) - v(x,t)$. Then if the result does not hold there exists a pair (x_0, t_0) with $x_0 \in \operatorname{int}\Omega$ such that

$$w(x_0,t_0) = 0, \qquad \Delta w(x_0,t_0) \geq 0, \quad \text{and} \quad w_t(x_0,t_0) < 0.$$

Equivalently, at (x_0, t_0) we have

$$u(x_0,t_0) = v(x_0,t_0), \qquad \Delta u(x_0,t_0) \geq \Delta v(x_0,t_0), \quad \text{and} \quad u_t(x_0,t_0) \leq v_t(x_0,t_0).$$

It follows that

$$[u_t - \Delta u - f(u,t)](x_0,t_0)] \leq [v_t - \Delta v - f(v,t)](x_0,t_0),$$

contradicting (iv).

In order to prove the result as stated we now consider $\tilde{u}(x,t) = u(x,t) + \epsilon \mathrm{e}^{Lt}$. Then

$$\tilde{u}_0 > u_0 \geq v_0, \qquad \tilde{u}|_{\partial\Omega} > u|_{\partial\Omega} \geq v|_{\partial\Omega},$$

and

$$\begin{aligned} \tilde{u}_t - \Delta\tilde{u} - f(\tilde{u},t) &= u_t + L\epsilon\mathrm{e}^{Lt} - \Delta u - f(u + \epsilon\mathrm{e}^{Lt}, t) \\ &> u_t - \Delta u - f(u,t), \end{aligned}$$

since $f(u + \epsilon\mathrm{e}^{Lt}, t) < f(u,t) + L\epsilon\mathrm{e}^{Lt}$. Nagumo's lemma implies that $\tilde{u}(x,t) > v(x,t)$ for all $t \geq 0$, and the result holds on taking the limit as $\epsilon \to 0$.

(Lemma 2.2). We note from the regularity results in Theorem 2.1 that u and v satisfy the smoothness requirements for all $s > t$. Set $f(u,t) = \lambda u - b(t)u^2$ and $g(u,t) = \lambda v - \beta(t)v^2$; clearly $f(u,t) \geq g(u,t)$ if $b(t) \leq \beta(t)$, and for $u, s \geq 0$

$$f(u+s,t) - f(u,t) = \lambda s - b(t)[s^2 + 2us] < \lambda s.$$

Finally we introduce sub- and super- trajectories as a generalisation of sub- and super- and equilibria in Hess (1991); see also Chueshov (2001) or Langa and Suárez (2002).

Definition 2.4 *We call* $\underline{u}$ *(*$\overline{u}$*)* $: R \to X$ *a sub-trajectory (super-trajectory) of (1) if it satisfies*

$$u(t,s;\underline{u}(s)) \geq \underline{u}(t) \text{ for all } t \geq s \quad (u(t,s;\overline{u}(s)) \leq \overline{u}(t) \text{ for all } t \geq s).$$

3 An attracting trajectory bifurcating from zero

In this section we prove our main result, namely that while all solutions tend to zero for $\lambda < \lambda_1$, when $\lambda > \lambda_1$ there exists a unique complete trajectory, i.e. a $u : R \to X$ such that

$$u(t,s;u(s)) = u(t) \qquad \text{for all} \qquad t \geq s,$$

that is bounded above and below, and that this trajectory is attracting.

Note that it is clear that when $b(t) \geq 0$ and $\lambda < \lambda_1$, the solution of (1) tends to zero as $t \to +\infty$. We therefore concentrate on the behaviour of solutions for $\lambda > \lambda_1$.

First we show that there exists at least one complete trajectory that is bounded above and below. The argument showing the uniqueness of this trajectory also shows that it attracts all other trajectories as $t \to +\infty$.

It is interesting to remark that although we are seeking a trajectory that attracts as $t \to +\infty$, the construction in fact involves the notion of 'pullback attraction'. This can be clearly illustrated by considering the non-autonomous logistic ODE

$$\dot{x} = \lambda x - b(t)x^2 \qquad \text{with} \qquad \lambda > 0 \quad \text{and} \quad 0 < b_0 \leq b(t) \leq B_0. \tag{7}$$

For the initial condition $x(s) = x_s$ this equation admits the explicit solution

$$x(t,s;x_s) = \frac{\mathrm{e}^{\lambda t}}{\mathrm{e}^{\lambda s} x_s^{-1} + \int_s^t \mathrm{e}^{\lambda r} b(r)\,\mathrm{d}r}.$$

Fixing x_s and letting $s \to -\infty$ we obtain the 'pullback limit'

$$x^*(t) = \frac{\mathrm{e}^{\lambda t}}{\int_{-\infty}^t \mathrm{e}^{\lambda r} b(r)\,\mathrm{d}r},$$

which is a complete trajectory of (7). (The lower bound on $b(t)$ ensures that $x^*(t)$ is well-defined, while the upper bound ensures that it is bounded away from zero.)

While it is clear by construction that $x^*(t)$ is 'pullback attracting', i.e. that

$$\lim_{s \to -\infty} x(t,s;x_s) = x^*(t)$$

for any x_s, it can also be shown by direct calculation that

$$|x(t,s;x_s) - x^*(t)| \to 0 \qquad \text{as} \qquad t \to +\infty$$

for any choice of s and x_s, i.e. that $x^*(t)$ attracts all trajectories forwards in time.

Our proof also uses this pullback technique to find an appropriate candidate solution that will attract forwards in time.

Theorem 3.1 *Suppose that $\lambda > \lambda_1$ and $B_0 < 2b_0$. Then there exists a unique complete trajectory $u(t)$ for (1) that satisfies*

$$\theta_{[\lambda,B_0]} \leq u(t) \leq \theta_{[\lambda,b_0]} \qquad \textit{for all} \qquad t \in R. \tag{8}$$

This trajectory attracts all other positive solutions as $t \to +\infty$, i.e.

$$\lim_{t\to+\infty} \|u(t,s;u_s) - u(t)\|_\infty = 0.$$

We note here that in fact the proof shows more, namely that the trajectory $u(t)$ is *uniformly* attracting forwards in time

$$\lim_{t\to+\infty} \sup_{s\in R} \|u(t+s,s;u_s) - u(t+s)\|_\infty = 0.$$

This implies uniform 'pullback convergence' (cf. Cheban et al., 2002),

$$\lim_{s\to+\infty} \sup_{t\in R} \|u(t,t-s;u_s) - u(t)\|_\infty = 0.$$

Observe that

$$\underline{u}(t) := \theta_{[\lambda,B_0]} \qquad \text{and} \qquad \overline{u}(t) := \theta_{[\lambda,b_0]},$$

are sub and super-trajectories of (1). Indeed, for $\underline{u}$ we have to prove that $u(t,s;\underline{u}(s)) \geq \underline{u}(t)$, or equivalently

$$\theta_{[\lambda,B_0]} \leq u(t,s;\theta_{[\lambda,B_0]}),$$

which holds since $\theta_{[\lambda,B_0]}$ is a sub-solution of (1) with $u_0 = \theta_{[\lambda,B_0]}$.

We now consider the 'pullback limit'

$$u(t) = \lim_{s\to+\infty} u(t,t-s;\theta_{[\lambda,b_0]}), \tag{9}$$

following the argument in Langa & Suárez (2002). To show that this limit exists, observe that

$$\begin{aligned} u(t,s;\theta_{[\lambda,b_0]}) &= S(t,s)\theta_{[\lambda,b_0]} \\ &= S(t,s+\epsilon)S(s+\epsilon,s)\theta_{[\lambda,b_0]} \\ &\leq S(t,s+\epsilon)\theta_{[\lambda,b_0]}, \end{aligned}$$

since $\theta_{[\lambda,b_0]}$ is a super-solution. It follows that $u(t,t-s;\theta_{[\lambda,b_0]})$ is pointwise non-increasing in s. Since it is bounded below by $\theta_{[\lambda,B_0]}$ this sequence converges pointwise (in x) to a limit $u^*(x,t)$. Now we also have

$$S(t,s)\theta_{[\lambda,b_0]} = S(t,t-1)[S(t-1,s)\theta_{[\lambda,b_0]}],$$

and we know that

$$\theta_{[\lambda,B_0]} \le S(t-1,s)\theta_{[\lambda,b_0]} \le \theta_{[\lambda,b_0]},$$

Therefore

$$S(t,s)\theta_{[\lambda,b_0]} \in S(t,t-1)\mathcal{I},$$

where $\mathcal{I}$ is the bounded set

$$\{u \in X : \ \theta_{[\lambda,B_0]} \le u \le \theta_{[\lambda,b_0]}\}.$$

Since $S(t,t-1)$ is a compact operator from X into itself (see Theorem 2.1), for each fixed t we know that $S(t,s)\theta_{[\lambda,b_0]}$ lies in a compact subset of X.

It follows that $S(t,s)\theta_{[\lambda,b_0]}$ converges uniformly (in x) to $u^*(x,t)$. Suppose not; then there exists an $\epsilon > 0$ and a sequence $s_n \to -\infty$ such that

$$\|S(t,s_n)\theta_{[\lambda,b_0]} - u^*(x,t)\|_\infty > \epsilon. \tag{10}$$

But by compactness of $S(t,t-1)\mathcal{I}$ there must be a subsequence s_{n_j} that converges uniformly to some $v^*(x)$. Since $S(t,s_{n_j})\theta_{[\lambda,b_0]} \to u^*(x,t)$ pointwise, we must have $v^*(x) = u^*(x,t)$, contradicting (10).

This trajectory $u(t)$ is a complete trajectory, since $S(t,s)$ is continuous from X into itself:

$$\begin{aligned}
S(t,\tau)u(\tau) &= S(t,\tau)\lim_{s\to-\infty} S(\tau,s)\theta_{[\lambda,b_0]} \\
&= \lim_{s\to-\infty} S(t,\tau)S(\tau,s)\theta_{[\lambda,b_0]} \\
&= \lim_{s\to-\infty} S(t,s)\theta_{[\lambda,b_0]} \\
&= u(t),
\end{aligned}$$

and it is clear that $u(t)$ satisfies

$$\theta_{[\lambda,B_0]} \le u(t) \le \theta_{[\lambda,b_0]} \quad \text{for all} \quad t \in R.$$

In order to prove both the uniqueness of this bounded trajectory and its forwards attraction property we consider two solutions of (1), $u(t,s;u_s)$ and $v(t,s;v_s)$ with $u_s, v_s \in \mathcal{V}_+$ and $u_s, v_s \ne 0$, and show that

$$\|u(t,s;u_s) - u(t,s;v_s)\|_\infty \to 0, \qquad \text{as} \qquad t \to +\infty \quad \text{or} \quad s \to -\infty. \tag{11}$$

Since (1) is order-preserving it follows that

$$p(t,s;p_s) \le u(t,s;u_s), v(t,s;v_s) \le q(t,s;q_s),$$

where

$$p_s := \min\{u_s, v_s\} \quad \text{and} \quad q_s := \max\{u_s, v_s\}$$

($p(\cdot)$ and $q(\cdot)$ are also solutions of (1)). Without loss of generality we will assume that $u_s > v_s$ and hence that

$$w := u - v > 0 \qquad \text{for all} \qquad t \ge s.$$

The function w satisfies the equation

$$\begin{cases} w_t - \Delta w = \lambda w - b(t)(u+v)w & \text{in } \Omega \times (s,\infty), \\ w = 0 & \text{on } \partial\Omega, \\ w(s) = u_s - v_s > 0 & \text{in } \Omega. \end{cases} \tag{12}$$

Now note that for any ε, it follows from (6) and the forward behaviour of the autonomous logistic equation, that there exists T_ε such that for $t - s \ge T_\varepsilon$ we have

$$\theta_{[\lambda,B_0]} - \varepsilon \le u(t,s;u_s), v(t,s;v_s) \le \theta_{[\lambda,b_0]} + \varepsilon, \tag{13}$$

and therefore for such t and s

$$w_t - \Delta w \le w(\lambda - 2b_0(\theta_{[\lambda,B_0]} - \varepsilon)).$$

We now show that

$$\lambda_1(2b_0\theta_{[\lambda,B_0]}) > \lambda, \tag{14}$$

for which it suffices to show that

$$\lambda_1(2b_0\theta_{[\lambda,B_0]} - \lambda) > 0. \tag{15}$$

But it is not hard to prove that $\theta_{[\lambda,B_0]} > 0$ is a supersolution of

$$-\Delta u + (2b_0\theta_{[\lambda,B_0]} - \lambda)u = 0$$

as required by (5). Indeed,

$$-\Delta\theta_{[\lambda,B_0]} + (2b_0\theta_{[\lambda,B_0]} - \lambda)\theta_{[\lambda,B_0]} = (2b_0 - B_0)\theta^2_{[\lambda,B_0]} > 0$$

since $2b_0 > B_0$.

Choosing ε sufficiently small that

$$\lambda_1(2b_0\theta_{[\lambda,B_0]}) > \lambda + 2\varepsilon b_0 \tag{16}$$

and using the corresponding value of T_ε in (13), the convergence property (11) follows.

To show that $u(t)$ is the unique trajectory bounded as in (8) take u, v two trajectories verifying (8). We can assume that $u(t) > v(t)$ for all $t \in \mathbb{R}$. Using that (1) is order-preserving and (8) we get

$$v(t,s;\theta_{[\lambda,B_0]}) \leq v(t,s;v(s)) = v(t) < u(t) = u(t,s;u(s)) \leq u(t,s;\theta_{[\lambda,b_0]}).$$

Now, it suffices to fix t and let s tend to $-\infty$.

That this trajectory attracts all other positive trajectories as $t \to +\infty$ is an immediate consequence of (11).

4 More general models

As we mentioned above, equation (1) is a prototype to which these results can be applied. In this section we want to mention some others examples for which our theory works.

Firstly, the Laplacian operator can be replaced by a general second order uniformly elliptic (not necessarily self-adjoint) operator, i.e

$$\mathcal{L}(\sqcap) = -\sum_{\rangle,|=\infty}^{\backslash} \dashv_{\rangle|}(\S)\partial_{\rangle}\partial_{|}\sqcap + \sum_{\rangle}^{\backslash} \lfloor_{\rangle}(\S)\partial_{\rangle}\sqcap + \rfloor(\S)\sqcap,$$

with $a_{ij} = a_{ji}$ and all coefficients sufficiently regular. In this case the system

$$\begin{cases} u_t + \mathcal{L}(\sqcap) = \lambda\sqcap - \lfloor(\sqcup)\sqcap^{\in} & \text{in } \Omega \times (s,\infty), \\ u = 0 & \text{on } \partial\Omega, \\ u(s) = u_0 & \text{in } \Omega, \end{cases} \tag{17}$$

has, once again, a unique uniformly bounded complete trajectory which bifurcates from zero when λ passes the first eigenvalue associated to $\mathcal{L}$.

On the other hand, our results also hold for the reaction term $\lambda u - b(t)u^p$ with $p > 1$ where we must now assume that

$$p\, b_0 > B_0. \tag{18}$$

Indeed, in this case instead of (12), we can write

$$w_t - \Delta w = \lambda w - b(t)(u^p - v^p),$$

so that, by the mean value theorem, $u^p - v^p = p\xi^{p-1}w,\ v \leq \xi \leq u$, and then

$$w_t - \Delta w \leq (\lambda - pb_0(\theta^{p-1}_{[\lambda,B_0]} - \varepsilon))w,$$

Now, by (18), we have that $\lambda_1(pb_0\theta^{p-1}_{[\lambda,B_0]}) > \lambda$.

Now, assume that we have a heterogeneous environment and so the function b depends on x and t. Specifically, $b \in C(\overline{\Omega} \times \mathbb{R})$ such that there exist two non-negative and non-trivial continuous functions a_0 and A_0 such that

$$0 \leq a_0(x) \leq b(x,t) \leq A_0(x) \qquad \text{for all } t \in \mathbb{R}. \tag{19}$$

In this case, when $\lambda < \lambda_1$ we can prove the existence of a unique positive solution $u_b(t,s;u_0)$ which goes to zero as $t \to +\infty$. Moreover, if

$$\min_{x\in\overline{\Omega}}\{a_0(x)\} := a_0 > 0$$

then we obtain the same type of attracting bifurcating trajectory under the condition

$$2a_0 > A_0,$$

where $A_0 := \max_{x\in\overline{\Omega}}\{A_0(x)\}$; or under the local condition

$$2a_0(x) > A_0(x) \qquad \text{for all } x \in \overline{\Omega}. \tag{20}$$

Now, assume that the set

$$\Omega_0 := int\{x \in \Omega : a_0(x) = 0\},$$

is non-empty and regular, and $\Omega_0 \subset \overline{\Omega}_0 \subset \Omega$. It is well-known (see Du & Huang (1999), Fraile et al. (1996), López-Gómez (2000) and references therein) that in the autonomous case, i.e., $b(x,t) = a_0(x)$ there exists a unique positive solution $U(t,s;u_0)$ of (17) for all $\lambda \in \mathbb{R}$. Moreover, there exists a value of λ_0 (the principal eigenvalue of the Laplacian in Ω_0 with homogeneous Dirichlet boundary conditions) such that:

i) If $\lambda < \lambda_1$ then $\|U(t,s;u_0)\|_\infty \to 0$ as $t \to +\infty$.

ii) If $\lambda \in (\lambda_1,\lambda_0)$ then $\|U(t,s;u_0)\|_\infty \to \omega_{[\lambda,a_0]}$ as $t \to +\infty$, where $\omega_{[\lambda,a_0]}$ is the unique positive solution of (3) with A replaced by $a_0(x)$.

iii) If $\lambda > \lambda_0$ then $\|U(t,s;u_0)\|_\infty \to \infty$ as $t \to +\infty$.

When b satisfies (19) and a_0 vanishes in Ω_0, we obtain the existence of a unique bounded trajectory for $\lambda \in (\lambda_1,\lambda_0)$ under condition (20), which implies evidently that A_0 also vanishes in Ω_0.

In all the above inhomogeneous cases, we can obtain a local bifurcation result under a more natural condition.

Theorem 4.1 *Assume that b satisfies* (19) *and*

$$2\int_\Omega a_0(x)\varphi_1^3(x)\,\mathrm{d}x > \int_\Omega A_0(x)\varphi_1^3(x)\,\mathrm{d}x, \tag{21}$$

where φ_1 is the positive eigenfunction associated to λ_1 normalised such that $\|\varphi_1\|_2 = 1$. Then, there exists $\delta > 0$ such that for $\lambda \in (\lambda_1,\lambda_1+\delta)$ there exists a unique trajectory as Theorem 3.1.

Observe that we can follow the proof of Theorem 3.1 provided that

$$\lambda_1(2a_0(x)\omega_{[\lambda,A_0]}) > \lambda. \tag{22}$$

Now, by Lemma 4.3 in Delgado et al. (2000) we have that

$$\lambda_1(2a_0(x)\omega_{[\lambda,A_0]}) = \lambda_1 + 2\,\frac{\displaystyle\int_\Omega a_0(x)\varphi_1^3(x)\,\mathrm{d}x}{\displaystyle\int_\Omega A_0(x)\varphi_1^3(x)\,\mathrm{d}x}(\lambda-\lambda_1) + o(\lambda-\lambda_1)^2, \qquad \lambda \approx \lambda_1.$$

It is clear that (22) holds for $\lambda \in (\lambda_1, \lambda_1+\delta)$ and some $\delta > 0$ provided that (21) is satisfied. The proof is complete.

5 Conclusions

We have extended the bifurcation theory for certain model logistic PDEs to treat some non-autonomous examples. It is at first sight surprising that the key to obtaining a trajectory which attracts all others as $t \to +\infty$ ('forwards in time') is the pullback construction.

However, pullback attraction is fundamental in the theory of attractors for non-autonomous equations (Cheban et al., 2002; Chepyzhov & Vishik, 2002) and for random dynamical systems (e.g. Crauel et al., 1997). It also plays a major rôle in the general bifurcation theory for non-autonomous scalar ODEs developed in Langa et al. (2002). As such it should be less surprising that it is a useful tool even when attention is finally restricted to notions of 'forwards attraction'.

For the particular model treated here, there remains the question of whether the condition that $2b_0 > B_0$ is required in general, or if it is simply a technical requirement of our method of proof. We hope to see this resolved in the future.

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References

D.N. Cheban, P.E. Kloeden, B. Schmalfu. The relationship between pullback, forward and global attractors of nonautonomous dynamical systems. *Nonlinear Dyn. Syst. Theory* **2** (2002) 125–144.

V.V. Chepyzhov, M.I. Vishik. *Attractors for equations of mathematical physics.* American Mathematical Society Colloquium Publications, 49. American Mathematical Society, Providence, RI, 2002.

I. Chueshov. Order-preserving skew-product flows and non-autonomous parabolic systems. *Acta Appl. Math.* **65** (2001) 185-205.

H. Crauel, A. Debussche, F. Flandoli. Random attractors. *J. Dynam. Differential Equations* **9** (1997) 307–341.

M. Delgado, J. López-Gómez, A. Suárez. On the symbiotic Lotka-Volterra model with diffusion and transport effects. *J. Differential Equations* **160** (2000) 175-262.

Y. Du, Q. Huang. Blow-up solutions for a class of semilinear elliptic and parabolic equations. *SIAM J. Math. Anal.* **31** (1999) 1–18.

J. M. Fraile, P. Koch-Medina, J. López-Gómez, S. Merino. Elliptic eigenvalue problems and unbounded continua of positive solutions of a semilinear elliptic equation. *J. Differential Equations* **127** (1996) 295-319.

P. Hess. *Periodic-Parabolic boundary value problems and positivity* (Pitman Research Notes in Mathematics 247, Harlow Longman.1991).

J. A. Langa, A. Suárez. Pullback permanence for non-autonomous partial differential equations, *Electronic J. Diff. Eqns.*, **72** (2002) 1-20.

J.A. Langa, J.C. Robinson, A. Suárez. Stability, instability, and bifurcation phenomena in non-autonomous differential equations. *Nonlinearity* **15** (2002) 887–903.

J.A. Langa, J.C. Robinson, A. Suárez. Pullback permanence in a non-autonomous competitive Lotka-Volterra model. *J. Differential Equations* **190** (2003) 214–238

J. López-Gómez, The maximum principle and the existence of principal eigenvalues for some linear weighted boundary value problems, *J. Differential Equations*, **127** (1996) 263-294.

J. López-Gómez, Large solutions, metasolutions, and asymptotic behaviour of the regular positive solutions of sublinear parabolic problems. *Electronic J. Diff. Eqns.* **Conf.5** (2000) 135-171.

X. Mora, Semilinear parabolic problems define semiflows on C^k spaces. *Trans. Amer. Math. Soc.* **278** (1983) 21–55.

J. Smoller. *Shock Waves and Reaction-Diffusion Equations.* Springer-Verlag, New York, 1983.

W. Walter. Nonlinear parabolic differential equations and inequalities. *Disc. Cont. Dyn. Sys.* **8** (2002) 451–468.