MELLIN TRANSFORMS FOR SOME FAMILIES OF q-POLYNOMIALS

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Abstract

By using Ramanujan's q-extension of the Euler integral representation for the gamma function, we derive the Mellin integral transforms for the families of the discrete q-Hermite II, the Al-Salam–Carlitz II, the big q-Laguerre, the big q-Legendre, the big q-Jacobi and the q-Hahn polynomials.

Key words: Mellin integral transforms, q-polynomials.

1 Introduction

Mellin integral transforms for some families of basic hypergeometric polynomials from the Askey scheme [15] were considered in [7]. Derivation of

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these Mellin transform pairs is essentially based on the use of Ramanujan's q-extension [17,4,5] of the Euler integral representation for the gamma function

$$\frac{\Gamma(x)\,\Gamma(1-x)}{\Gamma_q(1-x)} = \int_0^\infty \frac{t^{x-1}\,dt}{E_q((1-q)t)}\,, \qquad \Re x > 0\,, \tag{1}$$

where $\Gamma_q(z)$ is the q-gamma function

$$\Gamma_q(x) := \frac{(q;q)_\infty}{(q^x;q)_\infty}\,(1-q)^{1-x}\,, \qquad 0<q<1\,,$$

and $E_q(z)$ is Jackson's q-exponential function

$$E_q(z) := \sum_{n=0}^\infty \frac{q^{\binom{n}{2}}}{(q;q)_n}\,z^n = (-z;q)_\infty\,, \tag{2}$$

where $\binom{n}{2} = n(n-1)/2$. We employ the standard notation of the q-special functions theory, see e.g. [12] or [2]. In particular, the q-shifted factorials are given by

$$(a;q)_0 = 1,\ (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k),\ n=1,2,\dots,\ (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), \tag{3}$$

and we will use the notation

$${}_r\phi_p\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_p\end{matrix}\,\middle|\, q,z\right) := \sum_{k=0}^\infty \frac{(a_1;q)_k\cdots(a_r;q)_k}{(b_1;q)_k\cdots(b_p;q)_k}\,\frac{z^k}{(q;q)_k}\left[(-1)^k q^{\binom{n}{2}}\right]^{p-r+1} \tag{4}$$

for the basic hypergeometric series.

It is well known that Ramanujan evaluated a number of integrals that extend the classical beta integral of Euler (see [17], [13], [4]-[6]). These integrals have associated orthogonal polynomials that have played a significant role in the development of the q-special functions theory. Ramanujan's now classical q-extension of the beta integral of Euler is

$$\int_0^\infty x^c\,\frac{E_q(q^{b+c}x)}{E_q(x)}\,dx = \frac{\Gamma(c)\,\Gamma(1-c)\,\Gamma_q(b)}{\Gamma_q(1-c)\,\Gamma_q(b+c)}\,. \tag{5}$$

The formula (1) is an easy consequence of (5) and the limit relation

$$\lim_{b\to\infty} \frac{\Gamma_q(b)}{\Gamma_q(b+c)} = (1-q)^c \,,$$

when the change of variables $x=(1-q)t$ is made.

It was shown in [7] that by using a q-analogue of Euler's reflection formula

$$\Gamma_q(x)\,\Gamma_q(1-x) = \frac{iq^{1/8}\,(1-q)\,(q;q)_\infty^3}{q^{x/2}\,\theta_1(\ln q^{-ix/2}, q^{1/2})} \,,$$

$\theta_1(z,q)$ is the theta-function of Jacobi, one can represent (1) in the form

$$c_q(x)\,\Gamma_q(x) = \frac{1-q}{\ln q^{-1}}\, q^{x(x-1)/2} \int\limits_0^\infty \frac{t^{x-1}\,dt}{E_q((1-q)t)} \,, \qquad \Re\, x > 0 \,, \tag{6}$$

where $c_q(x)$ is some periodic factor, i.e. $c_q(x+n) = c_q(x)$ for any non-negative integer n (an explicit form of $c_q(x)$ can be found in [7]). It should be emphasized that this formula is simply Jacobi's triple product identity for the theta-function $\theta_1(z,q)$, rewritten in terms of the q-gamma function $\Gamma_q(z)$. The observation that Jacobi's triple product identity is a q-analogue of Euler's reflection formula for the gamma function $\Gamma(z)$ was known to G. Andrews and R. Askey since "the 1975-1976 academic year". Unfortunately, they never published anything about this fact. Besides, R. Askey believes that this observation "is due to George Andrews" (see the very end of [3]), although G. Andrews comments that "Dick modestly attributes it to me; this is more a measure of his generosity than his accurate memory" (e-mail communication, November 16, 2001). Anyway, the idea of regarding Jacobi's triple product identity as a q-extension of Euler's reflection formula is at least 25 years old. Two of us (MKA and NMA) regret that we were not already aware of this fact at the time of the writing of [7] (which was the starting point for using this idea in the derivation of Mellin integral transforms for some families of q-polynomials).

In view of the periodicity of $c_q(x)$, one can then derive from (6) a Mellin integral transform for the product $p_n(\lambda t;q)\,E_q^{-1}((1-q)t)$, where $p_n(z;q)$ is some polynomial in z of degree n. In this way the Mellin integral transforms were obtained for all those families of q-polynomials from the Askey scheme, in which independent variable is the argument of an appropriate terminating basic hypergeometric series. They consist of the Stieltjes–Wigert, the Rogers–Szegö, the q-Laguerre, the Wall, the alternative q-Charlier, and the little q-Jacobi polynomials.

In this paper we wish to apply the technique of [7] to the study of those families of q-polynomials from the Askey scheme, which contain the independent variable x in one of the parameters of the corresponding basic hypergeometric series. The simplest example of this type is the discrete q-Hermite II polynomials

$$\begin{aligned}\tilde{h}_n(x;q) &:= i^{-n}\, q^{-\binom{n}{2}}\, {}_2\phi_0\left(\begin{matrix} q^{-n}, ix \\ - \end{matrix}\,\middle|\, q, -q^n\right) \\ &= i^{-n}\, q^{-\binom{n}{2}} \sum_{k=0}^{n} \frac{(q^{-n};q)_k\,(ix;q)_k}{(q;q)_k}\, q^{nk-\binom{k}{2}}\,;\end{aligned}$$

there are also the Al-Salam–Carlitz II, the big q-Laguerre, the big q-Legendre, the big q-Jacobi, and the q-Hahn polynomials (see [15]).

Our motivation for the study of q-analogues of Mellin integral transforms comes from mathematical physics. It is well known that in nonrelativistic quantum mechanics the coordinate and momentum realizations are interrelated by the Fourier integral transform. But in some relativistic approaches to quantum mechanics the passage from the momentum to the configuration realization is accomplished by the Mellin integral transform. For instance, a relativistic quasipotential [14,16] model of the linear harmonic oscillator, studied in detail in [8,9,11], is governed by a difference Hamiltonian in the configuration x-realization. The passage from the configuration to the momentum realization is equivalent to the Mellin integral transform in the light-front variable $p^+ = p_0 + p$, rather than the Fourier transform as in the nonrelativistic case. Therefore we believe that our technique can be applied to constructing various q-extensions of such quantum-mechanical models, based on difference equations.

2 The Mellin transform for a particular family of q-polynomials

It is well known that Euler's integral representation

$$\Gamma(x) = \int_0^\infty t^{x-1}\, e^{-t}\, dt\,, \qquad \Re\, x > 0\,, \tag{7}$$

for the gamma function $\Gamma(x)$ is an instance of the Mellin integral transform

$$g(z) = \mathfrak{M}\{f(t); z\} = \int_0^\infty f(t)\, t^{z-1}\, dt$$

of the exponential function $f(t) = e^{-t}$. Similarly, (1) or (6) is the Mellin transform of the function $E_q^{-1}((1-q)t)$:

$$g_q(x) := \mathfrak{M}\{E_q^{-1}((1-q)t); x\} = \frac{\ln q^{-1}}{1-q}\, c_q(x)\, q^{-x(x-1)/2}\, \Gamma_q(x)\,, \qquad \Re x > 0\,.$$

Let

$$p_n(x;q) = \sum_{k=0}^{n} a_{nk}(q)\,(x;q)_k \tag{8}$$

be a q-polynomial of degree n in x (with coefficients $a_{nk}(q)$, which may depend, in addition to q, on some other parameters). Then

$$\mathfrak{M}\{p_n(\lambda t;q)\, E_q^{-1}((1-q)t); x\} = \sum_{k=0}^{n} a_{nk}(q)\, \mathfrak{M}\{(\lambda t;q)_k\, E_q^{-1}((1-q)t); x\},$$

where λ is a constant. Since the q-shifted factorial

$$(z;q)_k = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q q^{\binom{j}{2}}\,(-z)^j, \qquad \begin{bmatrix} k \\ j \end{bmatrix}_q := \frac{(q;q)_k}{(q;q)_{k-j}(q;q)_j}\,,$$

is a polynomial of degree k in z, we have

$$\begin{aligned} &\mathfrak{M}\{p_n(\lambda t;q)\, E_q^{-1}((1-q)t); x\} \\ &\quad = \sum_{k=0}^{n} a_{nk}(q) \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q q^{\binom{j}{2}} (-\lambda)^j\, \mathfrak{M}\{t^j\, E_q^{-1}((1-q)t); x\} \\ &\quad = \sum_{k=0}^{n} a_{nk}(q) \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q q^{\binom{j}{2}} (-\lambda)^j g_q(x+j). \end{aligned} \tag{9}$$

But

$$\begin{aligned} g_q(x+j) &= \frac{\ln q^{-1}}{1-q}\, c_q(x)\, q^{-(x+j)(x+j-1)/2}\Gamma_q(x+j) \\ &= q^{-jx-\binom{j}{2}}\, \frac{(q^x;q)_j}{(1-q)^j}\, g_q(x), \end{aligned} \tag{10}$$

because

$$\Gamma_q(x+j) = \frac{(q^x;q)_j}{(1-q)^j}\Gamma_q(x)$$

by the definition of the q-gamma function $\Gamma_q(x)$. Substituting (10) into the right-hand side of (9), one thus obtains that

$$\begin{aligned}
&\mathfrak{M}\{p_n(\lambda t;q)\,E_q^{-1}((1-q)t);x\} \\
&\quad = g_q(x)\sum_{k=0}^{n}\ a_{nk}(q)\ \sum_{j=0}^{k}\begin{bmatrix}k\\ j\end{bmatrix}_q (q^x;q)_j\left(-\frac{\lambda q^{-x}}{1-q}\right)^j.
\end{aligned} \tag{11}$$

Finally, the q-binomial coefficient $\left[\begin{smallmatrix}k\\ j\end{smallmatrix}\right]_q$ can be written as

$$\begin{bmatrix}k\\ j\end{bmatrix}_q = (-1)^j q^{kj-\binom{j}{2}}\frac{(q^{-k};q)_j}{(q;q)_j}. \tag{12}$$

Therefore the sum over the index j in the right-hand side of (11) represents a terminating basic hypergeometric series ${}_2\phi_0$ and we obtain the desired result

$$\begin{aligned}
&\mathfrak{M}\{p_n(\lambda t;q)\,E_q^{-1}((1-q)t);x\} \\
&\quad = g_q(x)\sum_{k=0}^{n}\ a_{nk}(q)\ {}_2\phi_0\left(\left.\begin{matrix}q^{-k},\ q^x\\ \text{—}\end{matrix}\ \right|\ q\ ,-\frac{\lambda q^{k-x}}{1-q}\right).
\end{aligned} \tag{13}$$

This formula gives an explicit form of the Mellin integral transform for the function $E_q^{-1}((1-q)t)$, multiplied by a polynomial $p_n(\lambda t;q)$ of the type (8) (with an arbitrary constant λ). Observe that the right-hand side of (13) is a polynomial of degree n in the variable q^{-x}, times the function $g_q(x)$. Also, the terminating basic hypergeometric series ${}_2\phi_0$ in (13) can be written as

$${}_2\phi_0\left(\left.\begin{matrix}q^{-k},\ q^x\\ \text{—}\end{matrix}\ \right|\ q\ ,-\frac{\lambda q^{k-x}}{1-q}\right) = C_k\left(q^{-x};\frac{1-q}{\lambda\,q^k};q^{-1}\right),$$

where the q^{-1}-Charlier polynomials $C_n(z;a;q^{-1})$ are defined (cf. [15], p. 112) as

$$C_n(z;a;q^{-1}) := {}_2\phi_0\left(\left.\begin{matrix}q^{-n},\ z^{-1}\\ \text{—}\end{matrix}\ \right|\ q\ ,-z/a\right). \tag{14}$$

For a particular choice of the constant λ, the sum over j in (11) is simplified; in other words, the terminating basic hypergeometric series ${}_2\phi_0$ in (13) reduces to a monomial in the variable q^{-x}. The point is that

$$ {}_2\phi_0\left(\begin{matrix} q^{-n},\ z^{-1} \\ - \end{matrix}\,\middle|\, q\ , zq^n\right) = z^n\,. \tag{15} $$

To verify (15), simply reverse the order of summation in the definition of ${}_2\phi_0$ and use the limit case of the q-Chu–Vandermonde sum

$$ {}_2\phi_1\left(\begin{matrix} q^{-n},\ b \\ c \end{matrix}\,\middle|\, q\ , q\right) = \frac{(c/b;q)_n}{(c;q)_n}\, b^n $$

with the vanishing parameter b. We note that the relation (15) can be expressed in terms of the q^{-1}-Charlier polynomials (14) as

$$ C_n(z;-q^{-n};q^{-1}) = z^n\,. $$

From (15) it follows that if one chooses $\lambda = q-1$, then the Mellin integral transform (13) reduces to

$$ \mathfrak{M}\{p_n((q-1)t;q)\, E_q^{-1}((1-q)t);x\} = g_q(x) \sum_{k=0}^{n}\ a_{nk}(q) q^{-kx}. \tag{16} $$

Notice that such simplification of (13) in the case when $\lambda = q-1$ is a consequence of the following property

$$ E_q(z) = (-z;q)_k\, E_q(q^k z) \tag{17} $$

of Jackson's q-exponential function (2). Indeed, by the definition (8),

$$ \begin{aligned} \mathfrak{M}\{p_n((q-1)t;q)\, E_q^{-1}((1-q)t);x\} &= \sum_{k=0}^{n}\ a_{nk}(q) \int_0^\infty \frac{((q-1)t;q)_k}{E_q((1-q)t)} t^{x-1} dt \\ &= \sum_{k=0}^{n}\ a_{nk}(q) \int_0^\infty \frac{t^{x-1}\, dt}{E_q((1-q)q^k t)}\,, \end{aligned} \tag{18} $$

where we have employed the property (17) with $z = (1-q)t$. The change of the variable $t \to q^{-k}t$ in (18) leads immediately to the Mellin transform (16).

Now it remains only to consider concrete examples of the aforementioned families of q-polynomials from the Askey scheme.

3 Concrete examples

1. We start with the Al-Salam–Carlitz II polynomials from the q-Askey scheme (see [15], p. 114)

$$\begin{aligned} V_n^{(a)}(x;q) &:= (-a)^n q^{-\binom{n}{2}} {}_2\phi_0 \left(\begin{matrix} q^{-n}, x \\ — \end{matrix} \,\middle|\, q, \frac{q^n}{a} \right) \\ &= (-a)^n q^{-\binom{n}{2}} \sum_{k=0}^{n} \frac{(q^{-n};q)_k (x;q)_k}{(q;q)_k} q^{k[n-(k-1)/2]} (-a^{-1})^k, \end{aligned} \tag{19}$$

which occupy the second (i.e. next-to-lowest) level in the Askey scheme of basic hypergeometric polynomials with the discrete orthogonality property (see [15], p. 62). From (19) it follows that the coefficients $a_{nk}(q)$ in (8) in this particular case are equal to

$$a_{nk}(q) = (-a)^n q^{-\binom{n}{2}} a^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q, \tag{20}$$

where we have used the relation (12). Substituting (20) into (16), we thus obtain a Mellin integral transform

$$\mathfrak{M}\{V_n^{(a)}((q-1)t;q)\, E_q^{-1}((1-q)t);x\} = (-a)^n q^{-\binom{n}{2}} H_n(q^{-x}/a;q) g_q(x), \tag{21}$$

where the Rogers–Szegö polynomials $H_n(z;q)$ are the q-analogue of Hermite polynomials on the unit circle (see [18,1,10]), defined as

$$H_n(z;q) := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q z^k = {}_2\phi_0 \left(\begin{matrix} q^{-n}, 0 \\ — \end{matrix} \,\middle|\, q, zq^n \right). \tag{22}$$

Notice that the special case of the Al-Salam–Carlitz II polynomials (19) with $a = -1$ is known as the discrete q-Hermite II polynomials $\widetilde{h}_n(x;q)$ (see [15], p. 119). Therefore, from (21) one obtains a Mellin transform

$$\mathfrak{M}\{\widetilde{h}_n(i(1-q)x;q)\, E_q^{-1}((1-q)t);x\} = i^{-n} q^{-\binom{n}{2}} H_n(-q^{-x};q) g_q(x),$$

which interrelates the discrete q-Hermite II and the Rogers–Szegö polynomials.

From (22) it is evident that

$$\lim_{q\to 1^-} H_n(z;q) = (z+1)^n.$$

The Mellin transform (21) in the limit as the parameter $q \to 1^-$ thus coincides with Euler's integral representation for the gamma function (7), both sides of which are multiplied by the constant factor $(-1)^n(1+a)^n$.

2. At the third level of the q-Askey scheme with the discrete orthogonality there is only one family of q-polynomials of the type (8), namely the big q-Laguerre polynomials (see [15], p. 91)

$$P_n(x;a,b;q) := {}_3\phi_2\left(\begin{matrix} q^{-n}, 0, x \\ aq, bq \end{matrix}\,\middle|\, q, q\right) = \sum_{k=0}^{n} \frac{(q^{-n};q)_k (x;q)_k\, q^k}{(aq;q)_k (bq;q)_k (q;q)_k}. \tag{23}$$

The coefficients $a_{nk}(q)$ in this case are equal to

$$a_{nk}(q) = \frac{(q^{-n};q)_k\; q^k}{(aq;q)_k (bq;q)_k (q;q)_k}. \tag{24}$$

Substituting (24) into (16), one obtains the following Mellin integral transform

$$\mathfrak{M}\{P_n((q-1)t;a,b;q)\,E_q^{-1}((1-q)t);x\} = {}_3\phi_2\left(\begin{matrix} q^{-n}, 0, 0 \\ aq, bq \end{matrix}\,\middle|\, q, q^{1-x}\right) g_q(x) \tag{25}$$

for the big q-Laguerre polynomials $P_n(x;a,b;q)$.

From the definition (23) it is clear that

$$\lim_{q\to 1^-} P_n((q-1)t;a,b;q) = [1-(1-a)^{-1}(1-b)^{-1}]^n.$$

Therefore the Mellin transform (25) in the limit as $q \to 1^-$ coincides with (7), multiplied by the constant factor $[1-(1-a)^{-1}(1-b)^{-1}]^n$.

3. The big q-Jacobi polynomials (see [15], p. 73)

$$P_n(x;a,b,c;q) := {}_3\phi_2\left(\begin{matrix} q^{-n}, abq^{n+1}, x \\ aq, cq \end{matrix}\,\middle|\, q, q\right) \tag{26}$$

occupy the fourth level in the Askey scheme for q-polynomials with the discrete orthogonality. Taking into account that the coefficients $a_{nk}(q)$ in this case are equal to

$$a_{nk}(q) = \frac{(q^{-n};q)_k\; (abq^{n+1};q)_k\; q^k}{(aq;q)_k (cq;q)_k (q;q)_k},$$

from (16) it follows that a Mellin integral transform for the big q-Jacobi polynomials $P_n(x; a, b, c; q)$ is of the form

$$\begin{aligned} &\mathfrak{M}\{P_n((q-1)t; a, b, c; q)\, E_q^{-1}((1-q)t); x\} \\ &\qquad = {}_3\phi_2\left(\begin{matrix} q^{-n}, abq^{n+1}, 0 \\ aq, cq \end{matrix} \,\middle|\, q, q^{1-x}\right) g_q(x). \end{aligned} \tag{27}$$

In view of the limit relation

$$\lim_{q\to 1^-} P_n((q-1)t; a, b, c; q) = \left[1 - \frac{1-ab}{(1-a)(1-c)}\right]^n ,$$

which follows directly from the definition (26), the Mellin transform (27) in the limit as $q \to 1^-$ coincides with (7), multiplied by $[1-(1-ab)(1-a)^{-1}(1-c)^{-1}]^n$.

Notice that there are also the q-Hahn polynomials (see [15], p. 76)

$$Q_n(x; \alpha, \beta, N; q) := {}_3\phi_2\left(\begin{matrix} q^{-n}, \alpha\beta q^{n+1}, x \\ \alpha q, q^{-N} \end{matrix} \,\middle|\, q, q\right) \tag{28}$$

at the same level of the q-Askey scheme. If one compares (28) with (26), then it becomes evident that the q-Hahn polynomials are merely the big q-Jacobi ones (26) with the parameters $a = \alpha$, $b = \beta$, and $c = q^{-N-1}$. The formula (27) thus provides an appropriate Mellin integral transform for the q-Hahn polynomials as well.

Finally, the special case of the big q-Jacobi polynomials (26) with $a = b = 1$ is known as the big q-Legendre polynomials $P_n(x; c; q)$ (see [15], p. 65). The Mellin integral transform in this case is

$$\mathfrak{M}\{P_n((q-1)t; c; q)\, E_q^{-1}((1-q)t); x\} = {}_3\phi_2\left(\begin{matrix} q^{-n}, q^{n+1}, 0 \\ cq, q \end{matrix} \,\middle|\, q, q^{1-x}\right) g_q(x).$$

Concluding remarks

We note in closing that there are some other instances of q-polynomials from the Askey scheme for which the independent variable x enters not in one, but two of the parameters of the corresponding basic hypergeometric series. For example, the continuous big q-Hermite polynomials $H_n(x;a|q)$ are of the form

$$H_n(x;a|q) := a^{-n}\, {}_3\phi_2 \left(\begin{matrix} q^{-n}, a\, e^{i\theta}, a\, e^{-i\theta} \\ 0,\ \ 0 \end{matrix} \middle| \, q, q \right), \quad x = \cos\theta.$$

We hope to discuss the possibility of applying the same technique of deriving Mellin transforms to this type of q-polynomials elsewhere.

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