$q-$Classical polynomials and the $q-$Askey and Nikiforov-Uvarov Tableaus

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Abstract

In this paper we continue the study of the $q-$ classical (discrete) polynomials (in the Hahn's sense) started in [18]. Here we will compare our scheme with the well known $q-$Askey Scheme and the Nikiforov-Uvarov Tableau. Also, new families of $q-$polynomials are introduced.

Introduction

The so-called $q-$polynomials constitute a very important and interesting set of special functions and more specifically of orthogonal polynomials. They appear in several branches of the natural sciences, e.g., continued fractions, Eulerian series, theta functions, elliptic functions,...; see [3, 9], quantum groups and algebras [14, 15, 25], among others (see also [10, 20]). They have been intensively studied in the last years by several people (see e.g. [13]) using several tools. One of them is the one reviewed in [13] which is based on the basic hypergeometric series [10] and was developed mainly by the American School starting by the works of Andrews and Askey (see e.g. [4], the literature on this method is so vast that we are not able to include it here, a very complete list is given in [13]) and lead to the so-called $q-$Askey Tableau of hypergeometric polynomials [13]. In other direction, the Russian (former Soviet) school, starting from the works by Nikiforov and Uvarov [21] and further developed by Atakishiyev and Suslov (see e.g. [5, 6, 20, 23, 24] and references contained therein), have considered the difference analog in non-uniform lattices of the hypergeometric differential equation [22], from where the hypergeometric representation of the $q-$polynomials follows in a very simple way [5, 23]. This schema leads to the Nikiforov-Uvarov tableau [20, 23] for the polynomial solutions of the difference hypergeometric equation on non-uniform lattices. A special mention deserves the paper by Atakishiev and Suslov [6] where a difference analog of the well known method of undeterminated coefficients have been developed for the hypergeometric equation on non-uniform lattices and also give a classification similar to the Nikiforov and Uvarov 1991 one but for the $q-$special functions (not only for the polynomials solutions).

Our main aims here are two: to continue the study started in [18] using the algebraic theory developed by Maroni [16] and to classify the $q-$classical polynomials and compare with the $q-$Askey and Nikiforov & Uvarov Tableaus. In fact, in [18] we have proven several characterization of the $q-$classical polynomials as well as a very simple computational algorithm for finding their main characteristics (e.g. the coefficients of the three-term recurrent relation, structure relation of Al-Salam Chihara, etc). Going further, we will give here a "very natural" classification of the $q-$classical polynomials introduced by Hahn in his paper [11], i.e., we will classify all orthogonal polynomial sequences such that their $q-$differences, defined by $\Theta f(x) = \frac{f(qx)-f(x)}{(q-1)x}$ are orthogonal in the widespread sence: the $q-$Hahn Tableau (a first step on this in the frame work of the $q-$Askey tableau was done in [15]). Notice that the aforesaid polynomials are instances of the

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$q-$polynomials on the linear exponential lattice $x(s) = c_1 q^s$. For several surveys on this lattice and their corresponding polynomials see [2, 4, 5, 8, 10, 13, 20, 24]. (see also section 3.2 from below). Furthermore, we will compare our classification ($q-$Hahn Tableau) with the aforesaid two Schemas. From this comparation we find that there are missing families in the $q-$Askey Schema (one of them is a non-positive definite family) and using the results of [18] we study them with details. Also the correspondence of this $q-$Hahn Schema and the Nikiforov & Uvarov one will be stablished. In such a way a complete correspondence between the $q-$classical families of the $q-$Askey and Nikiforov & Uvarov Tableaus for exponential linear lattices will be shown.

The structure of the paper is as follows. In Section 1 we introduce some notations and definitions useful for the next ones. In Section 2, the $q-$weight functions are introduced and computed for all $q-$classical families. This will allow to classify all orthogonal polynomial families of the $q-$Hahn tableau. Finally, in Section 3, several applications are considered: the classification of the $q-$classical polynomials ($q-$Hahn Tableau), the integral representation for the orthogonality, the hypergeometric representation of these $q-$classical polynomials as well as the detailed study of two new families of $q-$polynomials.

1 Preliminaries

In this section we will give a brief survey of the operational calculus and some basic concepts and results needed for the rest of the work.

Let $\mathbb{P}$ be the linear space of polynomial functions in $\mathbb{C}$ with complex coefficients and $\mathbb{P}^*$ be its algebraic dual space, i.e., $\mathbb{P}^*$ is the linear space of all linear applications $\mathbf{u} : \mathbb{P} \to \mathbb{C}$. In the following we will refer to the elements of $\mathbb{P}^*$ as functionals and we will denote them with bold letters ($\mathbf{u}$, $\mathbf{v}, \ldots$).

Since the elements of $\mathbb{P}^*$ are linear functionals, it is possible to determine them from their actions on a given basis $(B_n)_{n\geq 0}$ of $\mathbb{P}$, e.g. the canonical basis of $\mathbb{P}$, $(x^n)_{n\geq 0}$. In general, we will represent the action of a functional over a polynomial by formula $\langle \mathbf{u}, \pi \rangle$, $\mathbf{u} \in \mathbb{P}^*$, $\pi \in \mathbb{P}$, and therefore a functional is completely determined by a sequence of complex numbers $\langle \mathbf{u}, x^n \rangle = u_n$, $n \geq 0$, the so-called moments of the functional.

Definition 1.1 *Let $(P_n)_{n\geq 0}$ be a basis sequence of $\mathbb{P}$. We say that $(P_n)_{n\geq 0}$ is an orthogonal polynomial sequence (OPS in short), if and only if there exists a functional $\mathbf{u} \in \mathbb{P}^*$ such that $\langle \mathbf{u}, P_m P_n \rangle = k_n \delta_{mn}$, $k_n \neq 0$, $n \geq 0$, where δ_{mn} is the Kronecker delta. If $k_n > 0$ for all $n \geq 0$, we say that $(P_n)_{n\geq 0}$ is a positive definite OPS.*

Definition 1.2 *Let $\mathbf{u} \in \mathbb{P}^*$ be a functional. We say that $\mathbf{u}$ is a quasi-definite functional if and only if there exists a polynomial sequence $(P_n)_{n\geq 0}$, which is orthogonal with respect to $\mathbf{u}$. If $(P_n)_{n\geq 0}$ is positive definite, we say that $\mathbf{u}$ is a positive definite functional.*

Definition 1.3 *Given a polynomial sequence $(P_n)_{n\geq 0}$, we say that $(P_n)_{n\geq 0}$ is a monic orthogonal polynomial sequence (MOPS in short) with respect to $\mathbf{u}$, and we denote it by $(P_n)_{n\geq 0} = \operatorname{mops} \mathbf{u}$ if and only if $P_n(x) = x^n + \text{lower degree terms}$ and $\langle \mathbf{u}, P_m P_n \rangle = k_n \delta_{nm}$, $k_n \neq 0$, $n \geq 0$.*

Also the next theorem will be useful

Theorem 1.1 *(*Favard Theorem [7]*) Let $(P_n)_{n\geq 0}$ be a monic polynomial basis sequence. Then, $(P_n)_{n\geq 0}$ is an MOPS if and only if there exist two sequences of complex numbers $(d_n)_{n\geq 0}$ and $(g_n)_{n\geq 1}$, such that $g_n \neq 0$, $n \geq 1$ and*

$$xP_n = P_{n+1} + d_n P_n + g_n P_{n-1}, \quad P_{-1} = 0 \, , \; P_0 = 1, \quad n \geq 0 \, , \tag{1.1}$$

where $P_{-1}(x) \equiv 0$ and $P_0(x) \equiv 1$. Moreover, the functional $\mathbf{u}$ with respect to which the polynomials $(P_n)_{n\geq 0}$ are orthogonal is positive definite if and only if $(d_n)_{n\geq 0}$ is a real sequence and $g_n > 0$ for all $n \geq 1$.

In the following, we will use the notation:

Definition 1.4 *Let $\pi \in \mathbb{P}$ and $a \in \mathbb{C}$, $a \neq 0$. We call the operator $\mathrm{H}_a : \mathbb{P} \to \mathbb{P}$, $\mathrm{H}_a\pi(x) = \pi(ax)$, a dilation of ratio $a \in \mathbb{C} \setminus \{0\}$.*

This operator is linear on $\mathbb{P}$ and satisfies $\mathrm{H}_a(\pi\rho) = \mathrm{H}_a\pi \cdot \mathrm{H}_a\rho$. Also notice that for any complex number $a \neq 0$, $\mathrm{H}_a \cdot \mathrm{H}_{a^{-1}} = \mathrm{I}$, where I is the identity operator on $\mathbb{P}$, i.e., for all $a \neq 0$, H_a has an inverse operator. In the following we will omit any reference to q in the operators H_q and their inverse $\mathrm{H}_{q^{-1}}$. So, $\mathrm{H} := \mathrm{H}_q$, $\mathrm{H}^{-1} := \mathrm{H}_{q^{-1}}$.

Next, we will define the so called $q-$derivative operator [11]. We will suppose also that $|q| \neq 1$ (although it is possible to weak this condition).

Definition 1.5 *Let $\pi \in \mathbb{P}$ and $q \in \mathbb{C} \setminus \{0\}$, $|q| \neq 1$. The $q-$derivative operator Θ, is the operator $\Theta : \mathbb{P} \to \mathbb{P}$, defined by*

$$\Theta\pi = \frac{\mathrm{H}\pi - \pi}{\mathrm{H}x - x} = \frac{\mathrm{H}\pi - \pi}{(q-1)x}\,.$$

The $q^{-1}-$derivative operator $\Theta^\star$, is the operator $\Theta^\star : \mathbb{P} \to \mathbb{P}$ defined by

$$\Theta^\star\pi = \frac{\mathrm{H}^{-1}\pi - \pi}{\mathrm{H}^{-1}x - x} = \frac{\mathrm{H}^{-1}\pi - \pi}{(q^{-1}-1)x}\,.$$

In this way, $\Theta\pi$ and $\Theta^\star\pi$ will denote the $q-$derivative and $q^{-1}-$derivative of π, respectively.

The above two operators Θ and $\Theta^\star$ are linear operators on $\mathbb{P}$, and

$$\Theta x^n = \frac{\mathrm{H}x^n - x^n}{(q-1)x} = \frac{(q^n-1)x^n}{(q-1)x} = [n]x^{n-1}, \quad n > 0, \quad \Theta 1 = 0, \tag{1.2}$$

i.e., $\Theta\pi \in \mathbb{P}$. Here $[n]$, $n \in \mathbb{N}$, denotes the basic $q-$number n defined by

$$[n] = \frac{q^n - 1}{q-1} = 1 + q + \ldots + q^{n-1}, \quad n > 0, \quad [0] = 0\,. \tag{1.3}$$

Also the q^{-1} numbers $[n]^\star$, defined by $[n]^\star = \frac{q^{-n}-1}{q^{-1}-1} = q^{1-n}[n]$ will be used.

Notice that $\Theta^\star$ is not the inverse of Θ. In fact they are related by $\mathrm{H}\Theta^\star = \Theta$, $\mathrm{H}^{-1}\Theta = \Theta^\star$. The $q-$derivative satisfies the product rule $\Theta(\pi\rho) = \rho\Theta\pi + \mathrm{H}\pi \cdot \Theta\rho = \mathrm{H}\rho \cdot \Theta\pi + \pi\Theta\rho$.

Definition 1.6 *Let ω a derivable function at $x = 0$ such that $\forall a \in \operatorname{dom}\omega$, $aq \in \operatorname{dom}\omega$. Then, we will define the $q-$derivative of ω by the expression*

$$\Theta\omega = \frac{\mathrm{H}\omega - \omega}{\mathrm{H}x - x} = \frac{\mathrm{H}\omega - \omega}{(q-1)x}, \quad x \neq 0, \qquad \Theta\omega(0) = \omega'(0)\,. \tag{1.4}$$

Definition 1.7 *Let $\mathbf{u} \in \mathbb{P}^*$ and $\pi \in \mathbb{P}$. We define the action of a dilation H_a and the $q-$derivative Θ on $\mathbb{P}^*$ by the expressions $\mathrm{H}_a : \mathbb{P}^* \to \mathbb{P}^*$, $\langle \mathrm{H}_a\mathbf{u}, \pi\rangle = \langle \mathbf{u}, \mathrm{H}_a\pi\rangle$, $\Theta : \mathbb{P}^* \to \mathbb{P}^*$, $\langle \Theta\mathbf{u}, \pi\rangle = -\langle \mathbf{u}, \Theta\pi\rangle$, respectively.*

Definition 1.8 *Let $\mathbf{u} \in \mathbb{P}^*$ and $\pi \in \mathbb{P}$. We define a polynomial modification of a functional $\mathbf{u}$, the functional $\pi\mathbf{u}$, $\langle \pi\mathbf{u}, \rho\rangle = \langle \mathbf{u}, \pi\rho\rangle$, $\quad \forall\, \rho \in \mathbb{P}$.*

Notice that we use the same notation for the operators on $\mathbb{P}$ and $\mathbb{P}^*$. Whenever it is not specified on which linear space an operator acts, it will be understood that it acts on the polynomial space $\mathbb{P}$.

Definition 1.9 *Let $\mathbf{u} \in \mathbb{P}^*$ be a quasi-definite functional and $(P_n)_{n\geq 0} = \operatorname{mops}(\mathbf{u})$. We say that $\mathbf{u}$ or $(P_n)_{n\geq 0}$ are $q-$classic functional or MOPS, respectively, if and only if the sequence $(\Theta P_{n+1})_{n\geq 0}$ is also orthogonal.*

Notice that in the Hahn definition [11] q is a real parameter and here, in general, $q \in \mathbb{C} \setminus \{0\}$, $|q| \neq 1$.

In the following $(Q_n)_{n\geq 0}$ will denote the sequence of monic $q-$derivatives of $(P_n)_{n\geq 0}$, i.e., $Q_n = \frac{1}{[n+1]} P_{n+1}$, for all $n \geq 0$.

Theorem 1.2 *(Medem et al. [17, 18]) Let $\mathbf{u} \in \mathbb{P}^*$ be a quasi-definite functional. and $(P_n)_{n\geq 0} = \mathrm{mops}(\mathbf{u})$. Then, the following statements are equivalent:*

(a) $\mathbf{u}$ and $(P_n)_{n\geq 0}$ are, respectively, a $q-$classical functional and a $q-$classical MOPS.

(b) There exists a pair of polynomials ϕ and ψ, $\deg\phi \leq 2$, $\deg\psi = 1$, such that

$$\Theta(\phi\mathbf{u}) = \psi\mathbf{u}\,. \tag{1.5}$$

(c) $(P_n)_{n\geq 0}$ satisfies the $q-\mathcal{SL}$ difference equation

$$\phi\Theta\Theta^\star P_n + \psi\Theta^\star P_n = \widehat{\lambda}_n P_n\ ,\ n \geq 0, \tag{1.6}$$

i.e., P_n are the eigenfunctions of the Sturm-Liouville operator $\phi\Theta\Theta^\star + \psi\Theta^\star$ corresponding to the eigenvalues $\widehat{\lambda}_n$.

Moreover, if

$$\phi(x) = \widehat{a}x^2 + \bar{a}x + \dot{a}, \qquad \psi(x) = \widehat{b}x + \bar{b}, \qquad \widehat{b} \neq 0, \tag{1.7}$$

then, the quasi-definiteness of $\mathbf{u}$ implies $[n]\widehat{a} + \widehat{b} \neq 0$ and the following equivalences hold

$$[n]\widehat{a} + \widehat{b} \neq 0, \quad n \geq 0 \iff \widehat{\lambda}_n \neq \widehat{\lambda}_m, \quad \forall n, m \geq 1,\ n \neq m \iff \widehat{\lambda}_n \neq 0, \forall n \geq 1.$$

Theorem 1.3 *Let $\mathbf{u} \in \mathbb{P}^*$, be a quasi-definite functional, $(P_n)_{n\geq 0} = \mathrm{mops}\,\mathbf{u}$ and $Q_n^{(k)} = \frac{1}{[n+1]_{(k)}}\Theta^k P_{n+k}$, where $[n+1]_{(k)} \equiv [n+1][n+2]\ldots[n+k-1]$. The following statements are equivalent:*

(a) $(P_n)_{n\geq 0}$ is $q-$classical, (b) $(Q_n^{(k)})_{n\geq 0}$ is $q-$classical, $k \geq 1$.

Moreover, if $\mathbf{u}$ satisfies the equation $\Theta(\phi\mathbf{u}) = \psi\mathbf{u}$, $\deg\phi \leq 2$ and $\deg\psi = 1$, then $(Q_n^{(k)})$ is orthogonal with respect to $\mathbf{v}^{(k)} = \mathrm{H}^{(k)}\phi\cdot\mathbf{u}$, $\mathrm{H}^{(k)} = \prod_{i=1}^{k}\mathrm{H}^{i-1}\phi$, and it satisfies

$$\Theta(\phi^{(k)}\mathbf{v}^{(k)}) = \psi^{(k)}\mathbf{v}^{(k)}, \quad \deg\phi^{(k)} \leq 2 \quad \deg\psi^{(k)} = 1,$$

where $\phi^{(k)} = \mathrm{H}^k\phi$ and $\psi^{(k)} = \psi + \Theta\sum_{i=0}^{k-1}\mathrm{H}^i\phi$, and they are the polynomial solutions of the $q-\mathcal{SL}$ equation

$$\mathcal{SL}^{(k)}Q_n^{(k)} = \phi^{(k)}\Theta\Theta^\star Q_n^{(k)} + \psi^{(k)}\Theta^\star Q_n^{(k)} = \widehat{\lambda}_n^{(k)}Q_n^{(k)}, \tag{1.8}$$

where the polynomials $\phi^{(k)}$ and $\psi^{(k)}$ and the eigenvalues $\widehat{\lambda}_n^{(k)}$ are

$$\phi^{(k)} = q^{2k}\widehat{a}x^2 + q^k\overline{a}x + \dot{a}, \quad \psi^{(k)} = ([2k]\widehat{a} + \widehat{b})x + ([k]\overline{a} + \overline{b}), \quad \widehat{\lambda}_n^{(k)} = [n]^\star([2k+n-1]\widehat{a} + \widehat{b})\,. \tag{1.9}$$

Furthermore, in [17, 18] the following result was proven:

Theorem 1.4 *Let $\mathbf{u} \in \mathbb{P}^*$, be a quasi-definite functional, $(P_n)_{n\geq 0} = \mathrm{mops}\,\mathbf{u}$, $\phi, \phi^\star, \psi \in \mathbb{P}$, such that $\phi^\star = q^{-1}\phi + (q^{-1}-1)x\psi$, $\deg\phi \leq 2$, $\deg\phi^\star \leq 2$ and $\deg\psi = 1$. Then, the following statements are equivalent*

(a) $\mathbf{u}$ and $(P_n)_{n\geq 0} = \mathrm{mops}\,\mathbf{u}$ are $q-$classical and $\Theta(\phi\mathbf{u}) = \psi\mathbf{u}$,

(b) $\mathbf{u}$ and $(P_n)_{n\geq 0} = \mathrm{mops}\,\mathbf{u}$ are $q^{-1}-$classical and $\Theta^\star(\phi^\star\mathbf{u}) = \psi\mathbf{u}$.

(c) *There exist a polynomial* $\phi \in \mathbb{P}$, $\deg\phi \leq 2$ *and three sequences of complex numbers* a_n, b_n, c_n, $c_n \neq 0$, *such that*

$$\phi\Theta P_n = a_n P_{n+1} + b_n P_n + c_n P_{n-1}, \quad n \geq 1\ ; \tag{1.10}$$

(d) *there exist a complex numbers* e_n, h_n, *such that*

$$P_n = Q_n + e_n Q_{n-1} + h_n Q_{n-2}, \quad n \geq 2\ . \tag{1.11}$$

(e) *There exist a polynomial* $\phi \in \mathbb{P}$, $\deg\phi \leq 2$ *and a sequence of complex numbers* r_n, $r_n \neq 0$, $n \geq 1$ *such that*

$$P_n\mathbf{u} = r_n\Theta^n(\mathrm{H}^{(n)}\phi\cdot\mathbf{u}), \quad \mathrm{H}^{(n)}\phi = \prod_{i=1}^{n}\mathrm{H}^{i-1}\phi\ , \quad r_n = q^{\binom{n}{2}}\prod_{i=1}^{n}\Big([2n-i-1]\widehat{a}+\widehat{b}\Big)^{-1}, \quad n \geq 1. \tag{1.12}$$

2 The $q-$weight function ω

2.1 Definition and first properties

In this section we will consider the so-called weight functions for $q-$classical polynomials. The next proposition can be proven straightforward (see e.g. [12]).

Proposition 2.1 *Let* ω *a function such that if* $a \in \operatorname{dom}\omega$, $aq^{-1} \in \operatorname{dom}\omega$ *and that satisfies the difference equation*

$$\Theta^\star(\phi\omega) = q\psi\omega \quad \Longleftrightarrow \quad \phi\omega = q\mathrm{H}(\phi^\star\omega), \quad \phi,\psi \in \mathbb{P}, \quad \phi^\star = q^{-1}\phi + (q^{-1}-1)x\psi. \tag{2.1}$$

Then, the following two equations are equivalent

$$\phi\Theta\Theta^\star P_n + \psi\Theta^\star P_n = \widehat{\lambda}_n P_n\ , \quad \Longleftrightarrow \quad \Theta^\star(\phi\omega\Theta P_n) = q\widehat{\lambda}_n\omega P_n\ ,\ n \geq 1. \tag{2.2}$$

The above proposition allows us to generalize the classical procedure to the $q-$case for obtaining almost all the characteristics of the MOPS. The equation (2.1) is usually called the $q-$Pearson equation and its solution ω is known as the $q-$weight function and it allows to rewrite the Sturm-Liuoville equation (1.6) in its self-adjoint form (2.2). Moreover, the weight function ω allow us to obtain the "standard" $q-$Rodrigues formula and also justify the $q-$integral representation for the orthogonality relation. In such a way it is natural to give the following

Definition 2.1 *Let* $\mathbf{u} \in \mathbb{P}^*$, *be a quasi-definite functional satisfying the distributional equation (1.5), where* $\phi,\psi \in \mathbb{P}$, $\deg\phi \leq 2$, $\deg\psi = 1$ *and* $(P_n)_{n\geq 0} = \operatorname{mops}\mathbf{u}$. *We say that* ω *is the* $q-$*weight function associated to* $\mathbf{u}$ *(respectively to* $(P_n)_{n\geq 0}$*) if* ω *satisfies the equation (2.1)* $\Theta^\star(\phi\omega) = q\psi\omega$.

The last definition allows us to rewrite the $q-\mathcal{SL}$ equation (1.8) in its self-adjoint form. In fact, an straightforward calculations show that, if $\omega^{(k)}$ satisfies the $q-$Pearson equation

$$\Theta^\star(\phi^{(k)}\omega^{(k)}) = q\psi^{(k)}\omega^{(k)}, \tag{2.3}$$

where $\phi^{(k)}$ and $\psi^{(k)}$ are given in (1.9), then (1.8) can be rewritten in its self-adjoint form

$$\Theta^\star(\phi^{(k)}\omega^{(k)}\Theta Q_n^{(k)}) = q\widehat{\lambda}_n^{(k)}\omega^{(k)}Q_n^{(k)}\ ,\ n \geq 1,\ k = 0,1,\ldots,n. \tag{2.4}$$

Proposition 2.2 *Let* ω *the solution of (2.1) and* $\omega^{(k)}$ *the solution of (2.3). Then,*

$$\omega^{(k)} = \phi^{(n-1)}\omega^{(n-1)} = \cdots = \mathrm{H}^{(n)}\phi\cdot\omega, \quad \omega^{(0)} \equiv \omega. \tag{2.5}$$

Proof: We start from the $q-$Pearson equation (2.3) and rewrite it in its equivalent form $\phi^{(k)}\omega^{(k)} = q\mathrm{H}[\phi^{(k)}]^\star\mathrm{H}\omega^{(k)}$, where $[\phi^{(k)}]^\star = q^{-1}\phi^{(k)} + (q^{-1}-1)x\psi^{(k)} = \phi^\star$, for all $k \in \mathbb{N}$. Thus, by substituting $\omega^{(k)} = \mathrm{H}^{(n)}\phi\cdot\omega$ in $\phi^{(k)}\omega^{(k)} = q\mathrm{H}\phi^\star\mathrm{H}\omega^{(k)}$, we find

$$\phi^{(k)}\omega^{(k)} = q\mathrm{H}[\phi^{(k)}]^\star\mathrm{H}\omega^{(k)} \Longleftrightarrow \mathrm{H}^k\phi(\phi\mathrm{H}\phi\cdots\mathrm{H}^{k-1}\phi\cdot\omega) = q\mathrm{H}\phi^\star\mathrm{H}\phi\cdots\mathrm{H}^k\phi\mathrm{H}\omega \Longleftrightarrow$$

$$\phi\omega = q\mathrm{H}\phi^\star\mathrm{H}\omega \Longleftrightarrow \Theta^\star(\phi\omega) = q\psi\omega,$$

from where the proposition follows. □

Remark 2.1 *Notice that the polynomials $(\phi^{(k)})^\star$ and $(\phi^\star)^{(k)}$ are very different. In fact, the first one together with $\psi^{(k)}$ are the corresponding polynomials that appear in the $q^{-1}-$distributional equation satisfied by the functional $\mathbf{v}^{(k)}$, i.e., the functional with respect to which the $k-$th monic derivatives $Q_n^{(k)}$ are orthogonal, (see Proposition 1.4)*

$$\Theta(\phi^{(k)}\mathbf{v}^{(k)}) = \psi^{(k)}\mathbf{v}^{(k)} \quad \Longleftrightarrow \quad \Theta^\star(\phi^{(k)})^\star\mathbf{v}^{(k)} = \psi^{(k)}\mathbf{v}^{(k)}, \quad (\phi^{(k)})^\star = \phi^\star, \quad \forall k \in \mathbb{N},$$

whereas the second one joint with $(\psi^\star)^{(k)}$ are the polynomial coefficients of the $q^{-1}-\mathcal{SL}$ equation $(\psi^\star)^{(k)}\Theta^\star\Theta Q_n^{\star(k)}(\psi^\star)^{(k)} = (\widehat{\lambda}^\star)_n^{(k)}Q_n^{\star(k)}$, of the $n-$th $q^{-1}-$derivative $Q_n^{\star(k)}$ of the polynomials P_n, $Q_n^{\star(k)} = \frac{1}{[n+1]^\star_{(k)}}[\Theta^\star]^n P_{n+k}$ or the $q^{-1}-$distributional equation satisfied by the functional $\mathbf{v}^{\star(k)}$,

$$\Theta^\star[(\phi^\star)^{(k)}\mathbf{v}^{\star(k)}] = (\psi^\star)^{(k)}\mathbf{v}^{\star(k)}, \quad (\phi^\star)^{(k)} = \mathrm{H}^{-k}\phi^\star, \quad \forall k \in \mathbb{N},$$

i.e., the functional with respect to which the $k-$th monic derivatives $Q_n^{\star(k)}$ are orthogonal.

2.2 Computation of the $q-$weight functions

This section is devoted to obtain the $q-$weight function associated to all $q-$classical functionals, i.e., the quasi-definite functionals corresponding to the MOPS in the widespread sense $\langle \mathbf{u}, P_n^2\rangle \neq 0$, for all $n \geq 0$. In fact, Theorem 2.1 and 2.2 will give, in a very natural way, the key for the classification of all $q-$classical orthogonal polynomials.

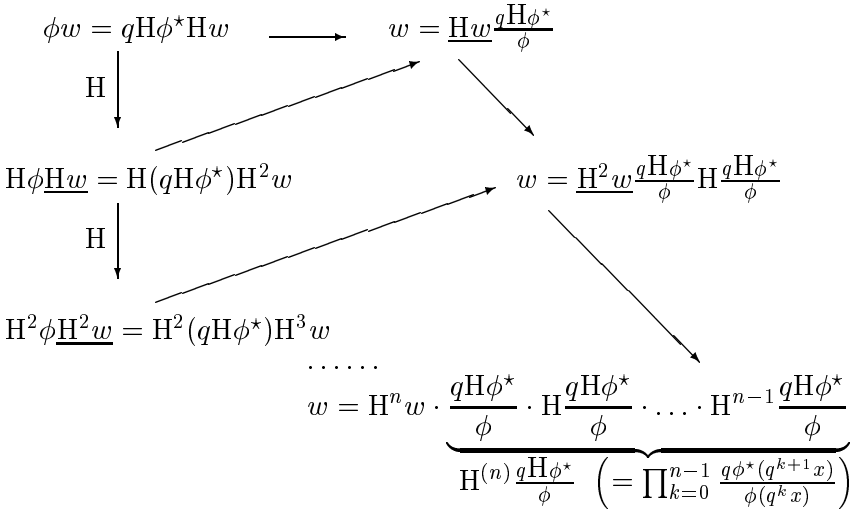
In the following we consider the case when $|q| < 1$ ($|q^{-1}| > 1$). Also we will use the standard notation $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n \geq 1$, $(a;q)_0 \equiv 1$ for the $q-$analogue of the Pocchammer symbol, and $(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n)$, for the absolutely convergent infinite product for $|q| < 1$.

First of all, we will rewrite the $q-$Pearson equation (2.1)

$$\Theta^\star(\phi\omega) = q\psi\omega \quad \Longleftrightarrow \quad \phi\omega = q\mathrm{H}\phi^\star\mathrm{H}\omega \quad \Longleftrightarrow \quad \phi^\star\omega = q^{-1}H^{-1}\phi\mathrm{H}^{-1}\omega, \tag{2.6}$$

and solve the resulting equation by the recurrent procedure shown in figure 1.

Figure 1. Recurrent schema using the $q-$dilation.



In the case when ω is continuous at 0 and $\omega(0) \neq 0$, taking the limit $n \to \infty$, we find, since $\lim_{n\to\infty} \mathrm{H}^n w = \lim_{n\to\infty} w(q^n x) = w(0)$,

$$\omega = \omega(0) \lim \mathrm{H}^{(\infty)} \frac{q\mathrm{H}\phi^\star}{\phi} = \omega(0) \lim_{n\to\infty} \mathrm{H}^{(n)} \frac{q\mathrm{H}\phi^\star}{\phi} = \omega(0) \prod_{n=0}^{\infty} \frac{q\mathrm{H}\phi^\star}{\phi}. \tag{2.7}$$

The next step is to obtain an explicit expression for the product $\mathrm{H}^{(\infty)} \frac{q\mathrm{H}\phi^\star}{\phi}$. For doing that we need a lemma which is interesting in its own right.

Lemma 2.1 *If π is an $n-$th degree polynomial with an independent term $\pi(0) = 1$, and zeros $a_i \in \mathbb{C}\backslash\{0\}$, $i = 1, 2, \ldots, n$, then*

$$\mathrm{H}^{(\infty)}\pi = (a_1^{-1}x; q)_\infty (a_2^{-1}x; q)_\infty \cdots (a_n^{-1}x; q)_\infty := (a_1^{-1}x, a_2^{-1}x, \cdots, a_n^{-1}x; q)_\infty,$$

is an entire function of x with zeros at $a_i q^{-k}$, $i = 1, 2, \ldots, n$ and $k \geq 0$. Furthermore, if π/ρ is a rational function such that $\pi(0) = \rho(0) \neq 0$ and with non-vanishing zeros of its numerator and denominator, then,

$$\mathrm{H}^{(\infty)} \frac{\pi}{\rho} = \frac{(a_1^{-1}x; q)_\infty (a_2^{-1}x; q)_\infty \cdots (a_n^{-1}x; q)_\infty}{(b_1^{-1}x; q)_\infty (b_2^{-1}x; q)_\infty \cdots (b_m^{-1}x; q)_\infty} = \frac{(a_1^{-1}x, a_2^{-1}x, \cdots, a_n^{-1}x; q)_\infty}{(b_1^{-1}x, b_2^{-1}x, \cdots, b_m^{-1}x; q)_\infty},$$

it is a meromorphic function with zeros at $a_i q^{-k}$, $i = 1, 2, \ldots, n$ and $k \geq 0$ and poles at $b_j q^{-l}$, $j = 1, 2, \ldots, m$ and $l \geq 0$, where $a_i \in \mathbb{C}$, $i = 1, 2, \ldots, n$ and $b_k \in \mathbb{C}$, $k = 1, 2, \ldots, m$, are the zeros of the numerator and denominator of π/ρ, respectively.

Proof: The proof is based on the fact that, if π is a polynomial of degree n with non vanishing zeros and $\pi(0) = 1$, then it admits the factorization

$$\pi = A(x - a_1)(x - a_2) \cdots (x - a_n) = \underbrace{(-1)^n A a_1 a_2 \cdots a_n}_{\pi(0)=1} (1 - a_1^{-1}x)(1 - a_2^{-1}x) \cdots (1 - a_n^{-1}x).$$

Then, $\mathrm{H}^{(k)}\pi = (a_1^{-1}x, a_2^{-1}x, \cdots, a_n^{-1}x; q)_k$ and so, $\mathrm{H}^{(\infty)}\pi = (a_1^{-1}x, a_2^{-1}x, \cdots, a_n^{-1}x; q)_\infty$. This function is an entire function due to the Weierstrass Theorem (see e.g. [1, §4.3]). The proof of the second statement is analogous and the function $\mathrm{H}^{(\infty)} \frac{\pi}{\rho}$ is meromorphic because is a quotient of two entire functions (see e.g. [1, §4.3]). □

Now, if $\phi(0) \neq 0$, the above lemma leads us to the following well known result [11]

Theorem 2.1 *Let $(P_n)_{n\geq 0} = \mathrm{mops}\,\mathbf{u}$ satisfying the $q-$Sturm-Liouville equation (1.6). If we denote by a_1 and a_2 the zeros of ϕ and by $a_1^\star$ and $a_2^\star$ the zeros of $\phi^\star$ (see Proposition 1.4), and all they are different from 0, then the following expressions for the $q-$weight functions ω hold*

ϕ	$\phi^\star$	$q-$weight function $\omega(x)$
$\widehat{a}(x - a_1)(x - a_2)$, $\widehat{a}a_1a_2 \neq 0$	$\widehat{a}^\star(x - a_1^\star)(x - a_2^\star)$, $\widehat{a}^\star a_1^\star a_2^\star \neq 0$	$\omega(x) = \dfrac{(a_1^{\star -1}qx, a_2^{\star -1}qx; q)_\infty}{(a_1^{-1}x, a_2^{-1}x; q)_\infty}$
	$\bar{a}^\star(x - a_1^\star)$, $\bar{a}^\star a_1^\star \neq 0$	$\omega(x) = \dfrac{(a_1^{\star -1}qx; q)_\infty}{(a_1^{-1}x, a_2^{-1}x; q)_\infty}$
	$\dot{a}^\star \neq 0$	$\omega(x) = \dfrac{1}{(a_1^{-1}x, a_2^{-1}x; q)_\infty}$
$\bar{a}(x - a_1)$, $\bar{a}a_1 \neq 0$	$\widehat{a}(x - a_1)(x - a_2)$, $\widehat{a}a_1a_2 \neq 0$	$\omega(x) = \dfrac{(a_1^{\star -1}qx, a_2^{\star -1}qx; q)_\infty}{(a_1^{-1}x; q)_\infty}$
$\dot{a} \neq 0$		$\omega(x) = (a_1^{\star -1}qx, a_2^{\star -1}qx; q)_\infty$

Proof: Since $\phi(x)=\widehat{a}(x-a_1)(x-a_2)$ and $\phi^\star=q^{-1}\phi+(q^{-1}-1)x\psi=\widehat{a}^\star(x-a_1^\star)(x-a_2^\star)$, we have $(q\mathrm{H}\phi^\star)(0)=q\phi^\star(0)=\phi(0)$, so the polynomials $q\mathrm{H}\phi^\star$ and ϕ have the same independent term. Using the power expansion of the polynomials ϕ and $\phi^\star$

$$\phi(x)=\widehat{a}x^2+\bar{a}x+\dot{a},\qquad \phi^\star(x)=\widehat{a}^\star x^2+\bar{a}^\star x+\dot{a}^\star,$$

we have $\widehat{a}^\star=q^{-1}\widehat{a}+(q^{-1}-1)\widehat{b}$, $\bar{a}^\star=q^{-1}\bar{a}+(q^{-1}-1)\bar{b}$ and $\dot{a}^\star=q^{-1}\dot{a}$, where, $\widehat{b},\bar{b}$ are the coefficient of the power expansion of ψ (see Eq. (1.7)). Thus,

$$\begin{cases}\deg\phi<2\Longrightarrow\widehat{a}=0\Longrightarrow\widehat{a}^\star\neq0\Longrightarrow\deg\phi^\star=2,\\ \deg\phi=2\Longrightarrow\widehat{a}\neq0\begin{cases}\widehat{b}\neq-\dfrac{\widehat{a}}{1-q}\Longrightarrow\widehat{a}^\star\neq0\Longrightarrow\deg\phi^\star=2,\\ \widehat{b}=-\dfrac{\widehat{a}}{1-q}\Longrightarrow\widehat{a}^\star=0\begin{cases}\bar{b}\neq-\dfrac{\bar{a}}{1-q}\Longrightarrow\deg\phi^\star=1,\\ \bar{b}=-\dfrac{\bar{a}}{1-q}\Longrightarrow\deg\phi^\star=1.\end{cases}\end{cases}\end{cases}$$

In all cases we can apply directly the above lemma which immediately leads us to the desired result. Notice also that all the obtained functions are meromorphic and so, they are continuous and non-vanishing at $x=0$, so we can suppose without any loss of generality that $\omega(0)=1$. □

In the case when $\phi(0)=0$, it is easy to see that $\phi^\star(0)=0$. This case requires a more detail study. In the following we should keep in mind that for the quasi-definiteness of $\mathbf{u}$ $\phi\not\equiv0$ and ϕ and ψ should be coprime polynomials (see [18]).

Proposition 2.3 *Let* $\mathbf{u}$ *be a q−classical functional satisfying the distributional equation (1.5) with* $\phi=\widehat{a}x^2+\bar{a}x$, $|\widehat{a}|+|\bar{a}|>0$, *and* $\psi=\widehat{b}x+\bar{b}$, $\widehat{b}\neq0$. *Then the following cases, compatible with the quasi-definiteness of* $\mathbf{u}$, *appear:*

(a) If $\phi=\widehat{a}x^2$, $\widehat{a}\neq0$, *then,* $\deg\phi^\star=2$ *and its two zeros are different, or* $\deg\phi^\star=1$.

(b) If $\phi=\widehat{a}x^2+\bar{a}x$, $\widehat{a}\bar{a}\neq0$, *then,* $\deg\phi^\star=2$, *or* $\deg\phi^\star=1$.

(c) If $\phi=\bar{a}x$, $\bar{a}\neq0$, *then,* $\deg\phi^\star=2$.

Proof:
(a) Since $\phi=\widehat{a}x^2$, then $\psi=\widehat{b}x+\bar{b}$, with $\bar{b}\neq0$, otherwise ψ divides ϕ. Therefore, $\phi^\star=(q^{-1}\widehat{a}+(q^{-1}-1)\widehat{b})x^2+(q^{-1}-1)\bar{b}x$ has a non-vanishing coefficient on x. If $\widehat{b}\neq-\frac{\widehat{a}}{1-q}$ then $\widehat{a}^\star\neq0$ and $\deg\phi^\star=2$ and $\phi^\star$ has two different zeros one of which is located at the origin. If $\widehat{b}=-\frac{\widehat{a}}{1-q}$ then $\deg\phi^\star=1$.

The other two cases are proven analogously. □

The next step is to find the $q-$weight functions for all possible cases according with the above proposition (remember that $\phi(0)=0=\phi^\star(0)$). There are two large classes. Class I corresponding to the case when ϕ and $\phi^\star$ have non-vanishing term on x and II when they have a vanishing term on x.

I. We start with the case when ϕ and $\phi^\star$ have not-vanishing term on x. In this case there are three different possibilities (subclasses):

(a) $\phi(x)=\widehat{a}x(x-a_1)$, $\widehat{a}a_1\neq0$ and $\phi^\star(x)=\widehat{a}^\star x(x-a_1^\star)$, $\widehat{a}^\star a_1^\star\neq0$,

(b) $\phi(x)=\widehat{a}x(x-a_1)$, $\widehat{a}a_1\neq0$ and $\phi^\star(x)=\bar{a}^\star x$, $\bar{a}^\star\neq0$,

(c) $\phi(x)=\bar{a}x$, $\bar{a}\neq0$ and $\phi^\star(x)=\widehat{a}^\star x(x-a_1^\star)$, $\widehat{a}^\star a_1^\star\neq0$.

To find the corresponding $q-$weight functions we will rewrite the quotient $q\mathrm{H}\phi^\star/\phi = xq\,(\mathrm{H}\phi^\star)_0/x\phi_0$, where $\phi = x\phi_0$ and $\mathrm{H}\phi^\star = x(\mathrm{H}\phi^\star)_0$. In general, $\phi_0(0) \neq (\mathrm{H}\phi^\star)_0(0)$, so, in order to apply a method, similar to the one used to prove Theorem 2.1, we will assume that ω can be rewritten on the form $\omega = |x|^\alpha\omega_0$, $\alpha \in \mathbb{C}\backslash\{0\}$, where α is a free parameter to be found. An straightforward calculations show that if ω satisfies a $q-$Pearson equation (2.6) then ω_0 satisfies the equation $\phi_0\omega_0 = aq\mathrm{H}\omega_0(\mathrm{H}\phi^\star)_0$, where $a = q^\alpha$. So,

$$\omega_0 = \mathrm{H}^n(\omega_0)\frac{aq(\mathrm{H}\phi^\star)_0}{\phi_0}, \quad a = q^\alpha, \quad \text{or} \quad \alpha = \mathrm{Log}_q(a),$$

where Log_q denotes the principal logarithm on the basis q, $|q| < 1$. In the following, we will use the notation $\bar{a} = -\widehat{a}a_1$ and $\bar{a}^\star = -\widehat{a}^\star a_1^\star$. Notice that, with this notation, $\phi = \widehat{a}x(x-a_1) = \widehat{a}x^2 + \bar{a}x$ and $\phi^\star = \widehat{a}^\star x(x-a_1^\star) = \widehat{a}^\star x^2 + \bar{a}^\star x$.

(a) In this case,

$$\frac{aq(\mathrm{H}\phi^\star)_0}{\phi_0} = \frac{aq^2\widehat{a}^\star a_1^\star(a_1^{\star\,-1}qx-1)}{\widehat{a}a_1(a_1^{-1}x-1)} = \frac{aq^2\bar{a}^\star(1-a_1^{\star\,-1}qx)}{\bar{a}(1-a_1^{-1}x)}.$$

If we choose now, a such that $aq(\mathrm{H}\phi^\star)_0(0) = \phi_0(0)$, i.e., $aq^2\bar{a}^\star = \bar{a}$, or equivalently, $\alpha = \mathrm{Log}_q(a) = -2 + \mathrm{Log}_q\frac{\bar{a}}{\bar{a}^\star}$, we can apply the Lemma 2.1 to get, $\omega_0 = \omega_0(0)\frac{(a_1^{\star\,-1}qx;q)_\infty}{(a_1^{-1}x;q)_\infty}$, which leads, without any loss of generality, to the following weight function (here we suppose that ω_0 is continuous and $\omega_0(0) \neq 0$)

$$\omega(x) = |x|^\alpha\frac{(a_1^{\star\,-1}qx;q)_\infty}{(a_1^{-1}x;q)_\infty}, \qquad \alpha = \mathrm{Log}_q(a) = -2 + \mathrm{Log}_q\frac{\bar{a}}{\bar{a}^\star}. \tag{2.8}$$

(b) In this case,

$$\frac{aq(\mathrm{H}\phi^\star)_0}{\phi_0} = \frac{aq^2\bar{a}^\star}{\bar{a}(1-a_1^{-1}x)} \quad \Longrightarrow \quad \omega_0 = \omega_0(0)\frac{1}{(a_1^{-1}x;q)_\infty}, \quad \alpha = \mathrm{Log}_q(a) = -2 + \mathrm{Log}_q\left(-\frac{\bar{a}}{\bar{a}^\star}\right),$$

so,

$$\omega(x) = \frac{|x|^\alpha}{(a_1^{-1}x;q)_\infty}, \quad \alpha = \mathrm{Log}_q(a) = -2 + \mathrm{Log}_q\left(-\frac{\bar{a}}{\bar{a}^\star}\right).$$

Finally, in the last case (c), we obtain

$$\omega(x) = |x|^\alpha(a_1^{\star\,-1}qx;q)_\infty, \quad \alpha = \mathrm{Log}_q(a) = -2 + \mathrm{Log}_q\left(-\frac{\bar{a}}{\bar{a}^\star}\right).$$

II. Let consider the other case, i.e., when ϕ and $\phi^\star$ have a vanishing term on x. In this case there are two possibilities:

(i) $\deg\phi \neq \deg\phi^\star$ which is divided in two subcases (a) $\phi = \widehat{a}x^2$, $\phi^\star = \bar{a}^\star x$, and (b) $\phi = \bar{a}x$, $\phi^\star = \widehat{a}^\star x^2$, and

(ii) $\deg\phi = \deg\phi^\star$, which also is divided in two subcases (a) $\phi = \widehat{a}x^2$, $\phi^\star = \widehat{a}^\star x(x-a_1^\star)$, $a_1^\star \neq 0$, and (b) $\phi = \widehat{a}x(x-a_1)$, $\phi^\star = \widehat{a}^\star x^2$, $a_1 \neq 0$.

In both cases, the method used in the case I of non-vanishing coefficients can not be used.

(i) In order to solve the problem for case II(i) we will generalize an idea by Häcker [12]. Let us define the function $h^{(\beta)} : [0,\infty) \to \mathbb{R}$ defined by

$$h^{(\beta)}(x) = \sqrt{x^{\log_q x^\beta - \beta}}, \quad \beta \neq 0,$$

which has the following property $\mathrm{H}h^\beta = x^\beta h^\beta$, or, equivalently, $h^\beta(qx) = x^\beta h^\beta(x)$, for all $x \geq 0$.

If we now define the function $\omega = x^\alpha h^{(1)}$, then, for the case II(i)a we have

$$\mathrm{H}\omega = \mathrm{H}x^\alpha h^{(1)} = q^\alpha x^\alpha x h^{(1)} = q^\alpha x\omega \quad \Longrightarrow \quad x\mathrm{H}\omega = q^\alpha x^2\omega,$$

then, comparing this resulting equation with the $q-$Pearson equation (2.6) for this choice of ϕ and $\phi^\star$, $\widehat{a}x^2\omega = q\bar{a}^\star x\mathrm{H}\omega$, we deduce that the function

$$\omega(x) = |x|^\alpha \sqrt{x^{\log_q x - 1}}, \qquad \alpha = -2 + \mathrm{Log}_q \frac{\widehat{a}}{\bar{a}^\star}, \qquad x \geq 0, \tag{2.9}$$

is the the solution of the $q-$Pearson equation and so, the corresponding $q-$weight function.

For the case II(i)b we have, in an analogous way, a similar solution but involving the function $h^{(-1)}$:

$$\omega(x) = |x|^\alpha \sqrt{x^{\log_q \frac{1}{x}+1}}, \qquad \alpha = -3 + \mathrm{Log}_q \frac{\bar{a}}{\widehat{a}^\star}, \qquad x \geq 0. \tag{2.10}$$

(ii) In this case the method developed for the above cases does not work. In fact, if we try to use the method for the case I, after some straightforward calculations, we arrive to an infinite divergent product. For this reason we will solve the $q-$Pearson equation using the equivalent equation (2.6) in q^{-1} dilation $q^{-1}\mathrm{H}^{-1}\phi\mathrm{H}^{-1}\omega = \phi^\star\omega$ (2.6), i.e., using a schema similar to the one given in figure 1 but when the recurrence is solved in the "opposite" direction to obtain the expression $\omega = \mathrm{H}^{-n}\omega\mathrm{H}^{(-n)}\frac{q^{-1}\mathrm{H}^{-1}\phi}{\phi^\star}$, which leads to the solution, by taking the limit $n \to \infty$, if there exists the value $\mathrm{H}^{-\infty}\omega = \omega(\infty)$. In such a way, we have for the case II(ii)a the expression

$$\omega = |x|^\alpha\omega_0\ ,\ \alpha \geq 0 \quad , \quad \phi^\star = x\phi_0^\star = x(\widehat{a}^\star x + \bar{a}^\star) \quad , \quad \mathrm{H}^{-1}\phi = x(\mathrm{H}^{-1}\phi)_0 = x(q^{-2}\widehat{a}x)\ .$$

hence, the $q^{-1}-$Pearson equation takes the form

$$x\phi_0^\star \cdot |x|^\alpha\omega_0 = q^{-1}\cdot x(\mathrm{H}^{-1}\phi)_0 \cdot \mathrm{H}^{-1}(|x|^\alpha\omega_0) \implies \phi_0^\star\omega_0 = q^{-1}(\mathrm{H}^{-1}\phi)_0\, q^{-\alpha}\mathrm{H}^{-1}\omega_0\ ,$$

and its solution is

$$\omega_0(0) = \mathrm{H}^{-n}w_0^\star\,\mathrm{H}^{(-n)}\frac{q^{-\alpha}q^{-1}(\mathrm{H}^{-1}\phi)_0}{\phi_0^\star} \overset{a:=q^\alpha}{=} \mathrm{H}^{-n}\omega_0\,\mathrm{H}^{(-n)}\frac{a^{-1}q^{-3}\widehat{a}x}{\widehat{a}^\star x + \bar{a}^\star}\ .$$

Now, choosing the value α, in such a way that $a^{-1}q^{-3}\widehat{a} = \widehat{a}^\star$, i.e., $\alpha = -3 + \mathrm{Log}_q\frac{\widehat{a}}{\widehat{a}^\star}$ we find,

$$\omega_0(0) = \mathrm{H}^{-n}w_0^\star\,\mathrm{H}^{(-n)}\frac{\widehat{a}^\star x}{\widehat{a}^\star x + \bar{a}^\star} = \mathrm{H}^{-n}\omega_0\mathrm{H}^{(-n)}\Big(1 - \frac{\bar{a}^\star}{\widehat{a}^\star x + \bar{a}^\star}\Big) = \mathrm{H}^{-n}\omega_0\prod_{i=0}^{n}\Big(1 - \frac{\bar{a}^\star q^i}{\widehat{a}^\star x + \bar{a}^\star q^i}\Big).$$

Obviously the above product is uniformly convergent in any compact subset of the complex plane that not contains the points $\{a_1^\star q^n, n \geq 0\}\bigcup\{0\}$, where $a_1^\star = -\bar{a}^\star/\widehat{a}^\star$ is the non-vanishing zero of $\phi^\star$ (in $x = 0$ the product diverges to zero). Furthermore, this product converges at ∞, so $\omega(\infty) = c \neq 0$, and thus

$$\omega(x) = |x|^\alpha w_0^\star = c|x|^\alpha\prod_{n=0}^{\infty}\Big(1 - \frac{\bar{a}^\star q^n}{\widehat{a}^\star x + \bar{a}^\star q^n}\Big) = c|x|^\alpha\mathrm{H}^{(-\infty)}\frac{\widehat{a}^\star x}{\widehat{a}^\star x + \bar{a}^\star} = c|x|^\alpha\frac{1}{(-\frac{\bar{a}^\star}{\widehat{a}^\star x};q)_\infty},$$

where $\alpha = \mathrm{Log}_q\frac{\widehat{a}q^{-3}}{\widehat{a}^\star}$, and which, without any loss of generality, leads to the following expression for the $q-$weight function

$$\omega(x) = |x|^\alpha\frac{1}{(a_1^\star/x;q)_\infty} = |x|^\alpha e_q(a_1^\star/x)\ , \qquad a_1^\star = -\frac{\bar{a}^\star}{\widehat{a}^\star x}, \qquad \alpha = -3 + \mathrm{Log}_q\frac{\widehat{a}}{\widehat{a}^\star}, \tag{2.11}$$

where e_q denotes the $q-$exponential function [10].

A similar situation happens in the II(ii)b subcase. In this case, we have

$$\omega_0(x) = \mathrm{H}^{-n}\omega_0\,\mathrm{H}^{(-n)}\frac{q^{-\alpha}q^{-1}(\mathrm{H}^{-1}\phi)_0}{\phi_0^\star} \overset{a:=q^\alpha}{=} \mathrm{H}^{-n}\omega_0\,\mathrm{H}^{(-n)}\frac{a^{-1}q^{-1}q^{-1}\widehat{a}(q^{-1}x - a_1)}{\widehat{a}^\star x}\ .$$

If we now choose $a^{-1}q^{-3}\widehat{a}=\widehat{a}^\star$, we find

$$\omega_0(x)=\mathrm{H}^{-n}\omega_0\,\mathrm{H}^{(-n)}\frac{\widehat{a}^\star x-a^{-1}q^{-2}\widehat{a}a_1}{\widehat{a}^\star x}=\mathrm{H}^{-n}\omega_0\,\mathrm{H}^{(-n)}\Big(1-\frac{\widehat{a}^\star q a_1}{\widehat{a}^\star x}\Big)=\mathrm{H}^{-n}\omega_0\,\mathrm{H}^{(-n)}\Big(1-\frac{a_1q}{x}\Big),$$

which is an absolute and uniformly convergent product in $\mathbb{C}\backslash\{0\}$. Finally, since $\omega(\infty)=c\neq 0$, and without any loss of generality we find the following expression for the $q-$weight function ω

$$\omega(x)=|x|^\alpha(a_1q/x;q)_\infty\quad,\quad \alpha=-3-\mathrm{Log}_q\frac{\widehat{a}}{\widehat{a}^\star}\,,$$

where a_1 is the non-vanishing zero of ϕ. All the above calculations can be summarize in the following theorem:

Theorem 2.2 *Let $(P_n)_{n\geq 0}=\mathrm{mops}\,\mathbf{u}$ satisfying the $q-$Sturm-Liouville equation (1.6). If we denote by a_1 and a_2 the zeros of ϕ and by $a_1^\star$ and $a_2^\star$ the zeros of $\phi^\star$ (see Proposition 1.4), and one of them are equal to 0, then the following expressions for the $q-$weight functions ω hold*

Case	ϕ	$\phi^\star$	$\omega(x)$	
II(ii)a	$\widehat{a}x^2$, $\widehat{a}\neq 0$	$\widehat{a}^\star x(x-a_1^\star)$, $\widehat{a}^\star a_1^\star\neq 0$	$\lvert x\rvert^\alpha\dfrac{1}{(a_1^\star/x;q)_\infty}$,	$\alpha=\mathrm{Log}_q\frac{\widehat{a}q^{-3}}{\widehat{a}^\star}$
II(i)a		$\bar{a}^\star x$, $\bar{a}^\star\neq 0$	$\lvert x\rvert^\alpha\sqrt{x^{\log_q x-1}}$,	$\alpha=\mathrm{Log}_q\frac{\widehat{a}q^{-2}}{\bar{a}^\star}$
I(a)	$\widehat{a}x(x-a_1)$, $\widehat{a}a_1\neq 0$	$\widehat{a}^\star x(x-a_1^\star)$, $\widehat{a}^\star a_1^\star\neq 0$	$\lvert x\rvert^\alpha\dfrac{(a_1^{\star-1}qx;q)_\infty}{(a_1^{-1}x;q)_\infty}$,	$\alpha=\mathrm{Log}_q\frac{\bar{a}q^{-2}}{\bar{a}^\star}$
I(b)		$\bar{a}^\star x$, $\bar{a}^\star\neq 0$	$\lvert x\rvert^\alpha\dfrac{1}{(a_1^{-1}x;q)_\infty}$,	$\alpha=\mathrm{Log}_q\frac{-\bar{a}q^{-2}}{\bar{a}^\star}$
II(ii)b		$\widehat{a}^\star x^2$, $\widehat{a}^\star\neq 0$	$\lvert x\rvert^\alpha(a_1q/x;q)_\infty$,	$\alpha=-\mathrm{Log}_q\frac{\widehat{a}q^3}{\widehat{a}^\star}$
I(c)	$\bar{a}x$, $\bar{a}\neq 0$	$\widehat{a}^\star x(x-a_1^\star)$, $\widehat{a}^\star a_1^\star\neq 0$	$\lvert x\rvert^\alpha(a_1^{\star-1}qx;q)_\infty$,	$\alpha=\mathrm{Log}_q\frac{-\bar{a}q^{-2}}{\bar{a}^\star}$
II(i)b		$\widehat{a}^\star x^2$, $\widehat{a}^\star\neq 0$	$\lvert x\rvert^\alpha\sqrt{x^{\log_q\frac{1}{x}+1}}$,	$\alpha=\mathrm{Log}_q\frac{\bar{a}q^{-3}}{\widehat{a}^\star}$

3 Applications

In this section we will consider some applications of the above theorems. In fact we will show how the $q-$weight functions can be used to give an integral representation for the orthogonality. Another interesting application is the already mentioned classification of all **orthogonal** families in the $q-$Hahn Tableau (in [23] the orthogonality was not considered). In fact Theorems 2.1 and 2.2 gives a natural classification of the $q-$classical orthogonal polynomials. Also by using the $q-$weights one can obtain an explicit formula of the polynomials satisfying a Rodrigues-type formula in terms of the polynomials coefficients ϕ and $\phi^\star$ from where the hypergeometric representation easily follows. The last have been done independently in [23] and [6] (see also [5, 20]) in the framework of the difference equations of hypergeometric type on the non-uniform lattices. Here we will show how all the $q-$classical families can be obtained by certain limiting processes from the most general case of $\emptyset-$Jacobi/Jacobi family. Finally, we will compare the Nikiforov & Uvarov and the $q-$Askey Tableaus with our $q-$Hahn Tableau and complete the $q-$Askey one with new families of orthogonal polynomials.

3.1 The $q-$integral representation for the orthogonality

In this section we will show how the $q-$weight functions and the $q-\mathcal{SL}$ equation lead to a $q-$integral representation for the orthogonality. The technique used here is very common in the theory of orthogonal polynomials (see e.g. [7, 13, 20]).

First of all we introduce the $q-$integral of Jackson [10, 25]. This integral is a Riemann sum on an infinite partition $\{aq^n,\, n\geq 0\}$,

$$\int_0^a f(x)\,d_qx=(1-q)a\sum_{n=0}^{\infty}f(aq^n)q^n\ ,\quad \text{and}\int_a^b f(x)\,d_qx=\int_0^b f(x)\,d_qx-\int_0^a f(x)\,d_qx,$$

so, it is valid the $q-$analogue of the Barrow rule (here $\Theta F(x)$ is continuous at $x=0$): $\int_a^b \Theta F(x)\,d_qx = F(b)-F(a)$, and the rules of integration by parts

$$\int_a^b f(x)\Theta g(x)\,d_qx=\mathrm{H}^{-1}f(x)\cdot g(x)\Big|_a^b-q\int_a^b g(x)\Theta^\star f(x)\,d_qx,$$

$$\int_a^b f(x)\Theta g(x)\,d_qx=fg\Big|_a^b-\int_a^b \mathrm{H}g(x)\Theta f(x)\,d_qx\ .$$

Obviously in the above expressions it is assumed that the function f is defined in the corresponding partition's points. This Jackson $q-$integral can be easily generalized to unbounded intervals and unbounded functions in a similar way as the Riemann integral [10, 25]. Furthermore, the Riemann-Stieltjes discrete integrals related with the $q-$classical polynomials can be represented as $q-$integrals (see e.g. [17, 19]).

Proposition 3.1 *Let ω be continuous function in $x=0$ satisfying the $q-$Pearson equation $\Theta^\star(\phi\omega)=q\psi\omega$, equivalent to the distributional equation $\Theta(\phi\mathbf{u})=\psi\mathbf{u}$ and let a, b complex numbers such that the boundary condition $\phi^\star\omega\Big|_a^b=0$, or equivalently $H^{-1}\phi\omega\Big|_a^b=0$ ($\phi w=q\mathrm{H}(\phi^\star w)$) holds. Then,*

$$\int_a^b P_n(x)P_m(x)\omega(x)d_qx=0,\qquad \forall n\neq m,\qquad (P_n)_{n\geq 0}=\mathrm{mops}\,\mathbf{u}.$$

Proof: The proof is straightforward. We start from the self-adjoint form of the $q-\mathcal{SL}$ equations for the polynomial P_n and P_m, respectively:

$$\Theta[\mathrm{H}^{-1}(\phi\omega)\Theta^\star P_n]=\widehat{\lambda}_n\omega P_n,\qquad \Theta[\mathrm{H}^{-1}(\phi\omega)\Theta^\star P_m]=\widehat{\lambda}_m\omega P_m.$$

If we multiply the first one by P_m, the second one by P_n, takes the $q-$integral over (a,b) and use the integration by part rules we find

$$\begin{aligned}(\widehat{\lambda}_n-\widehat{\lambda}_m)\int_a^b\omega P_nP_m\,d_qx&=\int_a^b(\omega\widehat{\lambda}_nP_n)P_m\,d_qx-\int_a^b(\omega\widehat{\lambda}_mP_m)P_n\,d_qx=\\&=\int_a^b\Theta\Big(\mathrm{H}^{-1}(\phi\omega)\Theta^\star P_n\Big)P_m\,d_qx-\int_a^b\Theta\Big(\mathrm{H}^{-1}(\phi\omega)\Theta^\star P_n\Big)P_nd_qx=\\&=\mathrm{H}^{-1}(\phi\omega)W_q^\star[P_m,P_n]\Big|_a^b+\int_a^b\Big[\mathrm{H}\Big(\mathrm{H}^{-1}(\phi\omega)\Theta^\star P_m\Big)\Theta P_n-\mathrm{H}(\mathrm{H}^{-1}(\phi\omega)\Theta^\star P_n\Big)\Theta P_m\Big]\,d_qx,\end{aligned}$$

where $W_q^\star[P_m,P_n]=P_m\Theta^\star P_n-P_n\Theta^\star P_m$ is the $q-$Wronskian. The first term in the last equation vanish since the boundary conditions. The second also vanish since $\mathrm{H}\Big(\mathrm{H}^{-1}(\phi w)\Theta^\star P_m\Big)\Theta P_n$ $=\phi w\Theta P_m\Theta P_n$. The result follows from the fact that for all $n\neq m$, $\widehat{\lambda}_n\neq\widehat{\lambda}_m$. □

Remark 3.1 *Notice that the choice of the integration interval (a,b) is conditioned to guarantee that $\int_a^b P_n^2 \omega d_q x \neq 0$, $n \geq 0$, for which, it is enough that ω be continuous function and does not vanish inside the interval of integration. This has a difficulty since, even in the simplest cases, i.e., $\emptyset-$families, ω has infinite zeros $a_i^\star q^{-n}$, $n \geq 1$, and infinite poles, $a_i q^{-n}$, $n \geq 0$. Notice also that natural values for (a,b) are the roots of $\phi^\star$ or the roots of $\phi(q^{-1}x)$.*

Of special interest is the study of the positive definite case, i.e., the case when $\int_a^b P_n^2 \omega d_q x > 0$ for all $n \geq 0$. For doing that we can use the Favard theorem. The detailed study of positive definite case will be considered in a forthcoming paper.

3.2 Classification of the $q-$classical polynomials

Since the equation (1.5) (and so the Sturm-Liouville equation (2.2)) gives all the information about the $q-$classical functional (and then about the corresponding MOPS), it is natural to use them for classifying the $q-$classical polynomials. Moreover, all this information is condensed in the polynomials ϕ and $\phi^\star$ instead of ϕ and ψ (and more exactly in their zeros) as it is shown in Theorems 2.1 and 2.2. So it is natural to use the zeros of ϕ and $\phi^\star$ to clasify all families of $q-$classical orthogonal polynomials [17, 19].

In such a way, since $\phi(0) = 0$ if and only if $\phi^\star(0) = 0$, it is natural, in a first step, to classify the $q-$classical polynomials into two wide groups: the $\emptyset-$families, i.e., the families such that $\phi(0) \neq 0$ and the $0-$families, i.e., the ones with $\phi(0) = 0$. Next, we classify each member in the aforesaid two wide classes in terms of the degree of the polynomials ϕ and $\phi^\star$ as well as the multiplicity of their roots in the case of $0-$families. In fact, if ϕ has two simple roots, the polynomials belong to the $0-$Jacobi/— family while if the roots are multiple, then they are $0-$Bessel/— family. So, we have the following scheme for the $q-$classical OPS:

$$\emptyset-\text{families} \begin{cases} \emptyset-\text{Jacobi/Jacobi} \\ \emptyset-\text{Jacobi/Laguerre} \\ \emptyset-\text{Jacobi/Hermite} \\ \\ \emptyset-\text{Laguerre/Jacobi} \\ \\ \emptyset-\text{Hermite/Jacobi} \end{cases} \qquad 0-\text{families} \begin{cases} 0-\text{Bessel/Jacobi} \\ 0-\text{Bessel/Laguerre} \\ \\ 0-\text{Jacobi/Jacobi} \\ 0-\text{Jacobi/Laguerre} \\ 0-\text{Jacobi/Bessel} \\ \\ 0-\text{Laguerre/Jacobi} \\ 0-\text{Laguerre/Bessel} \end{cases}$$

Notice that in this scheme can not appear the families $\emptyset-$Laguerre/Laguerre, $\emptyset-$Laguerre/Hermite $\emptyset-$Hermite/Laguerre and $\emptyset-$Hermite/Hermite since the connection between ϕ and $\phi^\star$, as well as the $0-$Bessel/Bessel case since they do not correspond to a quasi-definite functional (see Proposition 2.3).

3.2.1 Connection with the Nikiforov-Uvarov and the $q-$Askey Tableaus

Here we will identify our classification (scheme) of the $q-$classical polynomials with the two well known schemes by Nikiforov and Uvarov [23] and the $q-$Askey Tableau [13].

We start with the first one. The Nikiforov-Uvarov Tableau is based on the polynomial solutions of the *second order linear difference equation of hypergeometric type in the non-uniform lattice* $x(s)$:

$$\begin{gathered} \tilde{\sigma}(x(s)) \frac{\triangle}{\triangle x(s-\frac{1}{2})} \frac{\nabla y_n[x(s)]}{\nabla x(s)} + \frac{\tilde{\tau}(x(s))}{2} \left[\frac{\triangle y_n[x(s)]}{\triangle x(s)} + \frac{\nabla y_n[x(s)]}{\nabla x(s)} \right] + \lambda y_n[x(s)] = 0, \\ \nabla f(s) = f(s) - f(s-1), \ \triangle f(s) = f(s+1) - f(s), \ y_n[x(s)] \in \mathbb{P}[x(s)] \\ x(s) = c_1(q) q^s + c_2(q) q^{-s} + c_3(q), q \in \mathbb{C}, \end{gathered} \tag{3.1}$$

where $\tilde{\sigma}(x)$ and $\tilde{\tau}(x)$ are polynomials in $x(s)$ of degree at most 2 and 1, respectively, and λ_n is a constant, or, written in its equivalent form

$$\begin{gathered}\sigma(s)\frac{\triangle}{\triangle x(s-\frac{1}{2})}\frac{\nabla y_n[x(s)]}{\nabla x(s)}+\tau(s)\frac{\triangle y_n[x(s)]}{\triangle x(s)}+\lambda_n y_n[x(s)]=0,\\ \sigma(s)=\tilde{\sigma}(x(s))-\tfrac{1}{2}\tilde{\tau}(x(s))\triangle x(s-\tfrac{1}{2}),\quad \tau(s)=\tilde{\tau}(x(s)).\end{gathered}\tag{3.2}$$

Here $\mathbb{P}[x(s)]$ denotes the linear space of polynomials in $x(s)$. Notice that, if $x(s)=c_1q^s\equiv x$, i.e., we are in the so-called *linear exponential lattice*, then

$$\frac{\triangle y_n[x(s)]}{\triangle x(s)}=\Theta y_n(x)\quad\text{and}\quad\frac{\nabla y_n[x(s)]}{\nabla x(s)}=\Theta^\star y_n(x),\qquad y_n(x)\equiv y_n[x(s)].$$

Thus, using the fact that $\triangle x(s-\frac{1}{2})=q^{-\frac{1}{2}}\triangle x(s)$, the hypergeometric equation (3.2) in the linear lattice $x(s)=c_1q^s$ can be rewritten as

$$\sigma(s)\Theta\Theta^\star y_n(x)+q^{-\frac{1}{2}}\tau(s)\Theta y_n(x)=-\lambda_n q^{-\frac{1}{2}}y_n(x),\qquad y_n(x)\in\mathbb{P},$$

from where, and using the identity $\Theta=x(q-1)\Theta\Theta^\star+\Theta^\star$ we arrive to the equation

$$[\sigma+q^{-\frac{1}{2}}\tau(s)x(q-1)]\Theta\Theta^\star y_n(x)+q^{-\frac{1}{2}}\tau(s)\Theta^\star y_n(x)=-\lambda_n q^{-\frac{1}{2}}y_n(x),$$

which is nothing else that the $q-\mathcal{SL}$ equation (1.6) where

$$\sigma(s)=\phi+x(1-q)\psi=q\phi^\star,\qquad\tau(s)=q^{-\frac{1}{2}}\psi,\qquad\lambda_n=-q^{\frac{1}{2}}\widehat{\lambda}_n.\tag{3.3}$$

In other word, the $q-\mathcal{SL}$ equation (1.6) is a second order linear difference equation of hypergeometric type in the linear exponential lattice $x(s)=c_1q^s$. The above connection allows us to identify all the $q-$classical orthogonal polynomials (in the widespread Hahn's sense) with the $q-$polynomials in the exponential lattice in the Nikiforov et al. approach. In fact, using the explicit expression of the polynomials $\sigma(s)$ and $\sigma(s)+\tau(s)\triangle(x-\frac{1}{2})$ in the exponential lattice [23, Eqs. (84)-(85) page 241 and Table page 244], we can identify our 12 classes of $q-$polynomials with the ones given in [23] (see Table 3.2.1).

In order to identify the $q-$classical polynomials with the ones given in the $q-$Askey tableau [13] we rewrite the $q-\mathcal{SL}$ equation (1.6) in the following form:

$$\phi\cdot\mathrm{H}P_n-(\phi+q^2\phi^\star)P_n+q^2\phi^\star\cdot\mathrm{H}^{-1}P_n=(q-1)^2x^2\lambda_nP_n.$$

Then, a simple comparison of the above difference equation with those given in the $q-$Askey Tableau allows us to identify some of the families of the $q-$polynomials given in [13] with the corresponding $q-$classical ones, and so, with the ones in the Nikiforov-Uvarov Tableau. This will be given in Table 3.2.1.

From the above table 3.2.1 we see that the 0−Jacobi/Bessel and 0−Laguerre/Bessel families lead to new families of orthogonal polynomials. The reason for that they do not appear in the $q-$Askey tableau will be considered latter on. Notice also that the class N$^{\underline{o}}$8 from the Nikiforov-Uvarov tableau [23, page 244] do not lead to any orthogonal polynomial sequence even in the widespread sense considered here.

3.3 The Rodrigues formula and hypergeometric representation

For the sake of completeness we will include here the identification of the $q-$classical polynomials in terms of the basic hypergeometric series [10] defined by

$${}_r\varphi_p\left(\left.\begin{matrix}a_1,a_2,...,a_r\\ b_1,b_2,...,b_p\end{matrix}\right|q;\,z\right)=\sum_{k=0}^{\infty}\frac{(a_1;q)_k\cdots(a_r;q)_k}{(b_1;q)_k\cdots(b_p;q)_k}\frac{z^k}{(q;q)_k}\left[(-1)^kq^{\frac{k(k-1)}{2}}\right]^{p-r+1},\tag{3.4}$$

where, as before, $(a;q)_k=\prod_{m=0}^{k-1}(1-aq^m)$.

Table 3.2.1: Comparison of the Nikiforov-Uvarov, the $q-$Askey and the $q-$classical polynomial Tableaus

$q-$classical family	$\Longleftrightarrow$	Nikiforov-Uvarov Tableau [23]	$\Longrightarrow$	$q-$Askey Tableau [13]
$\emptyset-$Jacobi/Jacobi	$\Longleftrightarrow$	Eq. (86) page 242	$\Longrightarrow$	The Big $q-$Jacobi $q-$Hahn
$\emptyset-$Jacobi/Laguerre	$\Longleftrightarrow$	N$^{\underline{o}}$6 [23, page 244]	$\Longrightarrow$	$q-$Meixner Quantum $q-$Kravchuk
$\emptyset-$Jacobi/Hermite	$\Longleftrightarrow$	N$^{\underline{o}}$12 [23, page 244]	$\Longrightarrow$	Al-Salam-Carlitz II Discrete $q^{-1}-$Hermite II
$\emptyset-$Laguerre/Jacobi	$\Longleftrightarrow$	N$^{\underline{o}}$1 [23, page 244]	$\Longrightarrow$	Big $q-$Laguerre Affine $q-$Kravchuk
$\emptyset-$Hermite/Jacobi	$\Longleftrightarrow$	N$^{\underline{o}}$2 [23, page 244]	$\Longrightarrow$	Al-Salam-Carlitz I Discrete $q-$Hermite
$0-$Bessel/Jacobi	$\Longleftrightarrow$	N$^{\underline{o}}$4 [23, page 244]	$\Longrightarrow$	Alternative $q-$Charlier
$0-$Bessel/Laguerre	$\Longleftrightarrow$	N$^{\underline{o}}$11 [23, page 244]	$\Longrightarrow$	Stieltjes-Wigert
$0-$Jacobi/Jacobi	$\Longleftrightarrow$	N$^{\underline{o}}$3 [23, page 244]	$\Longrightarrow$	The Little $q-$Jacobi $q-$Kravchuk
$0-$Jacobi/Laguerre	$\Longleftrightarrow$	N$^{\underline{o}}$10 [23, page 244]	$\Longrightarrow$	$q-$Laguerre $q-$Charlier
$0-$Jacobi/Bessel	$\Longleftrightarrow$	N$^{\underline{o}}$7 [23, page 244]	$\Longrightarrow$	new OP family
$0-$Laguerre/Jacobi	$\Longleftrightarrow$	N$^{\underline{o}}$5 [23, page 244]	$\Longrightarrow$	Little $q-$Laguerre (Wall)
$0-$Laguerre/Bessel	$\Longleftrightarrow$	N$^{\underline{o}}$9 [23, page 244]	$\Longrightarrow$	new OP family
—		N$^{\underline{o}}$8 [23, page 244]		—

3.3.1 The Rodrigues formula

Let us first obtain the "standard" Rodrigues formula.

Proposition 3.2 *Let* $\mathbf{u}$, $\mathbf{u} \in \mathbb{P}^*$ *be a* $q-$*classical quasi-definite functional,* $(P_n)_{n\geq 0} = \operatorname{mops} \mathbf{u}$, *and* ω *the* $q-$*weight function defined by the* $q-$*Pearson equation (2.1). Then,*

$$P_n = q^{-n} r_n \frac{\Theta^{\star n}(\mathrm{H}^{(n)}\omega)}{\omega}. \tag{3.5}$$

Proof: The proof of this proposition is straightforward. In fact, using the definition of the $\omega^{(k)}$ we obtain $\omega^{(k)} = \phi^{(k-1)}\omega^{(k-1)}$, thus, using the equation (2.4) we have, for all $n \geq 1$,

$$\begin{aligned}\Theta^{\star n}(\mathrm{H}^{(n)}\omega) = \;& \Theta^{\star n}(\omega^{(n)} Q_0^{(n)}) = \frac{1}{[1]}\Theta^{\star n-1}[\Theta^{\star}(\phi^{(n-1)}\omega^{(n-1)}\Theta Q_1^{(n-1)}] \overset{(2.4)}{=} \\ = \;& \frac{q\widehat{\lambda}_1^{(n-1)}}{[1]}\Theta^{\star n-1}[\omega^{(n-1)}Q_1^{(n-1)}] = \cdots = q^n \frac{\widehat{\lambda}_1^{(n-1)}\cdots\widehat{\lambda}_n}{[1]\cdots[n]}\omega P_n.\end{aligned}$$

Finally, using the explicit expression for the coefficient r_n (1.12) the result follows. □

The Rodrigues formula is very useful for finding the explicit expression of the polynomials P_n. In fact, using the formula

$$\Theta^{\star n} f(x) = \frac{q^{\binom{n}{2}+n}}{(1-q)^n x^n} \sum_{k=0}^{n} (-1)^k q^{\frac{k(k+1)}{2}-nk} \begin{bmatrix} n \\ k \end{bmatrix}_q f(q^{k-n}x), \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}},$$

where $\binom{n}{2} = \frac{n(n-1)}{2}$, one easily obtains

$$P_n = \frac{q^{\binom{n}{2}} r_n}{(1-q)^n x^n} \sum_{k=0}^{n} (-1)^k q^{\frac{k(k+1)}{2}-nk} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{\mathrm{H}^{(n)}\phi(xq^{k-n})\omega(q^{n-k}x)}{\omega(x)},$$

or, equivalently,

$$P_n = \frac{r_n(-1)^n}{(1-q)^n x^n} \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{\mathrm{H}^{(n)}\phi(xq^{-k})\omega(q^{-k}x)}{\omega(x)}.$$

Now, taking into account the $q-$Pearson equation (2.6)

$$\frac{\mathrm{H}\omega}{\omega} = \frac{\phi}{q\mathrm{H}\phi^\star} \qquad \Longleftrightarrow \qquad \frac{\mathrm{H}^{-1}\omega}{\omega} = \frac{q\phi^\star}{\mathrm{H}^{-1}\phi},$$

we obtain the following explicit expression for the $q-$classical polynomials in terms of the polynomials ϕ and $\phi^\star$:

$$P_n = \frac{r_n(-1)^n}{(1-q)^n x^n} \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}+k} \begin{bmatrix} n \\ k \end{bmatrix}_q \prod_{i=0}^{k-1} \phi^\star(xq^{-i}) \prod_{i=0}^{n-k-1} \phi(xq^i). \tag{3.6}$$

This formula is equivalent to the one obtained in [5, Eq. (4.14)], [23, Eq. (33)] and [2, Eq. (2.24)] for the $q-$polynomials in the non-uniform lattice $x(s) = c_1 q^s$.

3.3.2 The hypergeometric representation

• We start with the $\emptyset-$Jacobi/Jacobi family, i.e., the case when $\phi = \widehat{a}(x-a_1)(x-a_2)$ and $\phi^\star = \widehat{a}^\star(x-a_1^\star)(x-a_2^\star)$ $\widehat{a}^\star a_1 a_2 \widehat{a}^\star a_1^\star a_2^\star \neq 0$. The other cases can be obtained in a similar way. Then, substituting in the above expression we find that the $q-$classical polynomials becomes

$$P_n = \frac{r_n(\widehat{a}a_1a_2)^n (x/a_1;q)_n (x/a_2;q)_n}{(1-q)^n x^n} {}_3\varphi_2 \left(\begin{matrix} q^{-n}, a_1^\star x^{-1}, a_2^\star x^{-1} \\ q^{1-n} a_1 x^{-1}, q^{1-n} a_2 x^{-1} \end{matrix} \middle| q;\ \frac{\widehat{a}^\star}{\widehat{a}} q^{-n+3} \right).$$

From the last formula it is not easy to see that P_n are polynomials on x of degree exactly equal n, thus, we will apply to the above equation the transformations (3.2.5) and (3.2.3) given in [10, page 61]. Notice that we can apply the transformation formula (3.2.5) [10, page 61] because the polynomials ϕ and $q\phi^\star$ have the same independent term, and then the condition $\widehat{a}a_1a_2 = q\widehat{a}^\star a_1^\star a_2^\star$ is fulfilled. So, the hypergeometric representation of the monic $q-$classical $\emptyset-$Jacobi/Jacobi polynomials is

$$P_n(x) = \frac{a_2^n (a_1^\star/a_2;q)_n (a_2^\star/a_2;q)_n}{(a_1^\star a_2^\star a_1^{-1} a_2^{-1} q^{n-1};q)_n} {}_3\varphi_2 \left(\begin{matrix} q^{-n}, a_1^\star a_2^\star a_1^{-1} a_2^{-1} q^{n-1}, x/a_2 \\ a_1^\star/a_2, a_2^\star/a_2 \end{matrix} \middle| q;\ q \right). \tag{3.7}$$

Notice that, since ϕ and $\phi^\star$ are invariant with respect to the change $a_1 \Longleftrightarrow a_2$ and $a_1^\star \Longleftrightarrow a_2^\star$, then we can obtain an equivalent hypergeometric representation

$$P_n(x) = \frac{a_2^n (a_1^\star/a_1;q)_n (a_2^\star/a_1;q)_n}{(a_1^\star a_2^\star a_1^{-1} a_2^{-1} q^{n-1};q)_n} {}_3\varphi_2 \left(\begin{matrix} q^{-n}, a_1^\star a_2^\star a_1^{-1} a_2^{-1} q^{n-1}, x/a_1 \\ a_1^\star/a_1, a_2^\star/a_1 \end{matrix} \middle| q;\ q \right). \tag{3.8}$$

Notice also that from any of the above two formulas follwos that P_n is a polynomial of degree exactly equal n. Before start with the detailed study of each case let us write another equivalent form for the $\emptyset-$Jacobi/Jacobi polynomials which can be obtained applying the transformation (III.12) from [10, page 241-242] to (3.7):

$$P_n(x) = \frac{q^{\binom{n}{2}}(-a_2^\star)^n(a_1^\star/a_2;q)_n(a_1^\star/a_1;q)_n}{(a_1^\star a_2^\star a_1^{-1}a_2^{-1}q^{n-1};q)_n}\,{}_3\varphi_2\left(\left.\begin{matrix} q^{-n}, a_1^\star a_2^\star a_1^{-1}a_2^{-1}q^{n-1}, a_1^\star/x \\ a_1^\star/a_2, a_1^\star/a_1\end{matrix}\right| q;\, qx/a_2^\star\right). \quad (3.9)$$

If we now choose $\phi = aq(x-1)(bx-c)$ and $\phi^\star = q^{-2}(x-aq)(x-cq)$, then Theorem 2.1 and Eq. (3.7) gives, for the weight function and the polynomials, respectively

$$\omega(x) = \frac{(x/a, x/c;q)_\infty}{(bx/c, x;q)_\infty}, \qquad p_n(x;a,b,c;q) = \frac{(aq;q)_n(cq;q)_n}{(abq^{n+1};q)_n}\,{}_3\varphi_2\left(\left.\begin{matrix} q^{-n}, abq^{n+1}, x \\ aq, cq\end{matrix}\right| q;q\right),$$

i.e., the Big $q-$Jacobi polynomials. If we now choose $c = q^{-N-1}$ they becomes the $q-$Hahn polynomials $Q_n(x;a,b,N|q)$ (usually they are written as polynomials in $x = q^{-s}$, see [13, 18]). Obviously, if we use instead of formula (3.7) the formulas (3.8) and (3.9) we obtain other representations for the Big $q-$Jacobi polynomials.

For the other 11 cases we can do the same, substitute the polynomials ϕ and $\phi^\star$ in (3.6) and make the corresponding calculations, but here we will show how, from the $q-$classical $\emptyset-$Jacobi/Jacobi polynomials, can be derived all other cases by taking the appropiate limits. A similar study have been done in [23]. Here we will complete it. We will give the details only in some special "difficult" cases or when the clarity and the accuracy are required.

• We continue with the $q-$classical $\emptyset-$Jacobi/Laguerre polynomials. To obtain them we take the limit $a_2^\star \to \infty$. Then, $\phi = \widehat{a}(x-a_1)(x-a_2)$ and

$$q\phi^\star = q\widehat{a}^\star(x-a_1^\star)(x-a_2^\star) = q\widehat{a}^\star a_2^\star(x-a_1^\star)(x/a_2^\star - 1) = \frac{\widehat{a}a_1a_2}{a_1^\star}(x-a_1^\star)(x/a_2^\star-1) \to -\frac{\widehat{a}a_1a_2}{a_1^\star}(x-a_1^\star),$$

where the relation $\widehat{a}a_1a_2 = q\widehat{a}^\star a_1^\star a_2^\star$ has been used. In this case and since

$$\lim_{a_2^\star\to\infty}\frac{(a_1^\star a_2^\star a_1^{-1}a_2^{-1}q^{n-1};q)_k}{(a_2^\star/a_2;q)_k} = q^{(n-1)k}\left(\frac{a_1^\star}{a_1}\right)^k,$$

Eq. (3.7) becomes

$$P_n(x) = \left(\frac{a_1a_2}{a_1^\star}\right)^n(a_1^\star/a_2;q)_n q^{-n(n-1)}\,{}_2\varphi_1\left(\left.\begin{matrix} q^{-n}, x/a_2 \\ a_1^\star/a_2\end{matrix}\right| q;\, q^n a_1^\star/a_1\right). \quad (3.10)$$

If we choose now $\phi = (x-1)(x+bc)$ and $\phi^\star = q^{-2}c(x-bq)$, then we obtain the $q-$Meixner polynomials

$$M_n(x;b,c;q) = (-c)^n(bq;q)_n q^{-n^2}\,{}_2\varphi_1\left(\left.\begin{matrix} q^{-n}, x \\ bq\end{matrix}\right| q;-\frac{q^{n+1}}{c}\right).$$

In this case $\omega(x) = \frac{(x/b;q)_\infty}{(-x/bc,x;q)_\infty}$. Putting in the above formulas $b = q^{-N-1}$ and $c = -p^{-1}$ we arrive to the Quantum $q-$Kravchuk polynomials $K_n^{qtm}(x;p,N;q)$.

• The next family is the $q-$classical $\emptyset-$Jacobi/Hermite one. In this case we take the limit $a_1^\star, a_2^\star \to \infty$. Then, $\phi = \widehat{a}(x-a_1)(x-a_2)$ and $q\phi^\star = \widehat{a}^\star a_1a_2$, thus (3.7) becomes

$$P_n(x) = (-a_2)^{-n}q^{\binom{n}{2}}\,{}_2\varphi_0\left(\left.\begin{matrix} q^{-n}, x/a_2 \\ -\end{matrix}\right| q;\, q^n a_2/a_1\right). \quad (3.11)$$

Choosing $\phi=(x-a)(x-1)$ and $q\phi^\star=a$ we obtain the Al-Salam & Carlitz II polynomials

$$V_n^{(a)}(x;q)=(-a)^n q^{-\binom{n}{2}}\,{}_2\varphi_0\left(\begin{matrix}q^{-n},x\\0\end{matrix}\,\middle|\,q;\frac{q^n}{a}\right),$$

If now $\phi=(x-i)(x+i)$ and $q\phi^\star=1$, we arrive to the Discrete $q-$Hermite polynomials II $\widetilde{h}_n(x;q)$

$$\widetilde{h}_n(x;q)=i^{-n}\,{}_2\varphi_0\left(\begin{matrix}q^{-n},ix\\-\end{matrix}\,\middle|\,q;-q^{-n}\right)=x^n\,{}_2\varphi_1\left(\begin{matrix}q^{-n},q^{-n+1}\\0\end{matrix}\,\middle|\,q^2;-\frac{q^2}{x^2}\right),$$

and for the weight function we have $\omega(x)=(ix,-ix;q)_\infty^{-1}=(-x^2;q^2)_\infty=\left(\prod_{k=0}^{\infty}(1+x^2q^{2k})\right)^{-1}$.

• The $q-$classical $\emptyset-$Laguerre/Jacobi polynomials. In this case $a_2\to\infty$. Then, $\phi=-q\widehat{a}^\star a_1^\star a_2^\star a_1^{-1}(x-a_1)$ and $\phi^\star=\widehat{a}^\star(x-a_1^\star)(x-a_2^\star)$, thus Eq. (3.9) gives

$$\begin{aligned}P_n(x)=&\ (-a_2^\star)^n q^{\binom{n}{2}}(a_1^\star/a_1;q)_n\,{}_2\varphi_1\left(\begin{matrix}q^{-n},a_1^\star/x\\a_1^\star/a_1\end{matrix}\,\middle|\,q;\,qx/a_2^\star\right)\\ =&\ a_1^n(a_1^\star/a_1;q)_n(a_2^\star/a_1;q)_n\,{}_3\varphi_2\left(\begin{matrix}q^{-n},x/a_1,0\\a_1^\star/a_1,a_2^\star/a_1\end{matrix}\,\middle|\,q;\,q\right).\end{aligned}\tag{3.12}$$

The last equality follows from the Jackson transformation formula (see [10, Eq. (III.5), page 241]), or, directly, taking the limit in formula (3.8). If we now choose $\phi=-acq(x-1)$ and $\phi^\star=q^{-2}(x-aq)(x-cq)$, we obtain the Big $q-$Laguerre polynomials

$$\begin{aligned}p_n(x;a,c;q)=&\ (aq;q)_n(cq;q)_n\,{}_3\varphi_2\left(\begin{matrix}q^{-n},0,x\\aq,cq\end{matrix}\,\middle|\,q;q\right)\\ =&\ (aq;q)_n(-cq)^n q^{\binom{n}{2}}\,{}_2\varphi_1\left(\begin{matrix}q^{-n},aqx^{-1}\\aq\end{matrix}\,\middle|\,q;\frac{x}{c}\right).\end{aligned}$$

Notice that they are nothing else that the Big $q-$Jacobi when $b=0$. Here $\omega(x)=\frac{(x/a,x/c;q)_\infty}{(x;q)_\infty}$. To this class also belong the $K_n^{aff}(x;p,N;q)$. In fact they are Big $q-$Laguerre polynomials with parameters $a=q^{-N-1}$ and $c=p$.

• The $q-$classical $\emptyset-$Hermite/Jacobi polynomials. In this case $a_1,a_2\to\infty$, thus $\phi=q\widehat{a}a_1^\star a_2^\star$ and $\phi^\star=\widehat{a}^\star(x-a_1^\star)(x-a_2^\star)$. Then, from Eq. (3.9) one easily find

$$P_n(x)=q^{\binom{n}{2}}(-a_2^\star)^n\,{}_2\varphi_1\left(\begin{matrix}q^{-n},a_1^\star/x\\0\end{matrix}\,\middle|\,q;\,qx/a_2^\star\right).\tag{3.13}$$

Now choosing $\phi=a$ and $q\phi^\star=(x-1)(x-a)$, (3.13) leads to the Al-Salam & Carlitz I polynomials

$$U_n^{(a)}(x;q)=(-a)^n q^{\binom{n}{2}}\,{}_2\varphi_1\left(\begin{matrix}q^{-n},x^{-1}\\0\end{matrix}\,\middle|\,q;\frac{x\,q}{a}\right).$$

In this case the $q-$weight function takes the form $\omega(x)=(qx/a,qx;q)_\infty$. If we put $a=-1$, the the Al-Salam & Carlitz I polynomials becomes the discrete $q-$Hermite polynomials I $h_n(x;q)$.

For the $0-$families the situation is more complicate and a new parameter δ should be included.

• To obtain the 0−Bessel/Jacobi polynomials we will take the limit $a_1, a_2, a_2^\star \to 0$. Thus, $\phi = \widehat{a}x^2$ and $\phi^\star = \widehat{a}^\star(x - a_1^\star)x$, but now we have a problem taking the limit in the expression $(a_1^\star a_2^\star a_1^{-1} a_2^{-1} q^{n-1}; q)_k$, so we will obliged the parameters $a_1, a_2, a_2^\star$ tend to zero such that $a_2^\star a_1^{-1} a_2^{-1} = q^\delta$, with δ a fixed constant such that $q^\delta = \widehat{a}/(q\widehat{a}^\star a_1^\star)$. Then, taking the limit in Eq. (3.7) we obtain

$$P_n(x) = \frac{q^{\binom{n}{2}}(-a_1^\star)^n}{(q^{n+\delta-1};q)_n} {}_2\varphi_1\left(\begin{matrix} q^{-n}, q^{n+\delta-1} \\ 0 \end{matrix} \middle| q;\, qx/a_1^\star \right), \qquad q^\delta = \frac{\widehat{a}}{q\widehat{a}^\star a_1^\star}. \tag{3.14}$$

To this class belongs the Alternative $q-$Charlier polynomials $K_n(x; a, q)$. In fact, putting $\phi = ax^2$ and $\phi^\star = q^{-2}x(1-x)$, thus $q^\delta = -aq$ and then

$$K_n(x; a; q) = \frac{(-1)^n q^{\binom{n}{2}}}{(-aq^n;q)_n} \, {}_2\varphi_1\left(\begin{matrix} q^{-n}, -aq^n \\ 0 \end{matrix} \middle| q; qx \right).$$

For them we have $\omega(x) = |x|^\alpha (x^{-1};q)_\infty^{-1}$, where $q^\alpha = -a/q$.

• For the 0−Bessel/Laguerre polynomials we have the limit $a_1, a_2, a_1^\star \to 0$ and $a_2^\star \to \infty$. Thus, $\phi = \widehat{a}x^2$ and $\phi^\star = \widehat{a}^\star(x - a_1^\star)(x - a_2^\star) = \widehat{a}^\star a_2^\star(x/a_2^\star - 1)(x - a_1^\star) = \widehat{a}a_1 a_2 a_1^{\star -1} q^{-1}(x/a_2^\star - 1)(x - a_1^\star)$. If we now take the limit in such a way that $a_1^\star a_1 a_2 = -q^\delta$ we arrive to the function $\phi^\star = \widehat{a}q^{-\delta-1}x$. In this case Eq. (3.9) immediately gives

$$P_n(x) = q^{-n(n+\delta-1)}(-1)^n {}_1\varphi_1\left(\begin{matrix} q^{-n} \\ 0 \end{matrix} \middle| q;\, -q^{n+\delta}x \right), \qquad q^\delta = -\frac{\widehat{a}}{\bar{a}^\star q}. \tag{3.15}$$

Now, setting $\phi = x^2$ and $\phi^\star = q^{-2}x$, we have $q^\delta = -q$ and we obtain the Stieltjes-Wigert polynomials

$$S_n(x; q) = (-1)^n q^{-n^2} {}_1\varphi_1\left(\begin{matrix} q^{-n} \\ 0 \end{matrix} \middle| q; -xq^{n+1} \right).$$

Here $\omega(x) = \sqrt{x^{\log_q x - 1}}$.

• The 0−Jacobi/Jacobi polynomials. In this case the limit is $a_2, a_2^\star \to 0$ providing that $a_2^\star/a_2 = q^\delta$, then $\phi = \widehat{a}x(x - a_1)$, $\phi^\star = \widehat{a}^\star x(x - a_1^\star)$ and (3.7) gives

$$P_n(x) = \frac{q^{\binom{n}{2}}(-a_1^\star)^n (q^\delta;q)_n}{(a_1^\star/a_1 q^{\delta+n-1};q)_n} {}_2\varphi_1\left(\begin{matrix} q^{-n}, a_1^\star/a_1 q^{n+\delta-1} \\ q^\delta \end{matrix} \middle| q;\, qx/a_1^\star \right), \qquad q^\delta = \frac{\widehat{a}a_1}{q\widehat{a}^\star a_1^\star}. \tag{3.16}$$

Putting $\phi = ax(bqx - 1)$ and $\phi^\star = q^{-2}x(x-1)$, $q^\delta = aq$, thus

$$p_n(x; a, b|q) = \frac{(-1)^n q^{\binom{n}{2}}(aq;q)_n}{(abq^{n+1};q)_n} {}_2\varphi_1\left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix} \middle| q; qx \right),$$

which are nothing else that the Little $q-$Jacobi polynomials. If now we take $\phi = px(1-x)$, $\phi^\star = q^{-2}x(x - q^{-N})$ we arrive to the following expression

$$K_n(x; p, N; q) = \frac{(-1)^n q^{-nN+\binom{n}{2}}(-pq^{N+1};q)_n}{(-pq^n;q)_n} {}_2\varphi_1\left(\begin{matrix} q^{-n}, -pq^n \\ -pq^{N+1} \end{matrix} \middle| q; xq^{N+1} \right),$$

thats constitutes an alternative definition for the $q-$Kravchuk polynomials which is equivalent to the "more" standard one just using the transformation formula (III.7) from [10, page 241]

$$K_n(x; p, N; q) = \frac{(q^{-N};q)_n}{(-pq^n;q)_n} {}_3\varphi_2\left(\begin{matrix} q^{-n}, x, -pq^n \\ q^{-N}, 0 \end{matrix} \middle| q; q \right).$$

Finally, we have $\omega(x)=|x|^{\alpha}\frac{(qx;q)_\infty}{(qbx;q)_\infty}$, $q^{\alpha}=a$ and $\omega(x)=|x|^{\alpha}\frac{(q^{N+1}x;q)_\infty}{(x;q)_\infty}$, $q^{\alpha}=pq^{N}$ for the weight functions of the Little $q-$Jacobi and $q-$Kravchuk polynomials, respectively.

• The 0−Jacobi/Laguerre polynomials. In this case we take the limit is $a_2,a_2^\star\to 0$ and $a_1^\star\to\infty$ in such a way that $a_2^\star/a_2=-q^{\delta}$, so $\phi=\widehat{a}x(x-a_1)$, $\phi^\star=\widehat{a}a_1q^{-\delta-1}x=\bar{a}^\star x$, and then

$$P_n(x)=(-a_1)^nq^{-n(n+\delta-1)}{}_2\varphi_1\left(\begin{matrix}q^{-n},x/a_1\\0\end{matrix}\,\middle|\,q;\,-q^{n+\delta}\right),\qquad q^{\delta}=\frac{\widehat{a}a_1}{q\bar{a}^\star}. \tag{3.17}$$

Putting $\phi=ax(x+1)$ and $\phi^\star=q^{-2}x$, then $q^{\delta}=-aq$, and we obtain the $q-$Laguerre polynomials $L_n^{\alpha}(x;q)\equiv L_n(x;a;q)$

$$L_n(x;a;q)=(-1)^nq^{-n^2}a^{-n}\ {}_2\varphi_1\left(\begin{matrix}q^{-n},-x\\0\end{matrix}\,\middle|\,q;aq^{n+1}\right),\qquad \omega(x)=\frac{|x|^{\alpha}}{(-x;q)_\infty},\quad q^{\alpha}=-a.$$

If we now choose $\phi=x(x-1)$ and $\phi^\star=q^{-2}ax$, we obtain $q^{\delta}=q/a$ and then we arrive to the $q-$Charlier polynomials

$$C_n(x;a;q)=(-1)^nq^{-n^2}a^{n}\ {}_2\varphi_1\left(\begin{matrix}q^{-n},x\\0\end{matrix}\,\middle|\,q;-\frac{q^{n+1}}{a}\right),\qquad \omega(x)=\frac{|x|^{\alpha}}{(x;q)_\infty},\quad q^{\alpha}=a^{-1}.$$

• The 0−Jacobi/Bessel polynomials. Here we take the limit is $a_2,a_1^\star,a_2^\star\to 0$ in such a way that $a_1^\star a_2^\star/a_2=q^{\delta}$, so $\phi=\widehat{a}x(x-a_1)$, $\phi^\star=\widehat{a}^\star x^2=\widehat{a}a_1q^{-\delta-1}x^2$, and then (3.7) gives

$$P_n(x)=q^{n(n+\delta-1)}(q^{n+\delta-1}/a_1;q)_n^{-1}{}_2\varphi_0\left(\begin{matrix}q^{-n},q^{n+\delta-1}/a_1\\—\end{matrix}\,\middle|\,q;\,xq^{1-\delta}\right),\qquad q^{\delta}=\frac{\widehat{a}a_1}{q\widehat{a}^\star}. \tag{3.18}$$

This family does not appear in the $q-$Askey Scheme unless they are not a trivial limit of a more general $q-$family. We will take the following parameterization $\phi=ax(x-b)$ and $\phi^\star=q^{-2}x^2$. Then, $q^{\delta}=abq$ and we obtain that this 0−Jacobi/Bessel polynomials, denoted by $j_n(x;a,b)$

$$j_n(x;a,b)=(ab)^nq^{n^2}(aq^n;q)_n^{-1}{}_2\varphi_0\left(\begin{matrix}q^{-n},aq^n\\—\end{matrix}\,\middle|\,q;\,x/(ab)\right),\quad \omega(x)=|x|^{\alpha}(bq/x;q)_\infty,\quad q^{\alpha}=a^{-1}q^{-5}.$$

They main data are shown in Table 3.3.2.

• The 0−Laguerre/Jacobi polynomials. In this case $a_2,a_2^\star\to 0$, $a_1\to\infty$, $q^{\delta}=-a_2^\star/a_2$, then $\phi=\bar{a}x=\widehat{a}^\star a_1^\star q^{\delta+1}x$, $\phi^\star=\widehat{a}^\star x(x-a_1^\star)$, and

$$P_n(x)=(-a_1^\star)^nq^{\binom{n}{2}}(-q^{\delta};q)_n{}_2\varphi_1\left(\begin{matrix}q^{-n},0\\-q^{\delta}\end{matrix}\,\middle|\,q;\,xq/a_1^\star\right),\qquad q^{\delta}=\frac{\bar{a}}{\widehat{a}^\star a_1^\star q}. \tag{3.19}$$

Putting $\phi=-ax$ and $\phi^\star=q^{-2}x(x-1)$, thus $q^{\delta}=-aq$, and we obtain the Little $q-$Laguerre or Wall polynomials

$$p_n(x;a|q)=(-1)^nq^{\binom{n}{2}}(aq;q)_n{}_2\varphi_1\left(\begin{matrix}q^{-n},0\\aq\end{matrix}\,\middle|\,q;qx\right),\qquad \omega(x)=|x|^{\alpha}(qx;q)_\infty,\quad q^{\alpha}=-a.$$

• Finally, the 0−Laguerre/Bessel family follows from Eq. (3.8) taking the limit $a_1,a_1^\star,a_2^\star\to 0$ and $a_2\to\infty$ providing that $a_1^\star a_2^\star/a_1=-q^{\delta}$, thus $\phi=\bar{a}x=\widehat{a}^\star q^{\delta+1}x$, $\phi^\star=\widehat{a}^\star x^2$ and

$$P_n(x)=(-1)^nq^{n(n+\delta-1)}{}_2\varphi_0\left(\begin{matrix}q^{-n},0\\—\end{matrix}\,\middle|\,q;\,-xq^{1-\delta}\right),\qquad q^{\delta}=\frac{\bar{a}}{q\widehat{a}^\star}. \tag{3.20}$$

As the case of 0−Jacobi/Bessel, this case leads to a new family which is not in the $q-$Askey Tableau. In this case we will adopt the parameterization $\phi = \bar{a}x = \widehat{a}^{\star}ax$, $\phi^{\star} = q^{-2}x^2$, $q^{\delta} = aq$, thus

$$P_n(x) \equiv l_n(x;a) = (-a)^n q^{n^2}\, {}_2\varphi_0\left(\begin{matrix} q^{-n}, 0 \\ — \end{matrix} \middle| q; -x/a \right), \qquad \omega(x) = |x|^{\alpha}\sqrt{x^{\log_q x^{-1}+1}}, \quad q^{\alpha} = a/q.$$

Remark 3.2 *Notice that in some examples the $q-$weight functions looks very different from the ones given in [13]. Sometimes the reason is the indeterminateness of the associated moment problem (e.g. the Stieltjes-Wieger polynomials of the $q-$Laguerre polynomials. Also, because sometimes instead the $q-$integrals, discrete sums are used (see e.g. the example of the Little $q-$Jacobi polynomials in [13]).*

Table 3.3.2: The $q-$classical polynomials $j_n(x;a,b)$ and $l_n(x;a)$

P_n	$j_n(x;a,b)$	$l_n(x;a)$
ϕ	$ax(x-b)$	ax
$\phi^{\star}$	$q^{-2}x^2$	$q^{-2}x^2$
ψ	$\frac{abq+(1-aq)x}{q(1-q)}$	$\frac{aq-x}{(q-1)q}$
$\widehat{\lambda}_n$	$-\frac{q^{-n}[n](a+q^n)}{1-q}$	$\frac{q^{-n}[n]}{1-q}$
r_n	$\frac{q^{\binom{n}{2}+n}(1-q)^n}{(aq^n;q)_n}$	$q^{\binom{n}{2}+n}(1-q)^n$
d_n	$\frac{abq^n\left(1-q^n+aq^{2n}-q^{n+1}\right)}{(1-aq^{2n-1})(1-aq^{2n+1})}$	$aq^n\left(q^n+q^{n+1}-1\right)$
g_n	$-\frac{a^2b^2q^{3n-1}(1-q^n)\left(1-aq^{n-1}\right)}{(1-aq^{2n-1})^2(1-aq^{2n})(1-aq^{2n-2})}$	$a^2q^{3n-1}\left(q^n-1\right)$
a_n	$a[n]$	0
b_n	$-\frac{ab[n](1-aq^n)\left(1+aq^{2n}\right)}{(1-aq^{2n-1})(1-aq^{2n+1})}$	$a[n]$
c_n	$\frac{a^2b^2q^{2n-1}[n](1-aq^n)\left(1-aq^{n-1}\right)}{(1-aq^{2n-1})^2(1-aq^{2n})(1-aq^{2n-2})}$	$a^2q^{2n-1}[n]$
e_n	$\frac{abq^n(1-q^n)\left(1+aq^{2n}\right)}{(1-aq^{2n-1})(1-aq^{1+2n})}$	$aq^n(q^n-1)$
h_n	$\frac{a^3b^2q^{4n-2}(1-q^n)\left(1-q^{n-1}\right)}{(1-aq^{2n-1})^2(1-aq^{2n})(1-aq^{2n-2})}$	0
d'_n	$\frac{abq^{n+1}\left(1-q^n-q^{n+1}+aq^{2n+2}\right)}{(1-aq^{2n+1})(1-aq^{2n+3})}$	$aq^{n+1}\left(q^n+q^{n+1}-1\right)$
g'_n	$-\frac{a^2b^2q^{3n+1}(1-q^n)\left(1-aq^{n+1}\right)}{(1-aq^{2n})(1-aq^{2n+1})^2(1-aq^{2n+2})}$	$a^2q^{3n+1}\left(q^n-1\right)$

Notice that for all $0 < q < 1$, the polynomials $l_n(x;a)$ never constitutes a positive definite family since $g_n < 0$ (see the Favard theorem (1.1)). The case if the $j_n(x;a,b)$ polynomials is more complicated. Nevertheless, choosing $a = q^{-N}$ it is easy to show that $j_n(x;a,b)$ constitute a finite family (similar to the $q-$Hahn polynomials) which is positive definite since $g_n > 0$ for all $n = 0, 1, \ldots, [N/2]$. The detailed study of the positive definite cases in dependence of the roots of

ϕ and $\phi^\star$ will be considered in a forthcoming paper.

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