CAT(k)-spaces, weak convergence and fixed points

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Abstract

In this paper we show that some of the recent results on fixed point for CAT(0) spaces still hold true for CAT(1) spaces, and so for any CAT(k) space, under natural boundedness conditions. We also introduce a new notion of convergence in geodesic spaces which is related to the Δ-convergence and applied to study some aspects on the geometry of CAT(0) spaces. At this point, two recently posed questions in [12] (W.A. Kirk and B. Panyanak, *A concept of convergence in geodesic spaces,* Nonlinear Anal. **68 (12)** (2008), 3689-3696) are answered in the negative. The work finishes with the study of the Lifšic characteristic and property (P) of Lim-Xu to derive fixed point results for uniformly lipschitzian mappings in CAT(k) spaces. A conjecture raised in [4] (S. Dhompongsa, W.A. Kirk and B. Sims, *Fixed points of uniformly lipschitzian mappings,* Nonlinear Anal., **65** (2006), 762–772) on the Lifšic characteristic function of CAT(k) spaces is solved in the positive.

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1 Introduction

Metric spaces of bounded curvature, and in particular CAT(k) spaces, can be understood as a generalization of Riemannian manifolds with bounded sectional curvature. In fact, it is very well-known that any complete simply connected Riemannian manifold with nonpositive sectional curvature is a CAT(0) space. The geometric idea behind CAT(k) spaces, as it is possible to appreciate in Section 2, is that geodesic triangles are somehow *thin* or, at least, not too fat. The term CAT(k) was introduced by M. Gromov to denote a distinguished class of geodesic metric spaces with curvature bounded above by $k \in \mathbb{R}$. In recent years, CAT(k) spaces have attracted the attention of many authors as they have played a very important role in different aspects of geometry. A very thorough discussion on these spaces and the role they play in geometry can be found in the book by M.R. Bridson and A. Haefliger [1] (see also [2, 8]).

As it was noted by W.A. Kirk in his fundamental works [10, 11], the geometry of CAT(k) spaces is rich enough to develop a very consistent theory on fixed point under metric conditions. These works were followed by a series of new works by different authors (see for instance [3, 4, 12, 14, 19])

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mainly focusing on CAT(0) spaces and $\mathbb{R}$-trees (see Section 2 for definitions) due to the particularly rich geometry of both classes of spaces. It was also noted in [11] that any CAT(k) space is uniformly convex in a certain sense but it turns out that CAT(0) spaces enjoy some other well-known and strong geometrical properties, such as an Euclidean-like law of cosines, the CN-inequality or the properties of the metric projection onto closed convex subsets (see [1] for details) which are helpful when dealing with their geometry. Also, since any CAT(k) space is a CAT(k') space for $k' > k$, all results for CAT(0) spaces immediately apply to any CAT(k) with $k \leq 0$. In this work, among other questions, we take up the question of finding out what can be said for CAT(k) spaces with $k > 0$ regarding the existence of fixed points under metric conditions on the considered mappings. Since any result on general CAT(1) spaces can be extended to any CAT(k) space with $k > 0$ without major changes we will mainly focus on CAT(1) spaces. We will start working from the uniform convexity of CAT(1) spaces to show how, in addition to the boundedness of the curvature, all the above-named properties of CAT(0) spaces such as the CN-inequality are, in some way, not required.

This work is organized as follows. In Section 2 we introduce some preliminary definitions and results regarding some basic questions about metric fixed point theory and spaces of bounded curvature. In Section 3 we recall some basic facts about the geometry of the spaces of bounded curvature of special relevance in metric fixed point theory such as those related to the uniform convexity or the normal structure in the sense of Brodskii and Milman. In Section 4 we prove that CAT(1) spaces enjoy the Kadec-Klee property by means of the Δ-convergence in a similar way as it has been recently shown for CAT(0) spaces in [12]. In this section we also show a fixed point result for convex type mappings in CAT(1) spaces. In Section 5 we take up some of the questions posed in [12] regarding the geometry of CAT(0) spaces, in particular we answer in the negative two of those questions and improve one result about the Δ-convergence of a sequence of interior points of geodesic segments when the sequences of the endpoints of such segments Δ-converge to the same point. In order to prove these results we need to introduce a new notion of convergence in geodesic spaces which is inspired by one of the two given by E.N. Sosov in [20] and which we relate to the notion of Δ-convergence. In Section 6, our last section, we follow the work [4] on the study of the Lifšic characteristic and the property (P) of Lim-Xu in CAT(0) spaces for CAT(k) spaces with $k \geq 0$. In particular we estimate the Lifšic characteristic for any CAT(k) space, answering in the positive a conjecture raised in [4], and show that CAT(1) spaces also enjoy property (P). Consequences on the existence of fixed points for uniformly lipschitzian mappings are also deduced, sharpening some of the results from [4].

2 Preliminaries

Let (X, d) be a metric space, then, for $D, E \subseteq X$ nonempty, set

$$\begin{aligned} r_x(D) &= \sup\{d(x,y) : y \in D\}, \quad x \in X; \\ \mathrm{rad}_E(D) &= \inf\{r_x(D) : x \in E\}; \\ \mathrm{diam}(D) &= \sup\{d(x,y) : x, y \in D\}; \\ \mathrm{cov}(D) &= \cap\, \{B : \ B \text{ is a closed ball and } D \subset B\}. \end{aligned}$$

The number $\mathrm{rad}_E(D)$ stands for the *Chebyshev radius* of D in E (if $E = X$ then we will rather write $\mathrm{rad}(D)$) and $\mathrm{cov}(D)$ the *admissible hull* of D (in X).

A subset A of X is said to be admissible if $\mathrm{cov}(A) = A$. The number

$$\tilde{N}(X) = \sup\Big\{\frac{\mathrm{rad}_A(A)}{\mathrm{diam}(A)}\Big\}$$

where the supremum is taken over all nonempty bounded admissible subsets A of X for which $\mathrm{diam}(A) > 0$ is called the *normal structure coefficient* of X. If $\tilde{N}(X) \leq c$ for some constant $c < 1$, then X is said to have *uniform normal structure* in the sense of Brodskii and Milman.

A mapping $T : X \to X$ is said to be *nonexpansive* if $d(Tx, Ty) \leq d(x, y)$ for any $x, y \in X$. The following theorem is known as the Kirk's Fixed Point Theorem for metric spaces (see [9, pg. 103] for more details on this theorem or [6] for a thorough exposition on metric fixed point theory).

Theorem 2.1 *Let X be a nonempty complete bounded metric space with uniform normal structure, then every nonexpansive mapping $T : X \to X$ has a fixed point, i.e., there is $x \in X$ such that $Tx = x$.*

A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map $c : [0, l] \subseteq \mathbb{R} \to X$ such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a *geodesic (or metric) segment* joining x and y. When it is unique this geodesic is denoted $[x, y]$. The space (X, d) is said to be a *geodesic space* (*D-geodesic space*) if every two points of X (every two points of distance smaller than D) are joined by a geodesic, and X is said to be *uniquely geodesic* (*D-uniquely geodesic*) if there is exactly one geodesic joining x and y for each $x, y \in X$ (for $x, y \in X$ such that $d(x, y) < D$). Let $Y \subset X$, we denote by $G_1(Y)$ the union of all geodesic segments in X with endpoints in Y. Then Y is said to be *convex* if $G_1(Y) = Y$ or, equivalently, if every pair of points $x, y \in Y$ can be joined by a geodesic in X and the image of any such geodesic is contained in Y. Y is said to be D-convex if this condition holds for all points $x, y \in Y$ with $d(x, y) < D$. For $n \geq 2$ we inductively define $G_n(Y) = G_1(G_{n-1}(Y))$; then

$$conv(Y) = \cup_{n=1}^{\infty} G_n(Y)$$

is the *convex hull* of Y.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). We will say that the triangle is *degenerate* if all three vertices belong to a same geodesic.

Next we introduce the Model Spaces M_k^n, for a more detailed description of them as well as for the proofs of results we state in this section the reader can check [1, Chapter I.2]. To begin we need to describe the spaces $\mathbb{E}^n, \mathbb{S}^n$ and $\mathbb{H}^n$.

Let $\mathbb{E}^n$ stand for the metric space obtained by equipping the vector space $\mathbb{R}^n$ with the metric associated to the norm arising from the Euclidean scalar product $(x|y) = \sum_{i=1}^{i=n} x_i y_i$, where $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$, i.e. $\mathbb{R}^n$ endowed with the usual Euclidean distance.

The n-dimensional sphere $\mathbb{S}^n$ is the set $\{x = (x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \ : \ (x|x) = 1\}$, where $(\cdot, \cdot)$ denotes the Euclidean scalar product.

Proposition 2.2 *Let $d : \mathbb{S}^n \times \mathbb{S}^n \to \mathbb{R}$ be the function that assigns to each pair $(A, B) \in \mathbb{S}^n \times \mathbb{S}^n$ the unique real number $d(A, B) \in [0, \pi]$ such that*

$$\cos d(A, B) = (A|B).$$

Then $(\mathbb{S}^n, d)$ is a metric space.

Geodesics in $\mathbb{S}^n$ coincide with sufficiently small arcs of great circles, i.e. intersections of $\mathbb{S}^n$ with a 2-dimensional vector subspace of $\mathbb{E}^{n+1}$. There is a natural way to parameterize arcs of great circles with respect to arc length which will be useful in this work: given a point $A \in \mathbb{S}^n$, a unit vector $u \in \mathbb{E}^{n+1}$ with $(u|A) = 0$ and a number $a \in [0, \pi]$, the path $c : [0, a] \to \mathbb{S}^n$ given by $c(t) = (\cos t)A + (\sin t)u$ is a geodesic and any geodesic in $\mathbb{S}^n$ can be parameterized this way. The next proposition summarizes some of the properties of the metric space $(\mathbb{S}^n, d)$.

Property 2.3 *Let $(\mathbb{S}^n, d)$ be as above and $A, B \in \mathbb{S}^n$, then:*

(1) If $d(A,B) < \pi$ then there is just one geodesic segment joining both points.

(2) If $B \neq A$ then the initial vector u of this geodesic is the unit vector, with the Euclidean norm, in the direction of $B - (A|B)A$.

(3) Balls of radius smaller than $\pi/2$ are convex sets.

By definition, the *spherical angle* between two geodesics from a point of $\mathbb{S}^n$, with initial vectors u and v, is the unique number $\alpha \in [0, \pi]$ such that $\cos\alpha = (u|v)$. Given $\triangle(A,B,C)$ a triangle in $\mathbb{S}^n$, the vertex angle at C is defined to be the spherical angle between the sides of $\triangle$ joining C to A and C to B. Then the Spherical Law of Cosines can be described as follows:

Proposition 2.4 *Let $\triangle$ be a spherical triangle with vertices A, B, C. Let $a = d(B,C)$, $b = d(C,A)$ and $c = d(A,B)$. Let γ denote the vertex angle at C. Then*

$$\cos c = \cos a \cos b + \sin a \sin b \cos\gamma.$$

Now, in order to introduce the Hyperbolic n-Space $\mathbb{H}^n$, let $\mathbb{E}^{n,1}$ denote the vector space $\mathbb{R}^{n+1}$ endowed with the symmetric bilinear form which associates to vectors $u = (u_1, \cdots, u_{n+1})$ and $v = (v_1, \cdots, v_{n+1})$ the real number $\langle u|v\rangle$ defined by

$$\langle u|v\rangle = -u_{n+1}v_{n+1} + \sum_{i=1}^{n} u_i v_i.$$

Then the *real hyperbolic n-space* $\mathbb{H}^n$ is

$$\{u \in \mathbb{E}^{n,1} : \langle u|u\rangle = -1, u_{n+1} \geq 1\}.$$

Proposition 2.5 *Let $d : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{R}$ be the function that assigns to each pair $(A,B) \in \mathbb{H}^n \times \mathbb{H}^n$ the unique non-negative number $d(A,B)$ such that*

$$\cosh d(A,B) = -\langle A, B\rangle.$$

Then $(\mathbb{H}^n, d)$ is a uniquely geodesic metric space.

Some of the most relevant properties of these spaces are summarized next.

Property 2.6 *Let $(\mathbb{H}^n, d)$ be as above and $A, B \in \mathbb{H}^n$, then:*

(1) If u is the unit vector, with respect to the bilinear form, in the direction $B + \langle A|B\rangle A$ then the geodesic segment joining A and B and starting at A is given by $c(t) = (\cosh t)A + (\sinh t)u$.

(2) Balls are convex sets.

(3) (Hyperbolic Law of Cosines) Under the same notation of Proposition 2.4,

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos\gamma,$$

where γ stands for the hyperbolic angle which can be defined in a similar way to the spherical angle.

The Model Spaces M_k^n are defined as follows.

Definition 2.7 *Given a real number k, we denote by M_k^n the following metric spaces:*

(1) if $k = 0$ then M_0^n is the Euclidean space $\mathbb{E}^n$;

(2) if $k > 0$ then M_k^n is obtained from the spherical space $\mathbb{S}^n$ by multiplying the distance function by the constant $1/\sqrt{k}$;

(3) if $k < 0$ then M_k^n is obtained from the hyperbolic space $\mathbb{H}^n$ by multiplying the distance function by the constant $1/\sqrt{-k}$.

Proposition 2.8 *M_k^n is a geodesic metric space. If $k \leq 0$ then M_k^n is uniquely geodesic and all balls in M_k^n are convex. If $k > 0$ then there is a unique geodesic segment joining $x, y \in M_k^n$ if and only if $d(x, y) < \pi/\sqrt{k}$. If $k > 0$, closed balls in M_k^n of radius smaller than $\pi/(2\sqrt{k})$ are convex.*

Let (X, d) be a geodesic metric space. A *comparison triangle* for a geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\triangle(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in M_k^2 such that $d_{M_k^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. If $k \leq 0$ then such a comparison triangle always exists in M_k^2. If $k > 0$ then such a triangle exists whenever $d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_1) < 2D_k$, where $D_k = \pi/\sqrt{k}$.

A geodesic triangle $\triangle$ in X is said to satisfy the CAT(k) *inequality* if, given $\bar{\triangle}$ a comparison triangle in M_k^2 for $\triangle$, for all $x, y \in \triangle$

$$d(x, y) \leq d_{M_k^2}(\bar{x}, \bar{y}),$$

where $\bar{x}, \bar{y} \in \bar{\triangle}$ are the respective *comparison points* of x, y, i.e., if $x \in [x_i, x_j]$ is such that $d(x, x_i) = \lambda d(x_i, x_j)$ and $d(x, x_j) = (1 - \lambda)d(x_i, x_j)$ then $\bar{x} \in [\bar{x}_i, \bar{x}_j]$ is such that $d(\bar{x}, \bar{x}_i) = \lambda d(\bar{x}_i, \bar{x}_j)$ and $d(\bar{x}, \bar{x}_j) = (1 - \lambda)d(\bar{x}_i, \bar{x}_j)$.

Definition 2.9 *If $k \leq 0$, then X is called a CAT(k) space if X is a geodesic space such that all of its geodesic triangles satisfy the CAT(k) inequality.*

If $k > 0$, then X is called a CAT(k) space if X is D_k-geodesic and all geodesic triangles in X of perimeter less than $2D_k$ satisfy the CAT(k) inequality.

$\mathbb{R}$-trees are a particular class of CAT(k) spaces for any real k which will be named at certain points of our exposition (see [1, pg. 167] for more details).

Definition 2.10 *An $\mathbb{R}$-tree is a metric space T such that:*

(1) it is a uniquely geodesic metric space;

(2) if x, y and $z \in T$ are such that $[y, x] \cap [x, z] = \{x\}$, then $[y, x] \cup [x, z] = [y, z]$.

Remark 2.11 *Notice that all triangles in an $\mathbb{R}$-tree are degenerate.*

Next we define the notion of comparison angle.

Definition 2.12 *Let p, q and r be three points in a metric space. The interior angle of $\bar{\triangle}(p, q, r) \subseteq \mathbb{E}^2$ at $\bar{p}$ is called the comparison angle between q and r at p and will be denoted $\overline{\angle}_p(q, r)$.*

The notion of angle in a geodesic space will be very important in our work.

Definition 2.13 *Let X be a metric space and let $c : [0, a] \to X$ and $c' : [0, a'] \to X$ be two geodesic paths with $c(0) = c'(0)$. Given $t \in (0, a]$ and $t' \in (0, a']$, we consider the comparison triangle $\triangle(\overline{c(0)}, \overline{c(t)}, \overline{c'(t')})$ and the comparison angle $\overline{\angle}_{c(0)}(c(t), c'(t'))$ in $\mathbb{E}^2$. The (Alexandrov) angle or the upper angle between the geodesic paths c and c' is the number $\angle_{c,c'} \in [0, \pi]$ defined by:*

$$\angle(c, c') = \limsup_{t,t' \to 0^+} \overline{\angle}_{c(0)}(c(t), c'(t')).$$

The angle between the geodesic segments $[p, x]$ and $[p, y]$ will be denoted $\angle_p(x, y)$.

Remark 2.14 *The Alexandrov angle coincides with the spherical angle on $\mathbb{S}^n$ and the hyperbolic angle on $\mathbb{H}^n$.*

A very important role in this work will be played by the notion of *uniform convexity* in a D-uniquely geodesic space. We define the modulus of convexity of (X, d) by

$$\delta_X(r, \varepsilon) = \inf\{1 - \frac{1}{r}(d(a, m))\},$$

where the infimum is taken over all points a, x, y and m the midpoint of $[x, y]$ in X satisfying that $d(a, x) < r$, $d(a, y) < r$ and $d(x, y) \geq \varepsilon$, with $\varepsilon, r < D$.

In this work we will need the estimation of the modulus of convexity of $\mathbb{S}^2$ with the spherical distance, remember that $D = D_1$ in this case. This can be found in [7, pg. 154] where the following is shown

$$\delta_{\mathbb{S}^2}(r, \varepsilon) = 1 - \frac{1}{r} \arccos\Big(\frac{\cos r}{\cos(\varepsilon/2)}\Big).$$

Definition 2.15 *A D-uniquely geodesic metric space (X, d) will be said to be uniformly convex if $\delta_X(r, \varepsilon) < 1$ for every $r \in (0, D)$ and $\varepsilon \in (0, D)$.*

We finish this section by introducing the notions of Lifšic characteristic and property (P) of Lim-Xu for metric spaces which will be used in the last section of this work for the study of uniformly l-lipschitzian mappings.

Definition 2.16 *A mapping $T : X \to X$ is said to be uniformly l-lipschitzian if there exists a constant l such that $d(T^n x, T^n y) \leq l d(x, y)$ for all $x, y \in X$ and $n \in \mathbb{N}$.*

Balls in X are said to be $c-$*regular* if the following holds: for each $l < c$ there exist $\mu, \alpha \in (0, 1)$ such that for each $x, y \in X$ and $r > 0$ with $d(x, y) \geq (1 - \mu)r$, there exists $z \in X$ such that

$$B(x; (1 + \mu)r) \bigcap B(y; l(1 + \mu)r) \subset B(z; \alpha r).$$

The Lifšic characteristic $\kappa(X)$ of X is defined as follows:

$$\kappa(X) = \sup\{c \geq 1 : \text{ balls in } X \text{ are } c\text{-regular}\}.$$

The above characteristic was applied by Lifšic in the following theorem.

Theorem 2.17 (Lifšic [16] (see also [6])) *Let (X, d) be a bounded complete metric space. Then every uniformly l-lipschitzian mapping $T : X \to X$ with $l < \kappa(X)$ has a fixed point.*

In [17], Lim and Xu introduced the so-called property (P) for metric spaces. A metric space (X, d) is said to have *property* (P) if given two bounded sequences $\{x_n\}$ and $\{z_n\}$ in X, there exists $z \in \bigcap_{n \geq 1} cov(\{z_j : j \geq n\})$ such that

$$\limsup_n d(z, x_n) \leq \limsup_j \limsup_n d(z_j, x_n).$$

The following theorem was proved in [17].

Theorem 2.18 *Let (X, d) be a complete bounded metric space with both property (P) and uniform normal structure. Then every uniformly l-lipschitzian mapping $T : X \to X$ with $l < \tilde{N}(X)^{-\frac{1}{2}}$ has a fixed point.*

3 Some basic facts

We begin this section with the study of the uniform convexity of CAT(1) spaces.

Proposition 3.1 *Let X be a complete CAT(1) space. If $diam(X) < \pi/2$, then X is uniformly convex and its modulus of convexity satisfies*

$$\delta_X(r, \varepsilon) \geq \delta_{\mathbb{S}^2}(r, \varepsilon).$$

Proof. This follows directly from the CAT(k) inequality for comparison triangles and the character of the module of convexity of the sphere. □

Notice that this result is optimal as the following example shows. Therefore, throughout this paper we will assume the condition $\mathrm{diam}(X) < \pi/2$ as a natural one when dealing with CAT(1) spaces.

Example 3.2 *Let $(\mathbb{S}^2, d)$ be the spherical space and $e_i \in \mathbb{S}^2$, for $i = 1, 2, 3$, be each of the elements of the canonical basis of $\mathbb{R}^3$. Let K be the closed convex hull over the sphere of $\{e_i : i = 1, 2, 3\}$, i.e, the positive octant of the sphere. Then we have that $diam(K) = \pi/2$ but K is not uniformly convex itself since $d(e_1, e_i) = \pi/2$ for $i = 2, 3$ and $d(e_1, m) = \pi/2$ for m the mid-point of the geodesic segment $[e_2, e_3]$.*

The following theorem, due to U. Lang and V. Schroeder [15], shows that a bit more can be said regarding the normal structure of a CAT(1) space.

Theorem 3.3 *Let X be a complete CAT(1) and S a nonempty bounded subset of X. If $rad_X(S) < \pi/2$, then there is a unique center for S and $diam(S) \geq \Psi(rad_X(S)) > rad_X(S)$, where*

$$\Psi(r) = 2 \arcsin(\frac{1}{\sqrt{2}} \sin r).$$

The next example shows that Theorem 3.3 is optimal with respect to the normal structure of the space.

Example 3.4 *Let us consider the unit sphere S_{ℓ_2} of the Hilbert space ℓ_2 provided with the intrinsic metric L_d. This space is a CAT(1) space. Consider the elements of the canonic basis $\{(e_i)\}_i^\infty$ of ℓ_2. Let $K = \{x = (x_n) \in S_{\ell_2} : x_n \geq 0$ for all $n \in \mathbb{N}\}$, i.e. K is the closed convex hull of $\{(e_i)\}_i^\infty$ in (S_{ℓ_2}, L_d).*

Since the intrinsic distance between two points x and y in S_{ℓ_2} coincides with the real number $d(x,y) \in [0,\pi]$ such that $(x|y)_{\ell_2} = \cos d(x,y)$, the diameter of K can be estimated as follows:

$$diam(K) = \sup_{i,j} d(e_i, e_j) = \sup_{i,j} \arccos(e_i|e_j) = \arccos 0 = \pi/2.$$

Now, given $x \in S_{\ell_2}$ we also have that $d(x,e_n) = \arccos(x|e_n) = \arccos x_n$. Thus,

$$\lim_{n\to\infty} d(x,e_n) = \lim_{n\to\infty} \arccos x_n = \arccos 0 = \pi/2.$$

Then, rad$(K) = \pi/2 =$diam(K).

The next proposition establishes very useful properties of the metric projection in CAT(1) spaces. Properties given by Statements (1) and (2), among others, are proved in [1] for CAT(0) spaces and proposed as an exercise (Exercise 2.6 (1)) for CAT(k) spaces with $k > 0$. Statement (3) follows as a consequence of (2), we include its proof.

Proposition 3.5 *Let X be a complete CAT(1) space, $x \in X$ and $C \subset X$ nonempty closed and π-convex such that dist$(x,C) < \pi/2$, then the following facts hold:*

(1) The metric projection $P_C(x)$ of x onto C is a singleton.

(2) If $x \notin C$ and $y \in C$ with $y \neq P_C(x)$ then $\angle_{P_C(x)}(x,y) \geq \pi/2$.

(3) If diam$(X) \leq \pi$, then, for any $y \in C$,

$$d(P_C(x), P_C(y)) = d(P_C(x), y) \leq d(x,y).$$

Proof of (3). It suffices to prove (3) for $x \in X \backslash C$ and $y \in C$. From (2), $\angle_{P_C(x)}(x,y) \geq \pi/2$, and so, by the Law of Cosines,

$$\begin{aligned} \cos d(y,x) &\leq \cos d(y,P_C(x)) \cos d(x,P_C(x)) + \sin d(y,P_C(x)) \sin d(x,P_C(x)) \cos\gamma \\ &\leq \cos d(y,P_C(x)) \cos d(x,P_C(x)) \\ &\leq \cos d(y,P_C(x)). \end{aligned}$$

Now, since diam$(X) \leq \pi$, we finally obtain $d(P_C(x), P_{C_k}(y)) = d(P_C(x), y) \leq d(x,y)$.
□

The following corollary, which will also be needed and follows by using similar techniques as those required in the proof of the previous proposition, allows us to say that CAT(1) spaces are in someway reflexive. Note that $r((c_n))$ stands for the asymptotic radius of the sequence (c_n) which is defined in the next section.

Corollary 3.6 *Let X be a complete CAT(1) space and (C_n) a decreasing sequence of nonempty closed and π-convex subsets of X. If there exists a sequence (c_n) such that $c_n \in C_n$ for all $n \in \mathbb{N}$ and $r((c_n)) < \pi/2$, then $\cap_n C_n \neq \emptyset$.*

In order to prove a counterpart of Kirk's Fixed Point Theorem (see Theorem 2.1) for CAT(1) spaces, we next define a new coefficient related to normal structure of a geodesic metric space X. The number

$$\hat{N}(X) = \sup\left\{\frac{\mathrm{rad}_A(A)}{\mathrm{diam}(A)}\right\}$$

where the supremum is taken over all nonempty bounded closed convex and admissible subsets A of X for which $\mathrm{diam}(A) > 0$ will be called the $\wedge$-*normal structure coefficient* of X. If $\hat{N}(X) \leq c$ for some constant $c < 1$, then X will be said to have $\wedge$-*uniform normal structure.*

The next lemma will be the key to show that CAT(1) spaces have the $\wedge$-uniform normal structure under natural conditions on the diameter. Notice that this lemma is closely related to Proposition 2 in [11].

Lemma 3.7 *Let C be a nonempty closed and convex subset of a complete $CAT(1)$ space X. If $rad_X(C) < \pi/2$ and $diam(X) \leq \pi$, then $rad_X(C)$ =$rad_C(C)$.*

Proof. Since the set C is bounded, Theorem 3.3 assures that there exists a unique point $x \in X$ such that $B(x, \mathrm{rad}_X(C)) \supset C$. In consequence, $\mathrm{dist}(x, C) \leq \mathrm{rad}_X(C) < \pi/2$. Now it directly follows from Proposition 3.5 that $P_C(x) = x$ which implies that $x \in C$ and so $\mathrm{rad}_X(C)$ =$\mathrm{rad}_C(C)$. □

Corollary 3.8 *If X is a complete $CAT(1)$ space with $rad(X) < \pi/2$ then X has $\wedge$-uniform normal structure.*

Proof. It follows as a direct combination of Theorem 3.3 and the above lemma. □

Next we prove Kirk's Fixed Point Theorem for CAT(1) spaces. We will follow the same patterns than the proof given in [9, pg. 103] of Theorem 2.1.

Theorem 3.9 *Let X be a complete nonempty $CAT(1)$ space such that $rad(X) < \pi/2$. Then every nonexpansive mapping $T : X \to X$ has at least one fixed point.*

Proof. By Corollary 3.6 and Zorn's Lemmas it follows that there exists a nonempty, convex and admissible subset D of X which is minimal with respect to being nonempty, convex, admissible and mapped into itself by T. Also, if $\mathrm{cac}(T(D))$ denotes the convex and admissible closure (defined in a natural way with respect to the set inclusion) of D in X, then $T : \mathrm{cac}(T(D)) \to \mathrm{cac}(T(D))$. So, the minimality of D implies that

$$D = \mathrm{cac}(T(D)).$$

Now assume $\mathrm{diam}(D) > 0$. From Lemma 3.7 and the fact that $\mathrm{rad}(X) < \pi/2$, it is possible to choose r so that

$$\mathrm{rad}_D(D) < r < \min\{\pi/2, \mathrm{diam}(D)\}.$$

It then follows that the set

$$C = \{x \in D \ : \ D \subseteq B(x, r)\} \neq \emptyset$$

is convex and, since

$$C = \big(\cap_{x_D} B(x, r) \big) \cap D,$$

also admissible.

Now the proof follows exactly the same steps than that of Theorem 5.1 in [9]. □

Remark 3.10 *W. A. Kirk in Theorem 11 of [11] also proved this last result but under the stronger assumption of $diam(X) < \pi/2$.*

As a consequence of Lemma 3.7 it also follows that Theorem 3.9 still holds true for convex subsets rather than for the whole space.

Corollary 3.11 *Let C be a nonempty closed and convex subset of a complete CAT(1) space X. If $rad_X(C) < \pi/2$ and $diam(X) \leq \pi$, then every nonexpansive mapping $T : C \to C$ has at least one fixed point.*

Remark 3.12 *Notice that neither Lemma 3.7 nor above corollary hold true if the condition $rad_X(C) < \pi/2$ is replaced by $rad_X(C) \leq \pi/2$. For that it is enough to consider C as any great circumference of $\mathbb{S}^2$.*

4 Δ-convergence and the Kadec-Klee property

In this section we show that Δ-convergence can be used in CAT(1) spaces in a similar way as it is used in [12] for CAT(0) spaces, obtaining a collection of similar results with the only difference that we have to impose the natural bound on the diameter of the CAT(1) space. To show this we begin with the definition of Δ-convergence.

Let X be a complete CAT(1) space and (x_n) a bounded sequence in X. For $x \in X$ set

$$r(x, (x_n)) = \limsup_{n\to\infty} d(x, x_n).$$

The *asymptotic radius* $r((x_n))$ of (x_n) is given by

$$r((x_n)) = \inf\{r(x, (x_n)) : x \in X\},$$

the *asymptotic radius* $r_C((x_n))$ *with respect to* $C \subseteq X$ of (x_n) is given by

$$r_C((x_n)) = \inf\{r(x, (x_n)) : x \in C\},$$

the *asymptotic center* $A((x_n))$ of (x_n) is given by the set

$$A((x_n)) = \{x \in X : r(x, (x_n)) = r((x_n))\},$$

and the *asymptotic center* $A_C((x_n))$ *with respect to* $C \subseteq X$ of (x_n) is given by the set

$$A_C((x_n)) = \{x \in C : r(x, (x_n)) = r_C((x_n))\}.$$

Proposition 4.1 *Let X be a complete CAT(1) space, $C \subseteq X$ nonempty closed and π-convex, and (x_n) a sequence in X. If $r_C(\{x_n\}) < \pi/2$, then $A_C((x_n))$ consists of exactly one point.*

Proof. Existence follows from Corollary 3.6. Uniqueness follows in a straightforward way from the uniform convexity of CAT(1) spaces as stated in Proposition 3.1. □

The next example shows the optimality of the last bound on the asymptotic radius.

Example 4.2 *As in Example 3.4, we consider the unit sphere S_{ℓ_2} of the Hilbert space ℓ_2 provided with the intrinsic metric L_d. Consider the sequence consisting of the canonic basis $\{(e_i)\}_i^\infty$ of ℓ_2. Let $y = (y_n) \in S_{\ell_2}$, then*

$$r(y, ((e_n)) = \limsup_n d(y, e_n) = \limsup_n \arccos y_n = \pi/2.$$

Thus, $r((e_n)) = \pi/2$ and $A((e_n)) = S_{\ell_2}$.

Definition 4.3 *A sequence (x_n) in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of (u_n) for every subsequence (u_n) of (x_n). In this case we write $\Delta - \lim_n x_n = x$ and call x the Δ-limit of (x_n).*

The next result follows as a consequence of the previous proposition.

Corollary 4.4 *Let X be a complete CAT(1) space and (x_n) a sequence in X. If $r(\{x_n\}) < \pi/2$, then (x_n) has a $\Delta -$ convergent subsequence.*

Proof. Reasoning as in [6, pg. 166] it follows that (x_n) has a regular subsequence (u_n) (i.e., a sequence such that all it subsequences have the same asymptotic radius). Then the corollary follows from the previous proposition. □

The next proposition gives a very important property of Δ-convergent sequences.

Proposition 4.5 *Let X be a complete CAT(1) space such that $diam(X) < \pi/2$. If a sequence (x_n) in X $\Delta -$ converges to $x \in X$, then*

$$x \in \bigcap_{k=1}^{\infty} \overline{conv}\{x_k, x_{k+1}, \ldots\},$$

where $\overline{conv}(A) = \bigcap\{B : B \supseteq A$ and B is closed and convex$\}$.

Proof. Let $C_k = \overline{conv}\{x_k, x_{k+1}, \ldots\}$ for $k \in \mathbb{N}$. Since $x_n \in C_k$ for all $n \geq k$, applying Proposition 3.5, it follows that

$$d(P_{C_k}(x), P_{C_k}(x_n)) = d(P_{C_k}(x), x_n) \leq d(x, x_n) \text{ for all } n \geq k.$$

Therefore,

$$r(P_{C_k}(x), (x_n)) = \limsup_{n\to\infty} d(P_{C_k}(x), x_n)$$
$$\leq \limsup_{n\to\infty} d(x, x_n) = r(x, (x_n)) = r((x_n)).$$

By Proposition 4.1, we have that $P_{C_k}(x) = x$ for all $k \in \mathbb{N}$ and so $x \in C_k$ for all $k \in \mathbb{N}$. □

Remark 4.6 *Note that the previous result is also true if we only assume that $diam(X) < \pi$ and $r(\{x_n\}) < \pi/2$.*

Next we prove the Kadec-Klee property for CAT(1) spaces. This property was proved for CAT(0) spaces in [12].

For a bounded sequence (x_n) in a metric space we denote,

$$sep(x_n) := \inf\{d(x_n, x_m) : n \neq m\}$$

the separation of the points of the sequence (x_n).

Theorem 4.7 (Kadec-Klee Property) *Let X be a complete CAT(1), let $p \in X$, and let $\varepsilon > 0$. Then there exists $\delta > 0$ such that $d(p, x) \leq 1-\delta$ for every sequence $(x_n) \subset X$ such that $d(p, x_n) \leq 1$, $sep(x_n) > \varepsilon$ and $\Delta - \lim_n x_n = x$.*

Proof. We may assume that $d(p, x_n) \equiv 1$ and by passing to a subsequence if necessary we may suppose $d(x_n, x) \geq \frac{\varepsilon}{2}$ for all n. Let $\triangle(\bar{p}, \bar{x}, \bar{x}_n) \subset \mathbb{S}^2$ be a comparison triangle for $\triangle(p, x, x_n)$. Since $d(\bar{x}_n, [\bar{p}, \bar{x}]) \leq 1 < \pi/2$, Proposition 3.5 applies and we can follow the same reasoning as in Theorem 3.9 of [12] to construct the sequences (u_n) and $(\bar{u}_n)$ such that $\bar{u}_n$ is the nearest point in $[\bar{p}, \bar{x}]$ to $\bar{x}_n$, u_n is the point in $[p, x]$ for which $d(p, u_n) = d(\bar{p}, \bar{u}_n)$, and $(\bar{u}_n)$ and (u_n) converge respectively to $\bar{u} \in [\bar{p}, \bar{x}]$ and to $u \in [p, x]$.

Let $a_n = d(\bar{p}, \bar{u}_n)$, $c_n = d(\bar{x}_n, \bar{u}_n)$ and $\gamma_n = \angle_{\bar{u}_n}(\bar{p}, \bar{x}_n)$, then, by the Law of Cosines in $\mathbb{S}^2$,

$$\cos 1 = \cos a_n \cos c_n + \sin a_n \sin c_n \cos \gamma_n.$$

Now, from $a_n \leq \pi$, $c_n \leq 1 < \pi$ and (2) of Proposition 3.5,

$$\cos 1 \leq \cos a_n \cos c_n.$$

Moreover, since $0 \leq c_n \leq 1 < \pi/2$,

$$\cos a_n \geq \frac{\cos 1}{\cos c_n}.$$

We can assume, due to the separation of the sequence (x_n), that there exits $\delta > 0$ such that $c_n \geq \delta$ for all n. Then we obtain that $\cos c_n \leq \cos \delta < 1$.

Thus, since $\cos a_n \geq \frac{\cos 1}{\cos c_n} \geq \frac{\cos 1}{\cos \delta} > \cos 1$, it follows

$$d(\bar{p}, \bar{u}_n) = a_n \leq \arccos\left(\frac{\cos 1}{\cos \delta}\right) < 1.$$

Now, since $d(\bar{p}, \bar{u}_n) = d(p, u_n)$ converges to $d(p, u)$,

$$d(p, u) \leq 1 - \eta$$

where $\eta = 1 - \arccos\left(\frac{\cos 1}{\cos \delta}\right)$.

To finish the proof we just need to show that $u = x$, which follows from the fact that $r(u, (x_n)) \leq r((x_n))$. $\square$

Next we show that we can give analogs in CAT(1) spaces to those other results in Section 3 of [12] for CAT(0) spaces. Notice that this shows that the CN inequality of Bruhat and Tits (see [1, pg. 163]) is not really required to obtain these results. In all the next definitions X is a CAT(1) space and $K \subseteq X$ convex.

Definition 4.8 *A mapping $T : K \to X$ is said to be of type Γ if there exits a continuous strictly increasing convex function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ with $\gamma(0) = 0$ such that, if $x, y \in K$ and if m and m' are the mid-points of the segments $[x, y]$ and $[T(x), T(y)]$ respectively, then*

$$\gamma(d(m', T(m))) \leq |d(x, y) - d(T(x), T(y))|.$$

Definition 4.9 *A mapping $T : K \to X$ is called $\alpha -$ almost convex for $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ continuous, strictly increasing, and $\alpha(0) = 0$, if for $x, y \in K$,*

$$J_T(m) \leq \alpha(\max\{J_T(x), J_T(y)\}),$$

where m is the mid-point of the segment $[x, y]$, and $J_T(x) := d(x, T(x))$.

Definition 4.10 *A mapping $T : K \to X$ is said to be of convex type on K if for (x_n),(y_n) two sequences in K and (m_n) the sequence of the mid-points of the segments $[x_n, y_n]$,*

$$\left.\begin{array}{l}\lim_{n\to\infty} d(x_n, T(x_n)) = 0 \\ \lim_{n\to\infty} d(y_n, T(y_n)) = 0\end{array}\right\} \Rightarrow \lim_{n\to\infty} d(m_n, T(m_n)) = 0.$$

Proposition 4.11 *Let K be a nonempty closed convex subset of a $CAT(1)$ space X and let $T : K \to X$. If $diam(K) < \pi/2$, then the following implications hold:*

$$\begin{array}{c} T \textit{ is nonexpansive} \Rightarrow T \textit{ is of type } \Gamma \Rightarrow \\ T \textit{ is } \alpha - \textit{almost convex} \Rightarrow T \textit{ is of convex type.}\end{array}$$

Proof. For the first implication, let m denote the mid-point of the segment $[x, y]$ for $x, y \in K$, and let m' denote the mid-point of the segment $[T(x), T(y)]$.

We first prove that $d(m', T(m)) < \pi/4$. From the non-expansivity of T we have that $d(T(m), T(x))$ and $d(T(m), T(y))$ are both smaller than or equal to $\pi/4$. Then it follows $T(x)$ and $T(y)$ are in $B(T(m), \pi/4)$, and, also, any point in the geodesic segment $[T(x), T(y)]$ is the ball by convexity. Therefore, it will suffice to find such a function γ defined on the interval $[0, \pi/4]$.

Now, from Proposition 3.1,

$$d(m', T(m)) \leq \arccos\left(\frac{\cos(\max\{d(T(m), T(x)), d(T(m), T(y))\})}{\cos(\frac{d(T(x),T(y))}{2})}\right) =$$

(without loss of generality)

$$= \arccos\left(\frac{\cos(d(T(m), T(x)))}{\cos(\frac{d(T(x),T(y))}{2})}\right).$$

Bearing in mind that both terms in the above inequality are less than $\pi/2$,

$$\cos(d(m', T(m))) \geq \frac{\cos(d(T(m), T(x)))}{\cos\left(\frac{d(T(x),T(y))}{2}\right)}$$

$$\geq \frac{\cos(d(m, x))}{\cos\left(\frac{d(T(x),T(y))}{2}\right)} = \frac{\cos\left(\frac{d(x,y)}{2}\right)}{\cos\left(\frac{d(T(x),T(y))}{2}\right)},$$

and so,

$$\cos^2(d(m', T(m))) \geq \frac{1 + \cos(d(x, y))}{1 + \cos(d(T(x), T(y)))}.$$

Hence,

$$\begin{aligned} \sin^2(d(m', T(m))) &\leq 1 - \frac{1 + \cos(d(x, y))}{1 + \cos(d(T(x), T(y)))} \\ &= \frac{\cos(d(T(x), T(y))) - \cos(d(x, y))}{1 + \cos(d(T(x), T(y)))} \\ &\leq \cos(d(T(x), T(y))) - \cos(d(x, y)) \\ \text{(for a certain } \xi \in (d(T(x), T(y)), d(x, y))) \quad & \\ &= (-\sin(\xi))(d(T(x), T(y)) - d(x, y)) \\ &= \sin(\xi)(d(x, y) - d(T(x), T(y))) \\ &\leq d(x, y) - d(T(x), T(y)). \end{aligned}$$

Thus it suffices to take $\gamma(t) = \sin^2(t)$ for $t \in [0, \pi/4]$ and extend it on $(\pi/4, \infty)$ so it fulfills all the required conditions to complete the first implication.

In order to prove the second implication we follow [5],

$$\begin{aligned} J_T(m) = d(m, T(m)) &\leq d(m, m') + d(m', T(m)) \\ &\leq d(m, m') + \gamma^{-1}(\mid d(x, y) - d(T(x), T(y)) \mid) \\ &\leq d(m, p) + d(p, m') + \gamma^{-1}(d(x, T(x)) + d(y, T(y))), \end{aligned}$$

where p is the mid-point of the segment $[x, T(y)]$.

We consider now the triangle $\triangle(x, y, T(y)) \subset X$ and its comparison triangle $\triangle\big(\bar{x}, \bar{y}, \overline{T(y)}\big)$ in $\mathbb{S}^2$. Let $\bar{m} \in [\bar{x}, \bar{y}]$ and $\bar{p} \in \big[\bar{x}, \overline{T(y)}\big]$ be the comparison points for m and p respectively. We want to prove that $d(m, p) \leq d(y, T(y))$, for which we will show that $d(\bar{m}, \bar{p}) \leq d\big(\bar{y}, \overline{T(y)}\big)$.

Let $c, c' : [0, 1] \to X$ be the geodesics that join $\bar{x}$ to $\bar{y}$ and $\overline{T(y)}$ parameterized proportionally with respect to the arc length, respectively. Then

$$c(t) = (\cos at)\bar{x} + (\sin at)\bar{u},$$

$$c'(t) = (\cos bt)\bar{x} + (\sin bt)\bar{v},$$

where $a = d(x, y)$, $b = d(x, T(y))$, and $\bar{u} = \frac{\bar{y} - (\bar{x}|\bar{y})\bar{x}}{\|\bar{y} - (\bar{x}|\bar{y})\bar{x}\|}$, $\bar{v} = \frac{\overline{T(y)} - \big(\bar{x}|\overline{T(y)}\big)\bar{x}}{\big\|\overline{T(y)} - \big(\bar{x}|\overline{T(y)}\big)\bar{x}\big\|}$ are the unitary vectors which define these geodesics. Since $\cos d(c(t), c'(t)) = (c(t)|c'(t))$, it will be enough to prove that the function $f(t) = (c(t)|c'(t))$ is decreasing.

Without loss of generality, we can assume that $\bar{x} = (1, 0, 0) \in \mathbb{R}^3$. Hence, if $\bar{u} = (u_1, u_2, u_3)$ and $\bar{v} = (v_1, v_2, v_3)$, then $u_1 = v_1 = 0$. Therefore,

$$f(t) = ((\cos at, (\sin at)u_2, (\sin at)u_3)|(\cos bt, (\sin bt)v_2, (\sin bt)v_3)) =$$

$$= \cos at \cos bt + \sin at \sin bt(\bar{u}|\bar{v}) =$$

$$= \frac{1}{2}(1 - (\bar{u}|\bar{v})) \cos(t(a + b)) + \frac{1}{2}(1 + (\bar{u}|\bar{v})) \cos(t(a - b)).$$

Then

$$f'(t) = -\frac{1}{2}(1 - (\bar{u}|\bar{v}))(a + b) \sin(t(a + b)) - \frac{1}{2}(1 + (\bar{u}|\bar{v}))(a - b) \sin(t(a - b)).$$

Since a and b are less than $\pi/2$ and $(\bar{u}|\bar{v}) \leq 1$, then $f'(t) \leq 0$.

In the same way, we can prove that $d(p, m') \leq d(x, T(x))$. Thus,

$$J_T(m) \leq d(y, T(y)) + d(x, T(x)) + \gamma^{-1}(d(x, T(x)) + d(y, T(y))) \leq$$

$$\leq \alpha(\max\{J_T(x), J_T(y)\}),$$

where $\alpha(t) = 2t + \gamma^{-1}(2t)$.

The third implication is immediate. □

We finish this section with the equivalent result of Theorem 3.14 in [12] for CAT(1) spaces.

Theorem 4.12 *Let K be a bounded closed convex subset of X a complete CAT(1) space, and let $T : K \to X$ be continuous and of convex type. Suppose*

$$\inf\{d(x, T(x)) : x \in K\} = 0$$

If diam$(X) < \pi/2$, then T has a fixed point in K.

Proof. Let $x_0 \in X$ be fixed and define

$$\rho_0 = \inf\{\rho > 0 : \inf\{d(x, T(x)) : x \in B(x_0; \rho) \cap K\} = 0\}.$$

Since $K \subseteq B(x_0,\text{diam}(X))$ we have that $\rho_0 < \pi/2 < \infty$. Moreover, if $\rho_0 = 0$ then $x_0 \in K$ and $T(x_0) = x_0$ by continuity of T. So we suppose $\rho_0 > 0$. Now choose $(x_n) \subset K$ such that $d(x_n, T(x_n)) \to 0$ and $d(x_n, x_0) \to \rho_0$. It suffices to show that (x_n) is convergent to prove the theorem. If (x_n) is not convergent, there exist $\varepsilon > 0$ and subsequences (u_k) and (v_k) of (x_n) such that $d(u_k, v_k) \geq \varepsilon$ for all k. Passing again to subsequences if necessary we may suppose $d(u_k, x_0) \leq \rho_0 + \frac{1}{k}$ and $d(v_k, x_0) \leq \rho_0 + \frac{1}{k}$. Let m_k be the mid-point of the segment $[u_k, v_k]$ and let $\bar{m}_k$ be the point corresponding to m_k on the comparison triangle $\triangle(\bar{x}_0, \bar{u}_k, \bar{v}_k) \subseteq \mathbb{S}^2$. Then, by the CAT(1) inequality and the module of convexity of $\mathbb{S}^2$,

$$d(x_0, m_k) \leq d(\bar{x}_0, \bar{m}_k) \leq \left(\rho_0 + \frac{1}{k}\right)\left(1 - \delta_{\mathbb{S}^2}\left(\rho_0 + \frac{1}{k}, \varepsilon\right)\right) =$$

$$= \arccos\left(\frac{\cos(\rho_0 + \frac{1}{k})}{\cos(\varepsilon/2)}\right).$$

Since $0 < \varepsilon < \pi/2$ and $(\rho_0 + \frac{1}{k})$ is decreasing and convergent, we consider k big enough so that

$$\cos(\frac{\varepsilon}{2}) < \frac{\cos\left(\rho_0 + \frac{1}{k}\right)}{\cos \rho_0} < 1.$$

Then, for $k' \geq k$,

$$\frac{\cos\left(\rho_0 + \frac{1}{k}\right)}{\cos(\frac{\varepsilon}{2})} \geq \frac{\cos\left(\rho_0 + \frac{1}{k'}\right)}{\cos(\frac{\varepsilon}{2})} > \cos \rho_0,$$

and so,

$$d(x_0, m_k) \leq \arccos\left(\frac{\cos(\rho_0 + \frac{1}{k})}{\cos(\varepsilon/2)}\right) \leq \rho' < \rho_0.$$

On the other hand, since T is of convex type, $\lim_{k\to\infty} d(m_k, T(m_k)) = 0$. This contradicts the definition of ρ_0. $\square$

Remark 4.13 *Notice that the same proof holds if the condition on the boundedness of X is replaced by the weaker one of the existence of such a sequence $(x_n) \subset X$ that $r((x_n)) < \pi/2$ and* $\lim d(x_n, Tx_n) = 0$.

5 A notion of weak convergence and an application

In [20] E. N. Sosov introduces two different notions of convergence in geodesic metric spaces. These notions coincide with Δ and weak convergence in Hilbert spaces. Next we use one of the notions given by Sosov to introduce a new one more adequate to our purposes. We will adopt the same notation used by Sosov.

Let X be a CAT(0) space and p a fixed point in X. Let S be the set of all the geodesic segments containing the point p. Given $I \in S$ and $x \in X$, we define the function $\phi_I : X \to \mathbb{R}$ as $\phi_I(x) = d(p, P_I(x))$ where $P_I(x)$ is the projection of x onto I. The set of all these ϕ_I is denoted by $\Phi_p(X)$.

Definition 5.1 *A bounded sequence $(x_n) \subseteq X$ ϕ_p-converges to a point $x \in X$ if*

$$\lim_{n\to\infty} \phi(x_n) = \phi(x)$$

for any $\phi \in \Phi_p(X)$.

The following proposition establishes an easy connection between Δ and ϕ convergence.

Proposition 5.2 *A sequence $(x_n) \subset X$ Δ-converges to p if, and only if, ϕ_p-converges to it.*

Proof. $\Rightarrow$: Let I be a geodesic segment containing p and $P_I(x_n)$ the projection of x_n onto I. Since $p \in I$, (x_n) ϕ_p-converges to p if, and only if, $P_I(x_n) \to p$ as $n \to \infty$ for each such I. So if (x_n) does not ϕ_p-converges to p then there exists I such that $P_I(x_n)$ does not converges to p in a strong sense. In this case there exists a subsequence of $P_I(x_n)$, which we denote the same, and $x \in I$ with $x \neq p$ such that $P_I(x_n) \to x$. Now, since $P_I(x_n)$ is the projection of x_n onto L, taking subsequences if necessary, we have that

$$\lim d(x_n, x) \leq \lim d(x_n, p)$$

which contradicts the uniqueness of the Δ-limit.

$\Leftarrow$: If (x_n) does not Δ-converges to p then there exists a subsequence of (x_n) which we denote the same and a point $x \neq p$ such that

$$\lim d(x_n, x) < \lim d(x_n, p).$$

Now it is enough to consider the segment determined by p and x to get a contradiction to the fact that (x_n) ϕ_p-converges to p. □

Remark 5.3 *Note that all we have just done remains valid for CAT(1) spaces of diameter bounded by $\pi/2$.*

In [12] a four point condition, the so-called (Q_4) condition, was studied for CAT(0) spaces. In that work it was asked if any CAT(0) space enjoys the (Q_4) condition as well as if this condition is necessary for their Proposition 4.2. We will answer both questions in the negative and also improve this latter proposition by means of a weaker geometrical condition than condition (Q_4).

Definition 5.4 *A complete CAT(0) space X is said to verify the (Q_4) condition if for any four points $x, y, p, q \in X$*

$$\left.\begin{array}{l} d(x,p) < d(x,q) \\ d(y,p) < d(y,q) \end{array}\right\} \Rightarrow d(m,p) \leq d(m,q)$$

for any point m on the segment $[x, y]$.

Remark 5.5 *Note that condition (Q_4) is also well defined for any uniquely geodesic metric space or even for D-uniquely geodesic spaces under some conditions on the points x and y.*

While it was asked in [12] if all complete CAT(0) spaces satisfy the (Q_4) condition, the only examples of such CAT(0) spaces explicitly named there were Hilbert spaces and $\mathbb{R}-$trees. Next we present a larger collection of CAT(0) spaces which satisfy this condition.

Definition 5.6 *Let $k \leq k'$, we will say that a CAT(k') space is of constant curvature equal to k if any non-degenerate triangle (with adequate boundedness condition if $k > 0$) in it is isometric to its comparison triangle in M_k^2.*

Then the following theorem, which we state for CAT(0) spaces for expository reasons, holds.

Theorem 5.7 *Any CAT*(0) *space of constant curvature satisfies the* (Q_4) *condition.*

Proof. The result follows for any model space M_k^2 in a similar way as it follows for $\mathbb{R}^2$. We write the proof for M_{-1}^2.

Let $x, y, p, q \in M_{-1}^2$ such that $d(x,p) < d(x,q)$ and $d(y,p) < d(y,q)$. By the definition of the hyperbolic metric, we have that

$$\operatorname{arccosh}(-\langle x|p\rangle) < \operatorname{arccosh}(-\langle x|q\rangle)$$

and

$$\operatorname{arccosh}(-\langle y|p\rangle) < \operatorname{arccosh}(-\langle y|q\rangle),$$

or equally, that

$$\langle x|p\rangle > \langle x|q\rangle \text{ and } \langle y|p\rangle > \langle y|q\rangle.$$

Let m be an interior point of the geodesic segment $[x,y]$. We need to prove that $\langle m|p\rangle \geq \langle m|q\rangle$. If $c : [0, d(x,y)] \to M_{-1}^2$ is the geodesic which joins the points x and y, we can describe each interior point m as

$$m = \cosh(\alpha d(x,y))x + \sinh(\alpha d(x,y))u,$$

where $\alpha \in (0,1)$ and $u = \frac{y+\langle x|y\rangle x}{\|y+\langle x|y\rangle x\|}$, where $\|y + \langle x|y\rangle x\| = \sqrt{\langle y + \langle x|y\rangle x | y + \langle x|y\rangle x\rangle} = \sinh d(x,y)$.

Then

$$\begin{aligned}
\langle m|p\rangle &= \cosh(\alpha d(x,y))\langle x|p\rangle + \sinh(\alpha d(x,y))\langle u|p\rangle \\
&= \left(\cosh(\alpha d(x,y)) + \frac{\sinh(\alpha d(x,y))\langle x|y\rangle}{\|y+\langle x|y\rangle x\|}\right)\langle x|p\rangle + \frac{\sinh(\alpha d(x,y))}{\|y+\langle x|y\rangle x\|}\langle y|p\rangle.
\end{aligned}$$

In the same way,

$$\langle m|q\rangle = \left(\cosh(\alpha d(x,y)) + \frac{\sinh(\alpha d(x,y))\langle x|y\rangle}{\|y+\langle x|y\rangle x\|}\right)\langle x|q\rangle + \frac{\sinh(\alpha d(x,y))}{\|y+\langle x|y\rangle x\|}\langle y|q\rangle.$$

It is obvious that $\dfrac{\sinh(\alpha d(x,y))}{\|y+\langle x|y\rangle x\|} \geq 0$, so it suffices to show that the factor of $\langle x|q\rangle$ is also nonnegative. So, we have,

$$\cosh(\alpha d(x,y)) + \frac{\sinh(\alpha d(x,y))\langle x|y\rangle}{\|y+\langle x|y\rangle x\|} \geq 0 \Leftrightarrow$$

(since $-\langle x|y\rangle > 0$ for $x, y \in M_{-1}^2$)

$$\tanh(\alpha d(x,y)) \leq \frac{\|y+\langle x|y\rangle x\|}{-\langle x|y\rangle} = \tanh d(x,y),$$

which holds due to the fact that tanh is an increasing function.

Now, for the general case it is enough to note that given the four point x, y, p and q we just take the comparison triangles for $\triangle(p,x,y)$ and $\triangle(q,x,y)$ in M_k^2 so that they have $[\bar{x}, \bar{y}]$ as a common side. Then the result follows by isometry to M_k^2. □

Remark 5.8 *A similar result holds for spaces of positive constant curvature.*

In contrast to this theorem, the next example shows that there exist in fact CAT(0) spaces without the (Q_4) condition.

Example 5.9 *Let $A = \{(x,y) \in \mathbb{R}^2 \ : \ x \geq 0\}$ endowed with the Euclidean distance d_1 and $B = \{(x,0) \in \mathbb{R}^2 \ : \ x \leq 0\}$ with the usual metric d_2 on $\mathbb{R}$. Let X be the gluing $A \sqcup_{(0,0)} B$ with the natural gluing metric d defined as*

$$d(x,y) = \begin{cases} d_i(x,y), & \textit{if } x,y \textit{ are both either in } A \textit{ or } B \\ d_1(x,0) + d_2(0,y), & \textit{if } x \in B \textit{ and } y \in A. \end{cases}$$

(See [1, pg. 67] for more details on gluings). By Reshetnyak gluing theorem ([1, pg. 347]) (X,d) is a CAT(0) space; however if we take $x = (0,1)$, $y = (0,-1)$, $p = (11/10,0)$ and $q = (-1,0)$ we have that $d(p,x) = d(p,y) < d(q,y) = d(q,x)$ but since m, the mid-point of the segment $[x,y]$, is equal to the pair $(0,0)$ we obtain that $d(p,m) > d(q,m)$, contradicting the (Q_4) condition.

The next theorem shows that this example is a particular case in a class of CAT(0) spaces missing the (Q_4) condition. Notice also that two spaces of constant curvature can be glued only through geodesic lines, geodesic segments or singletons so Reshetnyak gluing theorem can be applied. The following lemma will be needed.

Lemma 5.10 *Let $\triangle(x,y,z)$ be a triangle of constant curvature k and $\triangle(\bar{x},\bar{y},\bar{z})$ a comparison triangle for $\triangle(x,y,z)$ in $M^2_{k'}$ with $k < k'$. Then $d(x,m) < d(\bar{x},\bar{m})$ for any $m \in [y,z]$ and $\bar{m}$ its comparison point in $\triangle(\bar{x},\bar{y},\bar{z})$.*

Proof. By the comparison inequalities it follows that $d(x,m) \leq d(\bar{x},\bar{m})$. Now Proposition 9.1.19 in [2, pg. 314] says that if equality is reached then both triangles are isometric, which contradicts the fact that both triangles are of constant but different curvature. □

Theorem 5.11 *Any CAT(0) gluing space containing two spaces of constant but different curvature does not satisfy the (Q_4) condition.*

Proof. First we consider the case in which the gluing contains a geodesic segment. To illustrate this case we will only consider the particular gluing of M^2_{-1} and $\mathbb{R}^2$. For simplicity we will assume that the gluing segment supports non-degenerate triangles in both spaces, otherwise this can be reduced to the gluing through a singleton that we will see later. The general case follows then after applying some isometry techniques to triangles in the model spaces to fit them into the considered triangles of the gluing.

Let $[x,y] \subset M^2_{-1}$ and $[\bar{x},\bar{y}] \subset \mathbb{R}^2$ be two isometric geodesic segments, i.e, they have equal length. Let (X,d) be the metric space obtained by gluing the Euclidean plane and the hyperbolic plane along these segments. Let $\bar{z}$ be a point in $\mathbb{R}^2 \backslash [\bar{x},\bar{y}]$ such that $d(\bar{x},\bar{z}) = d(\bar{y},\bar{z})$. Using the existence of comparison triangles in M^2_{-1}, we can consider a point $z \in M^2_{-1}$ such that $d(z,x) = d(\bar{z},\bar{x}) = d(\bar{y},\bar{z}) = d(y,z)$.

Since we will reason in M^2_{-1} and $\mathbb{R}^2$ separately, we will treat the isometric segments as if they were different although they are not in X. Let $m \in [x,y]$ and $\bar{m} \in [\bar{x},\bar{y}]$ the mid-points of these segments. Then, from the above lemma, $d(z,m) < d(\bar{z},\bar{m})$. Now, by the formula of the cosines in $\mathbb{R}^2$,

$$d(\bar{z},\bar{m})^2 = \frac{d(\bar{y},\bar{z})^2}{2} + d(\bar{x},\bar{z})^2 - \Big(\frac{d(\bar{x},\bar{y})}{2}\Big)^2,$$

and so we can assure that $d(\bar{z},\bar{m})$ continuously depends on $d(\bar{z},\bar{x})$ and $d(\bar{z},\bar{y})$. Now we just need to shorten a little bit these distances to contradict the (Q_4) condition.

Let us suppose now that two spaces X and Y of constant curvature glue through a point w. Then $d(x,y) = d_X(x,w) + d_Y(w,y)$ for every $x \in X$ and $y \in Y$. We can assume that w is the vertex of a non-degenerate triangle in one of these spaces, say Y (notice that otherwise it would follow that both spaces X and Y are $\mathbb{R}$-trees and so they would not be of constant curvature as above defined). Consider $u, v \in Y$ so that $\triangle(w, u, v)$ is non-degenerate. Assume further that $d(w, v) = d(w, u)$ which imposes no restriction. Make p the projection of w onto the segment $[u, v]$, then $p \in (u, v)$. Let $c, c' : [0, 1] \to Y$ be proportionally parameterized geodesics with respect to the arc length of the segments $[w, v]$ and $[w, u]$ respectively. Then, from the reflection property of model spaces (see [1, Chapter I.2]) and the fact that Y is of constant curvature and so it is not an $\mathbb{R}$-tree, we have that $d(w, c(t)) = d(w, c'(t))$ for every t and that the segment $[w, p]$ intersects $[c(t), c'(t)]$ at its mid-point.

Now, guess for simplicity that $d(w, p) = 5/4$ (otherwise a simple re-scale would work the same), $q \in X$ is such that $d(w, q) = 3/4$, fix $t \in (0, 1)$ so that the mid-point m of $[c(t), c'(t)]$ satisfies $d(w, m) = 1/4$ and makes $x = c(t)$ and $y = c'(t)$. Now, a simple calculation with the corresponding Law of Cosines, implies that $d(x, p) = d(y, p) < d(x, q) = d(y, q)$ while $d(p, m) = d(q, m)$. The proof is finished after applying a continuity reasoning as in the above case. □

Condition (Q_4) was used in [12] to prove the following proposition.

Proposition 5.12 *Let X be a complete CAT(0) space with the (Q_4) condition, and suppose that (x_n) and (y_n) both Δ-converge to $p \in X$. Suppose $m_n \in [x_n, y_n]$ satisfies $d(x_n, m_n) = \lambda d(x_n, y_n)$ for fixed $\lambda \in (0, 1)$. Then (m_n) also Δ-converge to p.*

The authors of [12] ask if condition (Q_4) is necessary in this proposition. This question seems to make sense only in the absence of compactness since the above proposition trivially holds for proper CAT(0) spaces as in the case of Example 5.9. Of course, this answers in the negative this question. However we will see next that condition (Q_4) can be replaced by a weaker one which is still sufficient for a stronger version of Proposition 5.12.

Definition 5.13 *A complete CAT(0) space X has the property of the nice projection onto geodesics (property (N) for short) if, given any geodesic segment $I \subseteq X$ and P_I the metric projection onto I, it is the case that $P_I(m) \in [P_I(x), P_I(y)]$ for any x and y in X, and $m \in [x, y]$.*

Remark 5.14 *It is easy to see that among gluings given in Theorem 5.11, those which are obtained through singletons enjoy the (N) property if the original spaces do. The situation seems to be more complicated for gluings along geodesic segments. Still we do not know of any example of a CAT(k) space which does not enjoy the (N) property.*

Question. Does every complete CAT(0) space enjoy property (N)?

The following lemma shows the relation between the (Q_4) condition and the (N) property.

Lemma 5.15 *If a complete CAT(0) space enjoys the (Q_4) condition then it satisfies the (N) property.*

Proof. We first note that the (N) property trivially follows from the continuity of the projection P provided that whenever $u \in (x, y)$ with $P(u) = P(x)$ it is the case that $P(v) = P(x)$ for any $v \in [x, u]$. Now assume that X does not have the (N) property, then there exist $x, y \in X$ and $m \in (x, y)$, and a geodesic segment $I \subseteq X$ such that $P_I(x) = P_I(y) \neq P_I(m)$. Now make $p = P_I(x)$ and $q = P_I(m)$. Then, by Proposition 3.5, $d(p, x) < d(q, x)$, $d(p, y) < d(q, y)$ but $d(q, m) < d(p, m)$ which is a contradiction of the (Q_4) condition. □

Now we show that property (N) implies a stronger version of Proposition 5.12.

Theorem 5.16 *Let X be a complete CAT(0) space with property (N), and suppose that (x_n) and (y_n) both Δ-converge to $p \in X$. Suppose $m_n \in [x_n, y_n]$ for any $n \in \mathbb{N}$. Then (m_n) also Δ-converges to p.*

Proof. Since (x_n) and (y_n) both Δ-converge to p, Proposition 5.2 implies that both ϕ_p-converge to p. We will see that (m_n) also ϕ_p-converges to p. Let I be a geodesic segment containing p, then $\lim P_I(x_n) = \lim P_I(y_n) = p$ but since $P_I(m_n) \in [P_I(x_n), P_I(y_n)]$ for all n it also follows that $\lim P_I(m_n) = p$ which shows that m_n ϕ_p-converges to p and so it Δ-converges to p. □

6 The Lifšic characteristic and uniformly Lipschitzian mappings in CAT(k) spaces

In this section we first estimate the Lifšic characteristic for any CAT(k) space and second we study the property (P) in CAT(1) spaces. In both cases we obtain the corresponding fixed point results for uniformly lipschitzian mappings.

6.1 Lifšic characteristic in CAT(k) spaces

We begin with the estimation of the Lifšic characteristic in model spaces.

Proposition 6.1 *If $k < 0$, $\kappa(M_k^2) = \dfrac{\operatorname{arccosh}(\cosh^2 \sqrt{-k})}{\sqrt{-k}}$ for all $n \in \mathbb{N}$.*

Proof. For each $l \geq 1$, we are in the most unfavorable case to find some possible $\mu, \alpha \in (0,1)$ when the intersection of the balls is such that it contains two points that are at distance r one from each other. In order to find the smallest such $l \geq 1$ for which this happens, we consider the following situation:

Due to the isometry group on $\mathbb{H}^2$ (see, for instance, [18]) it will be enough to consider the points $x = (0,0,1) \in M_k^2$ and, for $u = (0,1,0) \in \mathbb{R}^3$, $y = (\cosh\sqrt{-k})x + (\sinh\sqrt{-k})u = (0, \sinh\sqrt{-k}, \cosh\sqrt{-k})$. By the definition of distance in M_k^2, we have $d(x,y) = 1$. Consider the points $c \in M_k^2$ which are at distance $r = 1$ from y. Then $c = (\cosh\sqrt{-k})y + (\sinh\sqrt{-k})v$, where v is such that $\langle v|v\rangle = 1$ and $\langle y|v\rangle = 0$. The geometry of the hyperbolic space shows us that the point c which gives us the smallest l is that corresponding to $v = (1,0,0)$ ($d(c, c^*) = 2$, if c^* is the point of the ball symmetric to c respect to $[x,y]$). Then

$$c = (\sinh\sqrt{-k}, \cosh(\sqrt{-k})\sinh\sqrt{-k}, \cosh^2\sqrt{-k}).$$

Thus, since

$$\cosh(\sqrt{-k}d(x,c)) = -\langle x|c\rangle = \cosh^2\sqrt{-k},$$

we have that

$$\kappa(M_k^2) = d(x,c) \leq \frac{\operatorname{arccosh}\ (\cosh^2\sqrt{-k})}{\sqrt{-k}}.$$

But since we were in the most unfavorable case, we deduce that in fact

$$\kappa(M_k^2) = \frac{\operatorname{arccosh}(\cosh^2\sqrt{-k})}{\sqrt{-k}}.$$

□

Proposition 6.2 *Let $k < 0$. If (X, d) is a complete CAT(k) space, then $\kappa(X) \geq \kappa(M_k^2)$.*

Proof. This basically follows in the same way as in the proof of Theorem 5 in [4], we write the proof for completeness. Let $r > 0$, choose $x, y \in X$ with $d(x, y) = r$ and let $\bar{x}, \bar{y} \in M_k^2$ be any two points with $d(x, y) = d(\bar{x}, \bar{y})$. Suppose that $l < \kappa(M_k^2)$. Then

$$\mathrm{rad}(B(\bar{x}, r) \cap B(\bar{y}, lr)) \leq \xi r$$

for some $\xi < 1$. Now choose $\alpha \in (\xi, 1)$. Then for $\mu \in (0, 1)$ sufficiently near 0 and α sufficiently near 1,

$$\mathrm{rad}(B(\bar{x}, (1 + \mu)r) \cap B(\bar{y}, l(1 + \mu)r)) \leq \alpha r,$$

with $d(\bar{x}, \bar{y}) \geq (1 - \mu)r$. Let

$$\bar{S} := B(\bar{x}, (1 + \mu)r) \cap B(\bar{y}, l(1 + \mu)r)$$

and

$$S := B(x, (1 + \mu)r) \cap B(y, l(1 + \mu)r).$$

Again by isometries in $\mathbb{H}^2$ (check [18], or, more precisely for this case, the remark on hyperplanes in $\mathbb{H}^n$ in [1, pg. 21]), the Chebyshev center $\bar{c}$ of $\bar{S}$ lies on the segment $[\bar{x}, \bar{y}]$. Also, if $u \in S$ and if $\triangle(\bar{y}, \bar{x}, \bar{u})$ is a comparison triangle for $\triangle(y, x, u)$ in M_k^2, then $\bar{u} \in \bar{S}$. Therefore $d(\bar{u}, \bar{c}) \leq \alpha r$. If c is the point of the segment $[x, y]$ for which $d(y, c) = d(\bar{y}, \bar{c})$, then $d(u, c) \leq d(\bar{u}, \bar{c}) \leq \alpha r$. From where the conclusion follows. □

Remark 6.3 *In [4] it was proved that $\kappa(X) \geq \sqrt{2}$ for any CAT(k) space with $k \leq 0$ and that $\kappa(X) = 2$ for X an $\mathbb{R}$-tree, then it was conjectured in Remark 1 that the Lifšic characteristic of a CAT(k) space for $k < 0$ is a continuous decreasing function on k which takes values in the interval $(\sqrt{2}, 2)$. Notice that the above two propositions together answer this conjecture in the positive.*

The next theorem sharpens Theorem 6 in [4].

Theorem 6.4 *Let $k < 0$. If (X, d) is a bounded complete CAT(k), then every uniformly l-lipschitzian mapping $T : X \to X$ with $l < \kappa(M_k^2)$ has a fixed point.*

Proof. It directly follows from Lifšic's Theorem (Theorem 2.17).□

Remark 6.5 *In this section we have only focused on the case CAT(k) with $k \leq 0$ for expository reasons. In a similar way it can be proved that, under adequate boundedness conditions,*

$$\kappa(X) = \frac{Arccos(\cos^2 \sqrt{k})}{\sqrt{k}}$$

for X a CAT(k) space with $k > 0$, where $Arccos(\cos^2(\sqrt{k}))$ must be understood as the value $\arccos(\cos^2(\sqrt{k}))$ *which varies in a continuous and increasing way with respect to k.*

6.2 Property (*P*) in CAT(1) spaces

In this section we show that every complete CAT(1) space under natural condition on the boundedness of its diameter has *property* (P).

Let $\{x_n\}$ be a bounded sequence in a metric space X. Define $\varphi : X \to \mathbb{R}$ by setting $\varphi(x) = \limsup_{n\to\infty} d(x, x_n), x \in X$.

Theorem 6.6 *Let X be a complete CAT(1) space. If $diam(X) < \pi/2$, then X has property (P).*

Proof. Let $\{x_n\}$ and $\{z_n\}$ be two bounded sequences in X and, as above, $\varphi(x) = \limsup_{n\to\infty} d(x, x_n)$ for $x \in X$. For each n, let

$$C_n := cov(\{z_j : j \geq n\}).$$

By Proposition 4.1 there exists a unique point $u_n \in C_n$ such that

$$\varphi(u_n) = \inf_{x \in C_n} \varphi(x).$$

Since $z_j \in C_n$ for all $j \geq n$, we have that $\varphi(u_n) \leq \varphi(z_j)$ whenever $j \geq n$. Therefore, $\varphi(u_n) \leq \limsup_{j\to\infty} \varphi(z_j)$ for all n. We claim that $\{u_n\}$ is a Cauchy sequence. To see this, suppose not. In this case, there exists $\varepsilon > 0$ such that for any $N \in \mathbb{N}$ there exist $i, j \geq N$ such that $d(u_i, u_j) \geq \varepsilon$. The sequence $\{\varphi(u_n)\}$ is increasing and bounded, and therefore convergent. Let $d := diam(X) < \pi/2$. Let $\xi > 0$ such that $\xi < \arccos\big(\cos(\frac{\varepsilon}{2}) \cos d\big) - d$, and choose N so large that $|\varphi(u_i) - \varphi(u_j)| \leq \xi$ if $i, j \geq N$. Now consider $i > j \geq N$ such that $d(u_i, u_j) \geq \varepsilon$. Let m_j be the mid-point of the geodesic segment joining u_i and u_j, and let $n \in \mathbb{N}$. Then, by the uniform convexity of X (see Proposition 3.1),

$$d(m_j, x_n) \leq \arccos\left(\frac{\cos(\max\{d(u_i, x_n), d(u_j, x_n)\})}{\cos(\frac{\varepsilon}{2})}\right),$$

or equally

$$\cos d(m_j, x_n) \geq \frac{\cos(\max\{d(u_i, x_n), d(u_j, x_n)\})}{\cos(\frac{\varepsilon}{2})}.$$

Then

$$\begin{aligned}
\liminf_n \cos d(m_j, x_n) &\geq \liminf_n \frac{\cos(\max\{d(u_i, x_n), d(u_j, x_n)\})}{\cos(\frac{\varepsilon}{2})} \\
&= \frac{1}{\delta} \liminf_n \cos(\max\{d(u_i, x_n), d(u_j, x_n)\}),
\end{aligned}$$

where $\delta := \cos(\frac{\varepsilon}{2}) < 1$.

Since the function *cosine* is decreasing in $[0, \pi/2]$, we have that

$$\cos(\limsup_n d(m_j, x_n)) = \cos \varphi(m_j) \geq \frac{1}{\delta} \cos(\limsup_n \max\{d(u_i, x_n), d(u_j, x_n)\}).$$

Thus,

$$\arccos(\delta \cos \varphi(m_j)) \leq \limsup_n \max\{d(u_i, x_n), d(u_j, x_n)\},$$

Since

$$\begin{aligned}
\limsup_n \max\{d(u_i, x_n), d(u_j, x_n)\} &= \max\{\limsup_n d(u_i, x_n), \limsup_n d(u_j, x_n)\} \\
&= \max\{\varphi(u_i), \varphi(u_j)\} \\
&= \frac{\varphi(u_i) + \varphi(u_j)}{2} + \frac{|\varphi(u_i) - \varphi(u_j)|}{2},
\end{aligned}$$

$$\arccos(\delta \cos \varphi(m_j)) \leq \varphi(u_j) + \xi.$$

Let $f(x) = \arccos(\delta \cos x) - x$, then $f'(x) \leq 0$ for all $x \in [0, d]$ and so

$$\arccos(\delta \cos x) - x \geq \arccos(\delta \cos d) - d = f(d)$$

for $x \in [0, d]$. Now, since $\xi < f(d)$, we have that

$$\varphi(m_j) \leq \arccos(\delta \cos \varphi(m_j)) - f(d) < \varphi(u_j),$$

which contradicts the definition of u_j. In consequence $\{u_n\}$ is a Cauchy sequence. Therefore, there exists $z \in X$ such that $\lim_{n\to\infty} u_n = z$ which obviously is in $\bigcap_{n=1}^{\infty} C_n$. Finally, from the continuity of φ and the fact that $\varphi(u_n) \leq \limsup_{j\to\infty} \varphi(z_j)$ for all n, we conclude that

$$\varphi(z) \leq \limsup_{j\to\infty} \varphi(z_j).$$

□

The corresponding fixed point theorem for uniformly lipschitzian mappings follows as immediate consequence of Theorem 2.18.

Theorem 6.7 *Let (X, d) be a complete bounded CAT(1) space. If diam$(X) < \pi/2$, then every uniformly k-lipschitzian mapping $T : X \to X$ with $k < \tilde{N}(X)^{-\frac{1}{2}}$ has a fixed point.*

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