On the modules of m-integrable derivations in non-zero characteristic

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Abstract

Let k be a commutative ring and A a commutative k-algebra. Given a positive integer m, or $m=\infty$, we say that a k-linear derivation δ of A is m-integrable if it extends up to a Hasse–Schmidt derivation $D=(\mathrm{Id}, D_1=\delta, D_2,\dots,D_m)$ of A over k of length m. This condition is automatically satisfied for any m under one of the following orthogonal hypotheses: (1) k contains the rational numbers and A is arbitrary, since we can take $D_i=\frac{\delta^i}{i!}$; (2) k is arbitrary and A is a smooth k-algebra. The set of m-integrable derivations of A over k is an A-module which will be denoted by $\mathrm{Ider}_k(A;m)$. In this paper we prove that, if A is a finitely presented k-algebra and m is a positive integer, then a k-linear derivation δ of A is m-integrable if and only if the induced derivation $\delta_{\mathfrak{p}}: A_{\mathfrak{p}}\to A_{\mathfrak{p}}$ is m-integrable for each prime ideal $\mathfrak{p}\subset A$. In particular, for any locally finitely presented morphism of schemes $f:X\to S$ and any positive integer m, the S-derivations of X which are locally m-integrable form a quasi-coherent submodule $\mathcal{I}der_S(\mathcal{O}_X;m)\subset \mathcal{D}er_S(\mathcal{O}_X)$ such that, for any affine open sets $U=\operatorname{Spec} A\subset X$ and $V=\operatorname{Spec} k\subset S$, with $f(U)\subset V$, we have $\Gamma(U,\mathcal{I}der_S(\mathcal{O}_X;m))=\mathrm{Ider}_k(A;m)$ and $\mathcal{I}der_S(\mathcal{O}_X;m)_p=\mathrm{Ider}_{\mathcal{O}_{S,f(p)}}(\mathcal{O}_{X,p};m)$ for each $p\in X$. We also give, for each positive integer m, an algorithm to decide whether all derivations are m-integrable or not.

Keywords: derivation; integrable derivation; Hasse–Schmidt derivation; differential operator
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Introduction

Let us start by recalling the algebraic interpretation of the integration of a vector field. Let X be a complex algebraic variety and χ an algebraic vector field on X, or, equivalently, a $\mathbb{C}$-derivation $\delta:\mathcal{O}_X\to\mathcal{O}_X$ of the sheaf of regular functions. Let us denote by $X[t]=\mathbb{A}^1_{\mathbb{C}}\times X$, $\mathbb{C}[\varepsilon]=\mathbb{C}[t]/(t^2)$, $X[\varepsilon]=\operatorname{Spec}\mathbb{C}[\varepsilon]\times X$ and $\overline{\delta}:X[\varepsilon]\to X$ the map of schemes determined by (and determining) δ: any section f of $\mathcal{O}_X$ is mapped to the section $f+\delta(f)\varepsilon$ of $\mathcal{O}_X[\varepsilon]$.

If X is nonsingular, we can consider the flow $\Theta:\mathcal{U}\to X^{\mathrm{an}}$ associated with χ^{an}, where $\mathcal{U}\subset X[t]^{\mathrm{an}}=\mathbb{C}\times X^{\mathrm{an}}$ is an open neighbourhood of $\mathcal{X}=\{0\}\times X^{\mathrm{an}}$.

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It turns out that for any holomorphic (or algebraic) function f on an open set $V \subset X^{\mathrm{an}}$, the function $\Theta^*(f) = f \circ \Theta$ is given by

$$(t,p) \in \Theta^{-1}(V) \subset \mathbb{C} \times X^{\mathrm{an}} \mapsto \sum_{i=0}^{\infty} t^i \frac{\delta^i(f)}{i!}(p) \in \mathbb{C}$$

for $|t|$ small enough. Hence, the formal completion of Θ along $\mathcal{X}$, $\widehat{\Theta} : \widehat{\mathcal{U}} = \widehat{X[t]^{\mathrm{an}}} \to X^{\mathrm{an}}$, comes from the purely (formal) algebraic map $\widehat{X[t]} \to X$ associated with the *exponential map* $e^{t\delta} : \mathcal{O}_X \to \mathcal{O}_X[[t]]$ attached to χ (or to δ) defined as

$$e^{t\delta}(f) = \sum_{i=0}^{\infty} t^i \frac{\delta^i(f)}{i!}$$

for any regular function f on some Zariski open set of X.

The exponential map $e^{t\delta}$ is a lifting of δ (it coincides with $\overline{\delta} \mod t^2$) and it can be regarded as the algebraic incarnation of the integration of the vector field χ.

The exponential map of a vector field makes sense not only over the complex numbers, but over any field of characteristic zero, and in fact it also works if X is eventually singular. However, it does not make sense over a field k of positive characteristic.

Nevertheless, the notion of *Hasse–Schmidt derivation* allows us to define what integrability means for a vector field in such a case (see [1, 9]). Given a commutative ring k and a commutative k-algebra A, a Hasse–Schmidt derivation of A over k (of length ∞) is a sequence $D = (\mathrm{Id}, D_1, D_2, D_3, \dots)$ of k-linear operators of A which appear as the coefficients of a k-algebra map $\Phi : A \to A[[t]]$ such that $\Phi(a) \equiv a \mod t$ for all $a \in A$: $\Phi(a) = a + D_1(a)t + D_2(a)t^2 + \cdots$. That property is equivalent to the fact that the D_i satisfy the Leibniz equality:

$$D_0 = \mathrm{Id}, \quad D_i(ab) = \sum_{r+s=i} D_r(a) D_s(b) \quad \forall a, b \in A,\ \forall i \geq 1.$$

A k-linear derivation $\delta : A \to A$ is said to be (∞-)integrable if there is a Hasse–Schmidt derivation D of A over k (of length ∞) such that $D_1 = \delta$, or in other words, if the k-algebra map $\overline{\delta} : a \in A \mapsto a + \delta(a)\varepsilon \in A[\varepsilon] = A[[t]]/(t^2)$ can be lifted up to a k-algebra map $\Phi : A \to A[[t]]$. The set of k-linear derivations of A which are integrable is a submodule of $\mathrm{Der}_k(A)$, which is denoted by $\mathrm{Ider}_k(A)$.

When A is a smooth k-algebra over an arbitrary commutative ring k or when k contains the rational numbers, any k-linear derivation $\delta : A \to A$ is (∞-)integrable. The modules $\mathrm{Ider}_k(A)$, and more generally, the Hasse–Schmidt derivations of A over k seem to play an important role among the differential structures in Commutative Algebra and Algebraic Geometry (see [17], [12]). They behave better in positive characteristic than $\mathrm{Der}_k(A)$ (see for instance [11] or [13]) and one expects that they can help to understand (some of) the differences between singularities in zero and non-zero characteristics, but they are difficult to deal with. For instance, it is not clear at all that (∞-)integrability is a local property (in the sense that can be tested locally at the primes ideals of A).

For a given positive integer m, the m-integrability of a k-linear derivation $\delta : A \to A$ is defined as the existence of a k-algebra map $\Phi : A \to A[[t]]/(t^{m+1})$ lifting the map $\overline{\delta}$ defined above. The set of k-linear derivations of A which are m-integrable is a submodule of $\operatorname{Der}_k(A)$, which is denoted by $\operatorname{Ider}_k(A; m)$. One obviously has $\operatorname{Der}_k(A) = \operatorname{Ider}_k(A; 1) \supset \operatorname{Ider}_k(A; 2) \supset \operatorname{Ider}_k(A; 3) \supset \cdots \supset \operatorname{Ider}_k(A; \infty) = \operatorname{Ider}_k(A)$.

This paper is devoted to the study of the modules $\operatorname{Ider}_k(A; m)$, for $m \geq 1$.

One of the main difficulties when dealing with m-integrability of a derivation is that one cannot proceed step by step: a derivation δ can be $(m+r)$-integrable, but it may have an intermediate m-integral $D = (\operatorname{Id}, D_1 = \delta, D_2, \ldots, D_m)$ which does not extends up to a Hasse–Schmidt derivation of length $(n+r)$ (cf. Example 3.7 in [12]).

Our main results are the following:

(I) If A is a finitely presented k-algebra and m is a positive integer, then the property of being m-integrable for a k-derivation δ of A is a local property, i.e. δ is m-integrable if and only if the induced derivation $\delta_{\mathfrak{p}} : A_{\mathfrak{p}} \to A_{\mathfrak{p}}$ is m-integrable for each prime ideal $\mathfrak{p} \subset A$. As a consequence, for any locally finitely presented morphism of schemes $f : X \to S$ and any positive integer m, the S-derivations of X which are locally m-integrable form a quasi-coherent submodule $\mathit{Ider}_S(\mathcal{O}_X; m) \subset \mathit{Der}_S(\mathcal{O}_X)$ such that, for any affine open sets $U = \operatorname{Spec} A \subset X$ and $V = \operatorname{Spec} k \subset S$, with $f(U) \subset V$, we have $\Gamma(U, \mathit{Ider}_S(\mathcal{O}_X; m)) = \operatorname{Ider}_k(A; m)$ and $\mathit{Ider}_S(\mathcal{O}_X; m)_p = \operatorname{Ider}_{\mathcal{O}_{S,f(p)}}(\mathcal{O}_{X,p}; m)$ for each $p \in X$ (see Theorem (2.3.6) and Corollary (2.3.7)). We have then a decreasing sequence of quasi-coherent modules

$$\mathit{Der}_S(\mathcal{O}_X) = \mathit{Ider}_S(\mathcal{O}_X; 1) \supset \mathit{Ider}_S(\mathcal{O}_X; 2) \supset \mathit{Ider}_S(\mathcal{O}_X; 3) \supset \cdots$$

and all the quotients $\mathit{Der}_S(\mathcal{O}_X)/\,\mathit{Ider}_S(\mathcal{O}_X; m)$ are supported by the non-smoothness locus of $f : X \to S$.

(II) For a given k-algebra A and for any positive integer m, there is a constructive procedure to see whether <u>all</u> k-derivations of A are m-integrable or not. In particular, if A and k are "computable" rings, then the above procedure becomes an effective algorithm (although of exponential complexity with respect to m) to decide whether the equality $\operatorname{Ider}_k(A; m) = \operatorname{Der}_k(A)$ is true or not (see 2.5).

Let us now comment on the content of this paper.

In section 1 we review the notion of Hasse–Schmidt derivation and its basic properties. We study logarithmic Hasse–Schmidt derivations with respect to an ideal I of some ambient algebra A and their relationship with Hasse–Schmidt derivations of the quotient A/I. In the last part we focus on the description of Hasse–Schmidt derivations on polynomial or power series algebras.

Section 2 contains the main results of this paper. First, we define m-integrability and logarithmic m-integrability and give a characterization of $(m+1)$-integrability for a Hasse–Schmidt derivation of length m. In section 2.2 we give some criteria for a derivation to be integrable, based on and extending previous results of [9] and [15]. Next, we study the behaviour of m-integrability under localization, for finite m, and prove (I) above. In the last part we prove the results needed to justify procedure (II) above.

In Section 3 we first compute some concrete examples and illustrate the nonlinear equations one encounters when computing systems of generators of the modules $\mathrm{Ider}_k(A;m)$. In the second part we state some questions, which seem to be important for understanding the relationship between the modules of m-integrable derivations and singularities.

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1 Notations and preliminaries

1.1 Notations

Throughout the paper we will use the following notations:

-) k will be a commutative ring and A a commutative k-algebra.

-) $\mathbb{N}_+ := \{n \in \mathbb{N} \mid n \geq 1\}$, $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$, $\overline{\mathbb{N}}_+ := \mathbb{N}_+ \cup \{\infty\}$.

-) If $n \in \mathbb{N}_+$, $[n] := \{0, 1, \ldots, n\}$, $[n]_+ := [n] \cap \mathbb{N}_+$ and $[\infty] := \mathbb{N}$.

-) If $n \in \mathbb{N}_+$, $A_n := A[[t]]/(t^{n+1})$ and $A_\infty = A[[t]]$. Each A_n is an augmented A-algebra, the augmentation ideal $\ker(A_n \to A)$ being generated by t.

-) For $n \in \overline{\mathbb{N}}_+$ and $m \in [n]_+$, let us denote by $\pi_{nm} : A_n \to A_m$ the natural epimorphism of augmented A-algebras.

-) If $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, $\operatorname{supp} \alpha = \{r \in \{1, \ldots, d\} \mid \alpha_r \neq 0\}$ and $|\alpha| := \alpha_1 + \cdots + \alpha_d$.

-) The ring of k-linear differential operators of A will be denoted by $\mathrm{Diff}_{A/k}$ (see [5]).

-) For $A = k[x_1, \ldots, x_d]$ or $A = k[[x_1, \ldots, x_d]]$, we will denote by $\partial_r : A \to A$ the partial derivative with respect to x_r.

1.2 Hasse-Schmidt derivations

In this section we remind the definition and basic facts of Hasse–Schmidt derivations (see [6],[10], §27, and [14], [17], [12] for more recent references). We also introduce the basic constructions that will be used throughout the paper.

(1.2.1) Definition. *A* Hasse–Schmidt derivation *of A (over k) of length $n \geq 1$ (resp. of length ∞) is a sequence $D = (D_i)_{i \in [n]}$ of k-linear maps $D_i : A \longrightarrow A$, satisfying the conditions:*

$$D_0 = \mathrm{Id}_A, \quad D_i(xy) = \sum_{r+s=i} D_r(x) D_s(y)$$

for all $x, y \in A$ and for all $i \in [n]$. We denote by $\mathrm{HS}_k(A;n)$ the set of all Hasse–Schmidt derivations of A (over k) of length $n \in \overline{\mathbb{N}}$ and $\mathrm{HS}_k(A) = \mathrm{HS}_k(A;\infty)$.

(1.2.2) The D_1 component of any Hasse-Schmidt derivation $D \in \mathrm{HS}_k(A;n)$ is a k-derivation of A. More generally, the D_i component is a k-linear differential operator of order $\leq i$ with $D_i(1) = 0$ for $i = 1, \ldots, n$.

(1.2.3) Any Hasse–Schmidt derivation $D \in \mathrm{HS}_k(A;n)$ is determined by the k-algebra homomorphism $\Phi_D : A \to A_n$ defined by $\Phi_D(a) = \sum_{i=0}^n D_i(a)t^i$ and satisfying $\Phi_D(a) \equiv a \mod t$. The k-algebra homomorphism Φ_D can be uniquely extended to a k-algebra automorphism $\widetilde{\Phi}_D : A_n \to A_n$ with $\widetilde{\Phi}_D(t) = t$:

$$\widetilde{\Phi}_D\left(\sum_{i=0}^n a_i t^i\right) = \sum_{i=0}^n \Phi(a_i)t^i.$$

So, there is a bijection between $\mathrm{HS}_k(A;n)$ and the subgroup of $\mathrm{Aut}_{k-\mathrm{alg}}(A_n)$ consisting of the automorphisms $\widetilde{\Phi}$ satisfying $\widetilde{\Phi}(a) \equiv a \mod t$ for all $a \in A$ and $\widetilde{\Phi}(t) = t$. In particular, $\mathrm{HS}_k(A;n)$ inherits a canonical group structure which is explicitly given by $D \circ D' = D''$ with $D''_l = \sum_{i+j=l} D_i \circ D'_j$, the identity element of $\mathrm{HS}_k(A;n)$ being $(\mathrm{Id}_A, 0, 0, \dots)$. It is clear that the map $(Id_A, D_1) \in \mathrm{HS}_k(A;1) \mapsto D_1 \in \mathrm{Der}_k(A)$ is an isomorphism of groups, where we consider the addition as internal operation in $\mathrm{Der}_k(A)$.

(1.2.4) For any $a \in A$ and any $D \in \mathrm{HS}_k(A;n)$, the sequence $a \bullet D$ defined by $(a \bullet D)_i = a^i D_i$, $i \in [n]$, is again a Hasse–Schmidt derivation of A over k of length n and $\Phi_{a \bullet D}(b)(t) = \Phi_D(b)(at)$ for all $b \in A$. We have $(aa') \bullet D = a \bullet (a' \bullet D)$, $1 \bullet D = D$ and $0 \bullet D =$ the identity element.

(1.2.5) For $1 \leq m \leq n \in \overline{\mathbb{N}}$, let us denote by $\tau_{nm} : \mathrm{HS}_k(A;n) \to \mathrm{HS}_k(A;m)$ the *truncation* map defined in the obvious way. One has $\Phi_{\tau_{nm} D} = \pi_{nm} \circ \Phi_D$. Truncation maps are group homomorphisms and they satisfy $\tau_{nm}(a \bullet D) = a \bullet \tau_{nm} D$. It is clear that the group $\mathrm{HS}_k(A;\infty)$ is the inverse limit of the groups $\mathrm{HS}_k(A;m)$, $m \in \mathbb{N}$.

(1.2.6) Definition. *Let $q \geq 1$ be an integer or $q = \infty$, and $D \in \mathrm{HS}_k(A;q)$. For each integer $m \geq 1$ we define $D[m]$ as the Hasse–Schmidt derivation (over k) of length mq determined by the k-algebra map obtained by composing the following maps:*

$$A \xrightarrow{\Phi_D} A_q = A[[t]]/(t^{q+1}) \xrightarrow{\bar{t} \mapsto \bar{t}^m} A_{mq} = A[[t]]/(t^{mq+1}).$$

In the case $q = 1$ and $D = (\mathrm{Id}_A, \delta)$, we simply denote $\delta[m] := D[m]$.

If $D = (\mathrm{Id}_A, D_1, D_2, \dots) \in \mathrm{HS}_k(A;q)$, then

$$D[m] = (\mathrm{Id}_A, 0, \dots, 0, \overbrace{D_1}^{m}, 0, \dots, 0, \overbrace{D_2}^{2m}, 0, \dots) \in \mathrm{HS}_k(A;mq).$$

The map $D \in \mathrm{HS}_k(A;q) \mapsto D[m] \in \mathrm{HS}_k(A;qm)$ is a group homomorphism and we have $(a^m \bullet D)[m] = a \bullet D[m]$, $(\tau_{qq'} D)[m] = \tau_{qm,q'm}(D[m])$ for $a \in A, 1 \leq q' \leq q$.

(1.2.7) Definition. *For each $n \in \overline{\mathbb{N}}_+$ and each $E \in \mathrm{HS}_k(A;n)$, we denote $\ell(E) = 0$ if $E_1 \neq 0$, $\ell(E) = n$ if E is the identity and $\ell(E) =$ maximun of the $r \in [n]$ such that $E_1 = \cdots = E_r = 0$ otherwise.*

(1.2.8) Definition. *Let $I \subset A$ be an ideal and $m \in \overline{\mathbb{N}}_+$. We say that:*

1) *A k-derivation $\delta : A \to A$ is* I-logarithmic *if $\delta(I) \subset I$. The set of k-linear derivations of A which are I-logarithmic is denoted by* $\mathrm{Der}_k(\log I)$.

2) *A Hasse–Schmidt derivation* $D \in \mathrm{HS}_k(A;m)$ *is called* I-logarithmic *if* $D_i(I) \subset I$ *for any* $i \in [m]$. *The set of Hasse–Schmidt derivations* $D \in \mathrm{HS}_k(A;m)$ *which are* I*-logarithmic is denoted by* $\mathrm{HS}_k(\log I;m)$. *When* $m=\infty$ *it will be simply denoted by* $\mathrm{HS}_k(\log I)$.

The set $\mathrm{Der}_k(\log I)$ is obviously a A-submodule of $\mathrm{Der}_k(A)$. Any $\delta \in \mathrm{Der}_k(\log I)$ gives rise to a unique $\overline{\delta} \in \mathrm{Der}_k(A/I)$ satisfying $\overline{\delta}\circ\pi = \pi\circ\delta$, where $\pi : A \to A/I$ is the natural projection. Moreover, if $A = k[x_1,\dots,x_d]$ or $A = k[[x_1,\dots,x_d]]$, the sequence of A-modules

$$0 \to I\,\mathrm{Der}_k(A) \xrightarrow{\text{incl.}} \mathrm{Der}_k(\log I) \xrightarrow{\delta\mapsto\overline{\delta}} \mathrm{Der}_k(A/I) \to 0$$

is exact.

(1.2.9) In the same vein, the set $\mathrm{HS}_k(\log I;m)$ is a subgroup of $\mathrm{HS}_k(A;m)$ and we have $A\bullet \mathrm{HS}_k(\log I;m) \subset \mathrm{HS}_k(\log I;m)$, $\mathrm{HS}_k(\log I;m)[n] \subset \mathrm{HS}_k(\log I;mn)$, $n\in\mathbb{N}$. A $D \in \mathrm{HS}_k(A;m)$ is I-logarithmic if and only if its corresponding k-algebra homomorphism $\Phi_D : A \to A_m$ satisfies $\Phi_D(I) \subset I_m := \ker \pi_m$, where $\pi_m : A_m \to (A/I)_m$ is the natural projection[1]. Moreover, a I-logarithmic Hasse–Schmidt derivation $D \in \mathrm{HS}_k(\log I;m)$ gives rise to a unique $\overline{D} \in \mathrm{HS}_k(A/I;m)$ such that $\overline{D}_i\circ\pi = \pi\circ D_i$ for all $i\in[m]$, and the following diagram is commutative

$$\begin{tikzcd} A \arrow[r, "\Phi_D"] \arrow[d, "\pi"] & A_m \arrow[d, "\pi_m"] \\ A/I \arrow[r, "\Phi_{\overline{D}}"] & (A/I)_m . \end{tikzcd}$$

The map $\Pi_m : D \in \mathrm{HS}_k(\log I;m) \to \overline{D} \in \mathrm{HS}_k(A/I;m)$ is clearly a homomorphism of groups and $\Pi_m(a\bullet D) = \pi(a)\bullet\Pi_m(D)$. So, its kernel contains the subgroup $I\bullet \mathrm{HS}_k(A;m)$ generated by the $a\bullet E$, with $a\in I$ and $E\in\mathrm{HS}_k(A;m)$. It is also clear that $\tau_{mn}\circ\Pi_m = \Pi_n\circ\tau_{mn}$ and $(\Pi_m D)[n] = \Pi_{mn}(D[n])$.

(1.2.10) Let $S\subset A$ be a multiplicative set. For each k-linear differential operator $P : A\to A$, let us denote by $\widetilde{P} : S^{-1}A \to S^{-1}A$ its canonical extension. We know that the map $P \in \mathrm{Diff}_{A/k} \mapsto \widetilde{P} \in \mathrm{Diff}_{S^{-1}A/k}$ is a ring homomorphism. Let $m\geq 1$ be an integer or $m=\infty$ and $\mathfrak{a}\subset A$ an ideal. Here is a summary of the basic facts of the behaviour of Hasse-Schmidt derivations under localization:

-) For any $D=(D_i)\in \mathrm{HS}_k(A;m)$, the sequence $\widetilde{D} := (\widetilde{D_i})$ is a Hasse-Schmidt derivation of $S^{-1}A$ (over k of length m) and the following diagram is commutative

$$\begin{tikzcd} A \arrow[r, "\Phi_D"] \arrow[d, "\text{can.}"] & A_m \arrow[d, "\text{can.}"] \\ S^{-1}A \arrow[r, "\Phi_{\widetilde{D}}"] & (S^{-1}A)_m . \end{tikzcd}$$

Moreover, if D is $\mathfrak{a}$-logarithmic, then $\widetilde{D}$ is $(S^{-1}\mathfrak{a})$-logarithmic.

[1]Observe that $\ker\pi_m = IA_m$ when I is finitely generated or m is finite.

-) The map $\Theta_m : D \in \mathrm{HS}_k(A;m) \to \widetilde{D} \in \mathrm{HS}_k(S^{-1}A;m)$ is a group homomorphism, $\Theta_m(a \bullet D) = \frac{a}{1} \bullet \Theta_m(D)$ and the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{HS}_k(\log \mathfrak{a};m) \arrow[r, "\Theta_m"] \arrow[d, "\Pi_m"] & \mathrm{HS}_k(\log(S^{-1}\mathfrak{a});m) \arrow[d, "\Pi_m"] \\
\mathrm{HS}_k(A/\mathfrak{a};m) \arrow[r, "\Theta_m"] & \mathrm{HS}_k(S^{-1}A/S^{-1}\mathfrak{a};m).
\end{tikzcd}$$

Moreover, $\tau_{mn} \circ \Theta_m = \Theta_n \circ \tau_{mn}$ and $(\Theta_m D)[n] = \Theta_{mn}(D[n])$.

The extension of Hasse–Schmidt derivations to rings of fractions is a particular case of the formally étale extensions (cf. [8] and [15], th. 1.5).

1.3 Hasse–Schmidt derivations of polynomial or formal power series algebras

Throughout this section we assume that $A = k[x_1,\dots,x_d]$ or $A = k[[x_1,\dots,x_d]]$. The *Taylor differential operators* $\Delta^{(\alpha)} : A \to A$, $\alpha \in \mathbb{N}^d$, are defined by:

$$g(x_1+T_1,\dots,x_d+T_d) = \sum \Delta^{(\alpha)}(g)T^\alpha, \quad \forall g \in A.$$

It is well known that $\{\Delta^{(\alpha)}\}_{|\alpha|\leq i}$ is a basis of the left (resp. right) A-module of k-linear differential operators of A of order $\leq i$. So, if $D \in \mathrm{HS}_k(A;m)$, there are unique $C^i_\alpha \in A$, $\alpha \in \mathbb{N}^d$, $0 < i \leq |\alpha| \in [m]_+$, such that $D_i = \sum_{0<|\alpha|\leq i} C^i_\alpha \Delta^{(\alpha)}$, $i \in [m]_+$. On the other hand, there are unique $c_{ri} \in A$, $i \in [m]_+$, $1 \leq r \leq d$, such that

$$\Phi_D(x_r) = x_r + \sum_{i=1}^m c_{ri}t^i, \quad 1 \leq r \leq d.$$

In fact, any system of $c_{ri} \in A$, $i \in [m]_+$, $1 \leq r \leq d$, determines uniquely such a homomorphism of k-algebras $A \to A_m$ and so a Hasse–Schmidt derivation $D \in \mathrm{HS}_k(A;m)$.

The following proposition gives the relationship between the C^i_α and the c_{ri} above. Its proof does not contain any surprise and it is left up to the reader.

(1.3.1) Proposition. *With the above notations, the following properties hold:*

1) $c_{ri} = D_i(x_r) = C^i_{e_r}$*, with* $e_r = (0,\dots,\overbrace{1}^{r},\dots,0)$*, for all* $i \in [m]_+$*,* $r = 1,\dots,d$*.*

2)

$$C^i_\alpha = \sum_{\substack{\{\varepsilon_r\}_{r\in\mathrm{supp}\,\alpha} \\ \varepsilon_r \geq \alpha_r, |\varepsilon|=i}} \left(\prod_{r\in\mathrm{supp}\,\alpha} \left(\sum_{\substack{\beta_1+\cdots+\beta_{\alpha_r}=\varepsilon_r \\ \beta_k>0}} \prod_{k=1}^{\alpha_r} c_{r,\beta_k} \right)\right)$$

for all $\alpha \in \mathbb{N}^d$*,* $|\alpha| \in [m]_+$*,* $0 < i \leq |\alpha|$*.*

The above proposition is a particular case of Theorem 2.8 in [3]. For the sake of completeness we include, without proof, the following result.

(1.3.2) Proposition. *Let $C^i_\alpha \in A$, $\alpha \in \mathbb{N}^d$, $0 < i \leq |\alpha| \in [m]_+$, be a system of elements of A and define $D_0 = \mathrm{Id}_A$, $D_i = \sum_{0<|\alpha|\leq i} C^i_\alpha \Delta^{(\alpha)}$, $i \in [m]_+$. The following properties are equivalent:*

(a) The sequence $D = (D_i)_{i\in[m]}$ is a Hasse–Schmidt derivation of A over k of length m.

(b) For all $i \in [m], i \geq 2$, for all $\varrho \in \mathbb{N}^d$ with $2 \leq |\varrho| \leq i$ and for all $\beta, \gamma \in \mathbb{N}^d$ with $\varrho = \beta + \gamma$, $|\beta|, |\gamma| > 0$ we have $C^i_\varrho = \sum C^j_\beta C^l_\gamma$, where the summation indexes are the (j, l) with $j \geq |\beta|, l \geq |\gamma|$ and $j + l = i$.

Let us notice that, if the equivalent properties of the preceding proposition hold, then the C^i_α with $2 \leq |\alpha| \leq i$ are determined by the C^j_β with $1 \leq |\beta| \leq j \leq i-1$. This applies in particular to the symbol of the D_i, $\sigma(D_i) = \sum_{|\alpha|=i} C^i_\alpha \xi^\alpha$, which only depend on D_1 (compare with Proposition 2.6 in [12]).

(1.3.3) Definition. *The* Taylor Hasse-Schmidt derivations *of A are the*

$$\Delta^{(s)} := (\mathrm{Id}_A, \Delta^{(s)}_1, \Delta^{(s)}_2, \Delta^{(s)}_3, \dots) \in \mathrm{HS}_k(A), \quad 1 \leq s \leq d,$$

where $\Delta^{(s)}_i = \Delta^{(0,\dots,\overbrace{i}^{s},\dots,0)}$ *for each* $i \geq 1$.

(1.3.4) Proposition. *Assume that $R = k[x_1, \dots, x_d]$, $S \subset R$ is a multiplicative set and $A = S^{-1}R$ or $A = k[[x_1, \dots, x_d]]$. For any ideal $I \subset A$, the group homomorphisms $\Pi_m : \mathrm{HS}_k(\log I; m) \to \mathrm{HS}_k(A/I; m)$, $m \in \overline{\mathbb{N}}$, (see (1.2.9)) are surjective.*

Proof. Let us prove the proposition in the case $A = S^{-1}R$, the case $A = k[[x_1, \dots, x_d]]$ being completely similar. Let us call $\sigma : R \to A, \pi : A \to A/I, \pi_m : A_m \to (A/I)_m$ the canonical maps and let $E \in \mathrm{HS}_k(A/I; m)$ be any Hasse–Schmidt derivation. Let $a_{ri} \in A$ be elements such that

$$\Phi_E(\pi(\sigma(x_r))) = \pi(\sigma(x_r)) + \sum_{i\in[m]} \pi(a_{ri}) t^i \in (A/I)_m, \quad r = 1, \dots, d,$$

and let $\Psi : R \to A_m$ be the k-algebra map defined by

$$\Psi(x_r) = \sigma(x_r) + \sum_{i\in[m]} a_{ri} t^i \in A_m, \quad r = 1, \dots, d.$$

Since $\Psi(f) \equiv \sigma(f) \mod t$ for each $f \in R$, we deduce that $\Psi(s)$ is invertible for all $s \in S$ and the map Ψ induces $\widetilde{\Psi} : A \to A_m$. It is clear that $\widetilde{\Psi}(a) \equiv a \mod t$ for each $a \in A$ and $\pi_m \circ \widetilde{\Psi} = \Phi_E \circ \pi$. So, $\widetilde{\Psi}$ induces a I-logarithmic Hasse-Schmidt derivation $D \in \mathrm{HS}_k(\log I; m)$ such that $\Pi_m(D) = E$ (see (1.2.9)). Q.E.D.

(1.3.5) Proposition. *Assume that $R = k[x_1, \dots, x_d]$, $S \subset R$ is a multiplicative set and let $\mathfrak{a} \subset R$ be a finitely generated ideal. For any (finite) integer $m \geq 1$, the map*

$$(s, D) \in S \times \mathrm{HS}_k(\log \mathfrak{a}; m) \mapsto \frac{1}{s} \bullet \Theta_m(D) \in \mathrm{HS}_k(\log(S^{-1}\mathfrak{a}); m)$$

is surjective.

PROOF. Let $E \in \mathrm{HS}_k(\log(S^{-1}\mathfrak{a}); m)$ be any $(S^{-1}\mathfrak{a})$-logarithmic Hasse–Schmidt derivation. Since m is finite, there are $a_{ij} \in R$, $1 \leq i = 1 \leq d$, $1 \leq j \leq m$ and $\sigma \in S$ such that

$$\Phi_E\left(\frac{x_i}{1}\right) = \frac{x_i}{1} + \left(\frac{a_{i1}}{\sigma}\right) t + \cdots + \left(\frac{a_{im}}{\sigma}\right) t^m \in (S^{-1}R)_m, \quad i = 1, \ldots, d.$$

Let us consider the k-algebra map $\Phi^0 : R \to R_m$ given by

$$\Phi^0(x_i) = x_i + a_{i1}t + \sigma a_{i2}t^2 + \cdots + \sigma^{m-1} a_{im} t^m \in R_m, \quad i = 1, \ldots, d$$

and the corresponding Hasse-Schmidt derivation $D^0 \in \mathrm{HS}_k(R; m)$ with $\Phi^0 = \Phi_{D^0}$. It is clear that $\left(\frac{\sigma}{1}\right) \bullet E = \Theta_m(D^0)$. Let $f_1, \ldots, f_u \in \mathfrak{a}$ be a finite system of generators. Since $\Theta_m(D^0)$ is $(S^{-1}\mathfrak{a})$-logarithmic, we deduce the existence of a $\tau \in S$ such that $\tau\Phi_{D^0}(f_l) \in A_m\mathfrak{a}$ for all $l = 1, \ldots, u$. So, $D := \tau \bullet D^0$ is $\mathfrak{a}$-logarithmic and $E = \left(\frac{1}{\sigma\tau}\right) \bullet \Theta_m(D)$. Q.E.D.

Proposition (1.3.5) is false for $m = \infty$, as shown for instance in example 1.4 in [15].

(1.3.6) COROLLARY. *Assume that A is a finitely presented k-algebra and let $T \subset A$ be a multiplicative set. Then, for any (finite) integer $m \geq 1$, the map*

$$(t, E) \in T \times \mathrm{HS}_k(A; m) \mapsto \frac{1}{t} \bullet \Theta_m(E) \in \mathrm{HS}_k(T^{-1}A; m)$$

is surjective.

PROOF. We may assume that $A = R/\mathfrak{a}$ with $R = k[x_1, \ldots, x_d]$ and $\mathfrak{a} \subset R$ a finitely generated ideal. Denote by $\pi : R \to A$ the natural projection and $S = \pi^{-1}(T)$. We have $T^{-1}A = S^{-1}R/S^{-1}\mathfrak{a}$. Let us look at the following commutative diagram

$$\begin{tikzcd}
S \times \mathrm{HS}_k(\log \mathfrak{a}; m) \arrow[r] \arrow[d, "\pi \times \Pi_m"] & \mathrm{HS}_k(\log(S^{-1}\mathfrak{a}); m) \arrow[d, "\Pi_m"] \\
T \times \mathrm{HS}_k(A; m) \arrow[r] & \mathrm{HS}_k(T^{-1}A; m).
\end{tikzcd}$$

The vertical arrows are surjective by Proposition (1.3.4). To conclude, we apply Proposition (1.3.5). Q.E.D.

2 Integrability

2.1 Integrable Hasse–Schmidt derivations

In this subsection, A will be again an arbitrary k-algebra.

(2.1.1) DEFINITION. *(Cf. [1, 9]) We say that a k-derivation $\delta : A \to A$ is n-*integrable *(over k), with $n \in \overline{\mathbb{N}}$, if there is a Hasse–Schmidt derivation $D \in \mathrm{HS}_k(A; n)$ such that $D_1 = \delta$. A such D will be called a n-*integral *of δ. The set of n-integrable k-derivations of A is denoted by* $\mathrm{Ider}_k(A; n)$. *We simply say that δ is* integrable *if it is ∞-integrable and we denote* $\mathrm{Ider}_k(A) = \mathrm{Ider}_k(A; \infty)$.

*More generally, we say that a Hasse–Schmidt derivation $D' \in \mathrm{HS}_k(A;m)$ is n-*integrable *(over k), with $m, n \in \overline{\mathbb{N}}, n \geq m$, if there is a Hasse–Schmidt derivation $D \in \mathrm{HS}_k(A;n)$ such that $\tau_{nm} D = D'$. A such D will be called a n-*integral *of D'. The set of n-integrable Hasse–Schmidt derivations of A over k of length m is denoted by $\mathrm{IHS}_k(A;m;n)$. We simply say that D' is* integrable *if it is ∞-integrable and we denote $\mathrm{IHS}_k(A;m) = \mathrm{IHS}_k(A;m;\infty)$.*

It is clear that the $\mathrm{Ider}_k(A;n)$ are A-submodules of $\mathrm{Der}_k(A)$, $\mathrm{Der}_k(A) = \mathrm{Ider}_k(A;1) \supset \mathrm{Ider}_k(A;2) \supset \mathrm{Ider}_k(A;3) \supset \cdots$ and

$$\mathrm{Ider}_k(A) \subset \bigcap_{n \in \mathbb{N}_+} \mathrm{Ider}_k(A;n). \tag{1}$$

It is also clear that the $\mathrm{IHS}_k(A;m;n)$ are subgroups of $\mathrm{IHS}_k(A;m)$, stable by the $\bullet$ operation, $\mathrm{IHS}_k(A;m) = \mathrm{IHS}_k(A;m;m) \supset \mathrm{IHS}_k(A;m;m+1) \supset \cdots$ and

$$\mathrm{IHS}_k(A;m) \subset \bigcap_{n \geq m} \mathrm{IHS}_k(A;m;n). \tag{2}$$

(2.1.2) Example. (1) Let $n \geq 1$ be an integer. If $n!$ is invertible in A, then any k-derivation δ of A is n-integrable: we can take $D \in \mathrm{HS}_k(A;n)$ defined by $D_i = \frac{\delta^i}{i!}$ for $i = 0, \ldots, n$. In the case $n = \infty$, if $\mathbb{Q} \subset A$, one proves in a similar way that any k-derivation of A is integrable.
(2) If A is 0-smooth (i.e. formally smooth for the discrete topologies) k-algebra, then any k-derivation of A is integrable (cf. [10], Theorem 27.1).

(2.1.3) Remark. A particularly important case of example (2.1.2) is $A = k[x_1, \ldots, x_d]$ or $A = k[[x_1, \ldots, x_d]]$. In this case we can do better than in example (2.1.2) and even exhibit a special integral for each $D \in \mathrm{HS}_k(A;m)$, $m \in \mathbb{N}_+$. Namely, consider the Hasse–Schmidt derivation $\varepsilon(D) \in \mathrm{HS}_k(A)$ determined by the k-algebra map $A = k[x_1, \ldots, x_d] \to A[[t]]$ sending each x_r to $\sum_{i \in [m]} D_i(x_r) t^i \in A[[t]]$. In other words, if $\varepsilon(D) = (D'_i)_{i \in \mathbb{N}}$, then $D'_i = D_i$ for all $i \in [m]$ and $D'_i(x_r) = 0$ for all $i > m$ and all $r = 1, \ldots, d$. It is clear that $\varepsilon(\mathrm{Id}_A, \partial_s)$ coincides with the "Taylor Hasse-Schmidt derivation" $\Delta^{(s)}$ defined in (1.3.3).

Definition (2.1.1) admits the following obvious logarihtmic version.

(2.1.4) Definition. *Let $I \subset A$ be an ideal and $n \in \overline{\mathbb{N}}$. We say that:*

1) *A I-logarithmic derivation $\delta \in \mathrm{Der}_k(\log I)$ is* I-logarithmically n-integrable *if there is a $D \in \mathrm{HS}_k(\log I;n)$ such that $D_1 = \delta$. A such D will be called a* I-logarithmic n-integral *of δ. The set of I-logarithmic k-linear derivations of A which are I-logarithmically n-integrable will be denoted by $\mathrm{Ider}_k(\log I;n)$. When $n = \infty$ it will be simply denoted by $\mathrm{Ider}_k(\log I)$.*

2) *A I-logarithmic Hasse–Schmidt derivation $D' \in \mathrm{HS}_k(\log I;m)$, with $m \leq n$, is* I-logarithmically n-integrable *if there is a $D \in \mathrm{HS}_k(\log I;n)$ such that $\tau_{nm} D = D'$. A such D will be called a* I-logarithmic n-integral *of D'. The set of I-logarithmically n-integrable I-logarithmic Hasse-Schmidt derivations of A over k of length m will be denoted by $\mathrm{IHS}_k(\log I;m;n)$. When $n = \infty$ it will be simply denoted by $\mathrm{IHS}_k(\log I;m)$.*

It is clear that the $\mathrm{Ider}_k(\log I;n)$ are A-submodules of $\mathrm{Der}_k(\log I)$ and $\mathrm{Der}_k(\log I) = \mathrm{Ider}_k(\log I;1) \supset \mathrm{Ider}_k(\log I;2) \supset \cdots$

$$\mathrm{Ider}_k(\log I) \subset \bigcap_{n\in\mathbb{N}_+} \mathrm{Ider}_k(\log I;n). \tag{3}$$

It is also clear that the $\mathrm{IHS}_k(\log I;m;n)$ are subgroups of $\mathrm{IHS}_k(\log I;m)$, stable by the $\bullet$ operation, $\mathrm{IHS}_k(\log I;m) = \mathrm{IHS}_k(\log I;m;m) \supset \mathrm{IHS}_k(\log I;m;m+1) \supset \cdots$ and

$$\mathrm{IHS}_k(\log I;m) \subset \bigcap_{n\geq m} \mathrm{IHS}_k(\log I;m;n). \tag{4}$$

The inclusions (3) and (4) seem not to be equalities in general (see question (3.6.1)). Nevertheless, we have the following proposition.

(2.1.5) Proposition. *The following properties hold:*

1) *Let $n \geq 1$ be an integer. If any k-derivation of A is n-integrable, then any Hasse–Schmidt derivation $D \in \mathrm{HS}_k(A;m)$ is also n-integrable, for all $m \leq n$.*

2) *If any k-derivation is n-integrable for all integers $n \geq 1$, then any Hasse–Schmidt derivation $D \in \mathrm{HS}_k(A;m)$ is also ∞-integrable, for all integers $m \geq 1$.*

Proof. For 1) we can mimic the proof of Proposition 1.4 in [12] by using Theorem 2.8 in [3] (see Remark 1.5 in [12]). For 2), we apply 1) and we obtain a sequence $E^n \in \mathrm{HS}_k(A;n)$, $n \geq m$, with $E^m = D$ and $\tau_{n+1,n}E^{n+1} = E^n$ for all $n \geq m$. It is clear that the inverse limit of the E^n (see (1.2.5)) is a ∞-integral of D. Q.E.D.

(2.1.6) Lemma. *Assume that $R = k[x_1,\dots,x_d]$, $S \subset R$ is a multiplicative set and $A = S^{-1}R$ or $A = k[[x_1,\dots,x_d]]$. Let $I \subset A$ be an ideal and $n \geq 1$ an integer. Then, any Hasse–Schmidt derivation D in the kernel of the group homomorphism Π_n (see (1.2.9)) is I-logarithmically (∞-)integrable.*

Proof. Let us prove the proposition in the case $A = S^{-1}R$, the case $A = k[[x_1,\dots,x_d]]$ being completely similar. Denote by $\widetilde{\delta_r} : A \to A$ the induced derivation by the partial derivative $\partial_r : R \to R$. We proceed by decreasing induction on $\ell(D)$ (see Definition (1.2.7)). If $\ell(D) = n$, then D is the identity and the result is clear. Let m be an integer with $0 \leq m < n$ and suppose that any $D' \in \ker \Pi_n$ with $m+1 \leq \ell(D')$ is I-logarithmically integrable, and let $D \in \ker \Pi_n$ with $\ell(D) = m$, i.e. D has the form $(\mathrm{Id}_A, 0,\dots,0, D_{m+1},\dots,D_n)$ with $D_{m+1} \neq 0$, and so D_{m+1} must be a k-derivation. Since $D \in \ker \Pi_n$, we deduce that $D_i(A) \subset I$ for all i. In particular, there are $a_1,\dots,a_d \in I$ such that $D_{m+1} = \sum_{r=1}^d a_r \widetilde{\delta_r}$.

The I-logarithmic Hasse-Schmidt derivation $E = (a_1 \bullet \widetilde{\Delta^{(1)}}) \circ \cdots \circ (a_d \bullet \widetilde{\Delta^{(d)}}) \in \ker \Pi_\infty$ is an (∞-)integral of D_{m+1}. Let us consider $D' = D \circ (\tau_{\infty n} E[m+1])^{-1} \in \ker \Pi_n$. It is clear that $\ell(D') \geq m+1$ and, by induction hypothesis, D' is I-logarithmically integrable. We conclude that $D = D' \circ (\tau_{\infty n} E[m+1])$ is also I-logarithmically integrable. Q.E.D.

(2.1.7) REMARK. The proof of the above lemma shows that $\ker \Pi_n$ is generated by the n-truncations of the $(a \bullet E)[m]$, with $a \in I$, $E \in \mathrm{HS}_k(A)$, $m \in [n]$. In fact, for $n = \infty$ we obtain that $\ker \Pi_\infty$ is the closure of subgroup of $\mathrm{HS}_k(\log I)$ generated by the $(a \bullet E)[m]$, with $a \in I$, $E \in \mathrm{HS}_k(A)$ and $m \in \mathbb{N}_+$, where we consider in $\mathrm{HS}_k(A)$ the inverse limit topology of the discrete topologies in the $\mathrm{HS}_k(A; m)$, $m \in \mathbb{N}$ (see (1.2.5)). Namely, for $D \in \ker \Pi_\infty$, by the same procedure as in the proof of the lemma we construct inductively a sequence $E^q = (\widetilde{a_1^q \bullet \Delta^{(1)}}) \circ \cdots \circ (\widetilde{a_d^q \bullet \Delta^{(d)}})$, $q \geq 1$, $a_r^s \in I$, such that $\ell(D \circ (F^q)^{-1}) \geq q$, where $F^q = E^q[q] \circ \cdots \circ E^1[1]$. So $D \circ (F^q)^{-1}$ tends to the identity element as $q \to \infty$ and D is the limit of F^q as $q \to \infty$.

(2.1.8) PROPOSITION. *Assume that $R = k[x_1, \ldots, x_d]$, $S \subset R$ is a multiplicative set and $A = S^{-1}R$ or $A = k[[x_1, \ldots, x_d]]$. Let $I \subset A$ be an ideal, $m \geq 1$ an integer, $n \in \overline{\mathbb{N}}$ with $n \geq m$ and $E \in \mathrm{HS}_k(A/I; m)$. The following properties are equivalent:*

- *(a) E is n-integrable.*
- *(b) Any $D \in \mathrm{HS}_k(\log I; m)$ with $\overline{D} = E$ is I-logarithmically n-integrable.*
- *(c) There is a $D \in \mathrm{HS}_k(\log I; m)$ with $\overline{D} = E$ which is I-logarithmically n-integrable.*

PROOF. The implication (b) $\Rightarrow$ (c) is an obvious consequence of Proposition (1.3.4) and (c) $\Rightarrow$ (a) comes from (1.2.9). For the remaining implication (a) $\Rightarrow$ (b), let $Z \in \mathrm{HS}_k(A/I; n)$ be an n-integral of E and let $D \in \mathrm{HS}_k(\log I; m)$ be a logarithmic Hasse-Schmidt derivation with $\overline{D} = E$. From Proposition (1.3.4), there is a $U \in \mathrm{HS}_k(\log I; n)$ such that $\overline{U} = Z$. Since $\overline{\tau_{nm} U} = \tau_{nm} \overline{U} = \tau_{nm} Z = E = \overline{D}$, we have $D \circ (\tau_{nm} U)^{-1} \in \ker \Pi_m$ and so, by Lemma (2.1.6), we deduce that D is I-logarithmically n-integrable. Q.E.D.

(2.1.9) COROLLARY. *Under the hypotheses of Proposition (2.1.8), the map $\Pi_m : \mathrm{IHS}_k(\log I; m; n) \to \mathrm{IHS}_k(A/I; m; n)$ is surjective.*

(2.1.10) COROLLARY. *Under the hypotheses of Proposition (2.1.8), the following properties are equivalent:*

- *(a) $\mathrm{IHS}_k(A/I; m; n) = \mathrm{HS}_k(A/I; m)$.*
- *(b) $\mathrm{IHS}_k(\log I; m; n) = \mathrm{HS}_k(\log I; m)$.*

PROOF. It is a straightforward consequence of the proposition. Q.E.D.

(2.1.11) EXAMPLE. *(Normal crossings)* Let us take $f = \prod_{i=1}^{e} x_i \in A = k[x_1, \ldots, x_d]$ and $I = (f) \subset A$. The A-module $\mathrm{Ider}_k(\log I)$ is generated by

$$\{x_1 \partial_1, \ldots, x_e \partial_e, \partial_{e+1}, \ldots, \partial_d\}$$

and any of these I-logarithmic derivations are integrable I-logarithmically, since $\Delta^{(j)}, x_i \bullet \Delta^{(i)} \in \mathrm{HS}_k(\log I)$ for $i = 1, \ldots, e$ and $j = e+1, \ldots, n$. In particular $\mathrm{Ider}_k(\log I) = \mathrm{Der}_k(\log I)$ and $\mathrm{Ider}_k(A/I) = \mathrm{Der}_k(A/I)$.

(2.1.12) Proposition. *Let A be an arbitrary k-algebra, $I \subset A$ an ideal with generators f_l, $l \in L$, and $n \geq 1$ an integer. Let $D \in \mathrm{HS}_k(\log I; n)$ be a I-logarithmic Hasse–Schmidt derivation and assume that D is $(n+1)$-integrable and let $(\mathrm{Id}_A, D_1, \ldots, D_n, D_{n+1}) \in \mathrm{HS}_k(A; n+1)$ be an $(n+1)$-integral of D. The following properties are equivalent:*

(a) D is I-logarithmically $(n+1)$-integrable.

(b) There is a derivation $\delta \in \mathrm{Der}_k(A)$ such that $D_{n+1}(f_l) + \delta(f_l) \in I$ for all $l \in L$

Proof. It comes from the fact that any other $(n+1)$-integral of D must be of the form $(\mathrm{Id}_A, D_1, \ldots, D_n, D_{n+1} + \delta)$ with $\delta \in \mathrm{Der}_k(A)$. Q.E.D.

(2.1.13) Corollary. *Assume that $A = k[x_1, \ldots, x_d]$ or $A = k[[x_1, \ldots, x_d]]$. Let $I = (f_1, \ldots, f_p) \subset A$ be an ideal and $n \geq 1$ an integer. Let $D \in \mathrm{HS}_k(\log I; n)$ be a I-logarithmic Hasse–Schmidt derivation and let us consider its integral $D' = \varepsilon(D)$ (see remark (2.1.3)). The following properties are equivalent:*

(a) D is I-logarithmically $(n+1)$-integrable.

(b) There are $\alpha_r, a_{st} \in A$, $r = 1, \ldots, d$, $s, t = 1, \ldots, p$, such that

$$D'_{n+1}(f_s) = \alpha_1 \, (f_s)'_{x_1} + \cdots + \alpha_d \, (f_s)'_{x_d} + a_{s1} f_1 + \cdots + a_{sp} f_p \quad \forall s = 1, \ldots, p.$$

Moreover, if (b) holds, an explicit I-logarithmic $(n+1)$-integral of D is given by $(\mathrm{Id}_A, D_1, \ldots, D_n, D'_{n+1} - \delta)$, with $\delta = \sum_{r=1}^d \alpha_r \partial_r$.

(2.1.14) Remark. (1) In the case of a "computable" base ring k (for instance, any finitely generated extension of $\mathbb{Z}, \mathbb{Q}$ or of any finite field) and a finitely presented k-algebra A, Proposition (2.1.8) and Corollary (2.1.13) give an effective way to decide whether a given Hasse–Schmidt derivation $D \in \mathrm{HS}_k(A; n)$ of finite length n is $(n+1)$-integrable or not and, if yes, to compute an explicit $(n+1)$-integral of D.

(2) Nevertheless, the question of deciding whether a given Hasse–Schmidt derivation $D \in \mathrm{HS}_k(A; n)$ of finite length n is $(n+r)$-integrable or not, with $r \geq 2$, is much more involved. First of all, we cannot proceed "step by step", since D can be $(n+r)$-integrable and simultaneously admit an $(n+1)$-integral which is not $(n+r)$-integrable (cf. example 3.7 in [12]). On the other hand, the condition of $(n+r)$-integrability of D, $r \geq 2$, gives rise to nonlinear equations which seem not obvious to treat in general with the currently available methods, either theoretical or computational (see for instance Lemmas (3.1.1), (3.3.3), (3.5.5)).

(3) The following example is a very particular case of a general result, but it also serves to illustrate the nonlinear nature of integrability and the difficulties that come from: Let $A = k[x_1, \ldots, x_d]$, $f \in A$, $I = (f)$ and $\delta = \sum_{r=1}^d a_r \partial_r$ any k-derivation of A. The following properties are equivalent:

(a) δ is a I-logarithmic derivation and it is I-logarithmically 2-integrable.

(b) $\sum_{r=1}^d f'_{x_r} a_r \in I$ and $\sum_{|\alpha|=2} \Delta^{(\alpha)}(f) \; \underline{a}^\alpha \in (f, f'_{x_1}, \ldots, f'_{x_d})$.

So, in order to compute a system of generators of the A-module $\mathrm{Ider}_k(\log I; 2)$, we have to deal with nonlinear homogeneous equations of degree 2 (see examples in sections 3.1, 3.3).

2.2 Jacobians and integrability

Let k be an arbitrary (commutative) ring and assume that $R = k[x_1, \ldots, x_d]$ or $R = k[[x_1, \ldots, x_d]]$. Let $I = (f_1, \ldots, f_u) \subset R$ be a finitely generated ideal and $A = R/I$. For each $e = 1, \ldots, \min\{d, u\}$ let J^0_e be the ideal generated by all the $e \times e$ minors of the Jacobian matrix $(\partial f_j/\partial x_i)$, and $J_e = (J^0_e + I)/I$. We have $J_1 \supset J_2 \supset \cdots$. Let c be the maximum index e with $J_e \neq 0$ (or equivalently with $J^0_e \not\subset I$), in case it exists. The ideal J_c only depends on the k-algebra A and is called the *Jacobian ideal* of A over k and denoted by $J_{A/k}$. It is nothing else but the smallest non-zero Fitting ideal of the module of k-differentials $\Omega_{A/k}$ (see [7]).

(2.2.1) PROPOSITION. *Under the above hypotheses, any $\delta \in \mathrm{Der}_k(\log I) \cap (J^0_c + I)\,\mathrm{Der}_k(R)$ is I-logarithmically integrable.*

PROOF. The proof follows the same lines that the proof of Theorem 11 in [9]. Let us write $J^0 = J^0_c$. Since $I\,\mathrm{Der}_k(R) \subset \mathrm{Ider}_k(\log I)$, we can assume that $\delta = \sum_{r=1}^d c_{r1}\partial_r$ with $c_{r1} \in J^0$. Let us consider $D^1 = (\mathrm{Id}_A, \delta) \in \mathrm{HS}_k(\log I; 1)$ and $E^1 = \varepsilon(D^1) \in \mathrm{HS}_k(R; \infty)$ (see (2.1.3)). We have that $E^1_2 = \sum_{|\alpha|=2}\left(\prod_{r=1}^d c_{r1}^{\alpha_r}\right)\Delta^{(\alpha)} \in (J^0)^2\,\mathrm{Diff}_{R/k}$, and so $E^1_2(f_j) \in (J^0)^2$ for all $j = 1, \ldots, u$. From Lemma (2.2.2) there is $(c_{12}, \ldots, c_{d2}) \in R^d$, with $c_{r2} \in J^0$, such that

$$(c_{12}, \ldots, c_{d2})\,((\partial f_j/\partial x_i)_{i,j}) \equiv (E^1_2(f_1), \ldots, E^1_2(f_u)) \mod I,$$

i.e. $E^1_2(f_j) - \sum_{r=1}^d c_{r2}(f_j)'_{x_r} \in I$, and so we deduce that D^1 is I-logarithmically 2-integrable, an I-logarithmic 2-integral being $D^2 = (\mathrm{Id}_A, \delta, D^2_2)$ with $D^2_2 = E^1_2 - \sum_{r=1}^d c_{r2}\partial_r \in J^0\,\mathrm{Diff}_{R/k}$ (see Corollary (2.1.13)).

Assume that we have found a $D^m = (\mathrm{Id}_A, \delta, D^2_2, \ldots, D^m_m) \in \mathrm{HS}_k(\log I; m)$ with $D^s_s \in J^0\,\mathrm{Diff}_{R/k}$, $s = 1, \ldots, m$, hence with $c_{rs} := D^s_s(x_r) \in J^0$, $r = 1, \ldots, d$. Let us consider $E^m = \varepsilon(D^m) \in \mathrm{HS}_k(R; \infty)$. From Proposition (1.3.1), 2) we deduce that $E^m_{m+1} \in (J^0)^2\,\mathrm{Diff}_{A/k}$ and so $E^m_{m+1}(f_j) \in (J^0)^2$ for all $j = 1, \ldots, u$. From Lemma (2.2.2), there is $(c_{1,m+1}, \ldots, c_{d,m+1}) \in R^d$, with $c_{r,m+1} \in J^0$, such that

$$(c_{1,m+1}, \ldots, c_{d,m+1})\,((\partial f_j/\partial x_i)_{i,j}) \equiv (E^m_{m+1}(f_1), \ldots, E^m_{m+1}(f_u)) \mod I,$$

i.e. $E^m_{m+1}(f_j) - \sum_{r=1}^d c_{r,m+1}(f_j)'_{x_r} \in I$, and so we deduce again that D^m is I-logarithmically $m{+}1$-integrable, an I-logarithmic $(m{+}1)$-integral being $D^{m+1} = (\mathrm{Id}_A, \delta, D^2_2, \ldots, \underline{D}^m_m, D^{m+1}_{m+1})$ with $D^{m+1}_{m+1} = E^m_{m+1} - \sum_{r=1}^d c_{r,m+1}\partial_r \in J^0\,\mathrm{Diff}_{R/k}$ (see Corollary (2.1.13)).

In that way, we construct inductively the D^m_m, $m \geq 2$, such that $(\mathrm{Id}_A, \delta, D^2_2, \ldots) \in \mathrm{HS}_k(\log I; \infty)$ and so δ is I-logarithmically integrable. Q.E.D.

(2.2.2) LEMMA. *Let $\mathbf{X} = (X_{ij})$, $i = 1, \ldots, d$, $j = 1, \ldots, u$, be variables, $W = \mathbb{Z}[\mathbf{X}]$, $\mathfrak{a}_e \subset W$ the ideal generated by the $e \times e$ minors of $\mathbf{X}$ and $U = W/\mathfrak{a}_{c+1}$.*

Then, for each $c \times c$ minor μ of $\mathbf{X}$ and for each $j = 1, \ldots, u$, the system

$$(u_1, \ldots, u_d)\mathbf{X} = (0, \ldots, 0, \overbrace{\mu}^{j}, 0, \ldots, 0)$$

has a solution in U.

Proof. We know that U is an integral domain (cf. [2], Theorem (2.10) and Remark (2.12)). Denote by K its field of fractions and by $\pi : W \to U$ the natural projection. The lemma is an easy consequence of the fact that the matrix $\pi(\mathbf{X}) \otimes K$ has rank c. Q.E.D.

The following corollary of Proposition (2.2.1) generalizes Theorem 11 in [9], which was only stated and proved for k a perfect field.

(2.2.3) Corollary. *Under the above hypotheses, we have*

$$J_{A/k} \subset \operatorname{ann}_A \left(\operatorname{Der}_k(A)/\operatorname{Ider}_k(A)\right).$$

The proof of the following result is similar to the proof of Proposition (2.2.1).

(2.2.4) Proposition. *Let $f \in R$, $I = (f)$, and $J^0 = (f'_{x_1}, \ldots, f'_{x_d})$ the gradient ideal. If $\delta : R \to R$ is a I-logarithmic k-derivation with $\delta \in J^0 \operatorname{Der}_k(R)$, then δ admits a I-logarithmic integral $D \in \operatorname{HS}_k(\log I)$ with $D_i(f) = 0$ for all $i > 1$. In particular, if $\delta(f) = 0$, the integral D can be taken with $\Phi_D(f) = f$.*

(2.2.5) We quote here Theorem 1.2 in [15]: Let $I \subset A = k[x_1, \ldots, x_d]$ be an ideal generated by quasi-homogeneous polynomials with respect to the weights $w(x_r) \geq 0$. Then, the Euler vector field $\chi = \sum_{r=0}^d w(x_r)\partial_r$ is I-logarithmically (∞-)integrable. In fact, a I-logarithmic integral of χ is the Hasse–Schmidt derivation associated with the map $A \to A[[t]]$ given by

$$x_r \mapsto x_r \left(\tfrac{1}{1-t}\right)^{w(x_r)}, \quad r = 1, \ldots, d.$$

(2.2.6) Proposition. *Let $f \in A = k[x_1, \ldots, x_d]$ be a quasi-homogeneous polynomial with respect to the weights $w(x_r) > 0$ and $I = (f) \subset A$. Assume that the weight of f is a unit in k and that all the partial derivatives of f are non-zero and form a regular sequence. Then $\operatorname{Der}_k(\log I) = \operatorname{Ider}_k(\log I)$.*

Proof. From the hypotheses we deduce that the A-module $\operatorname{Der}_k(\log I)$ is generated by the Euler vector field χ and the crossed derivations $\theta_{rs} = f'_{x_s}\partial_r - f'_{x_r}\partial_s$, $1 \leq r < s \leq d$. But χ is I-logarithmically integrable by (2.2.5) and θ_{rs} is I-logarithmically integrable by Proposition (2.2.1). Q.E.D.

2.3 Behaviour of integrability under localization

Throughout this section, k will be an arbitrary commutative ring.

The proof of the following proposition is clear from (1.2.10).

(2.3.1) Proposition. *Let A be a k-algebra, $S \subset A$ a multiplicative set, $\mathfrak{a} \subset A$ be an ideal, $m \geq 1$ an integer, $n \in \overline{\mathbb{N}}$ with $n \geq m$ and $D \in \operatorname{HS}_k(\log \mathfrak{a}; m)$. If D $\mathfrak{a}$-logarithmically n-integrable, then $\widetilde{D} \in \operatorname{HS}_k(S^{-1}A; m)$ is $(S^{-1}\mathfrak{a})$-logarithmically*

m-integrable. In particular, the map Θ_m sends $\mathrm{IHS}_k(\log\mathfrak{a};m;n)$ to $\mathrm{IHS}_k(\log(S^{-1}\mathfrak{a});m;n)$.

The two following propositions are straightforward consequences of Proposition (1.3.5) and Corollary (1.3.6) respectively.

(2.3.2) PROPOSITION. *Assume that $A=k[x_1,\dots,x_d]$ and let $S\subset A$ be a multiplicative set and $\mathfrak{a}=(f_1,\dots,f_u)\subset A$ be a finitely generated ideal. Then, for any integers $m\geq q\geq 1$, the map*

$$(s,F)\in S\times \mathrm{IHS}_k(\log\mathfrak{a};q;m)\mapsto \frac{1}{s}\bullet\Theta_q(F)\in \mathrm{IHS}_k(\log(S^{-1}\mathfrak{a});q;m)$$

is surjective.

(2.3.3) PROPOSITION. *Assume that A is a finitely presented k-algebra and let $T\subset A$ be a multiplicative set. Then, for any integers $m\geq q\geq 1$ the map*

$$(t,G)\in T\times \mathrm{IHS}_k(A;q;m)\mapsto \frac{1}{t}\bullet\Theta_q(G)\in \mathrm{IHS}_k(T^{-1}A;q;m)$$

is surjective.

Proposition (2.3.3) can be also obtained form Proposition (2.3.2) and Corollary (2.1.9).

(2.3.4) COROLLARY. *Assume that $A=k[x_1,\dots,x_d]$ and let $S\subset A$ be a multiplicative set, $\mathfrak{a}=(f_1,\dots,f_u)\subset A$ be a finitely generated ideal. Then, for any integer $m\geq 1$ the canonical map*

$$\frac{\delta}{s}\in S^{-1}\operatorname{Ider}_k(\log\mathfrak{a};m)\mapsto \frac{1}{s}\widetilde{\delta}\in \operatorname{Ider}_k(\log(S^{-1}\mathfrak{a});m)$$

is an isomorphism of $(S^{-1}A)$-modules.

PROOF. The injectivity is a consequence of the fact that, under the above assumptions, the canonical map $S^{-1}\operatorname{Der}_k(A)\to\operatorname{Der}_k(S^{-1}A)$ is an isomorphism. The surjectivity is given by Proposition (2.3.2) in the case $q=1$. Q.E.D.

(2.3.5) COROLLARY. *Assume that A is a finitely presented k-algebra and let $T\subset A$ be a multiplicative set. Then, for any integer $m\geq 1$ the canonical map*

$$T^{-1}\operatorname{Ider}_k(A;m)\to\operatorname{Ider}_k(T^{-1}A;m)$$

is an isomorphism of $(T^{-1}A)$-modules.

PROOF. The injectivity goes as in the proof of Corollary (2.3.4). The surjectivity is given by Proposition (2.3.3) in the case $q=1$. Q.E.D.

(2.3.6) THEOREM. *Assume that A is a finitely presented k-algebra, $m\geq 1$ is an integer and let $\delta\in\operatorname{Der}_k(A)$. The following properties are equivalent:*

(a) $\delta\in\operatorname{Ider}_k(A;m)$.

(b) $\delta_{\mathfrak{p}}\in\operatorname{Ider}_k(A_{\mathfrak{p}};m)$ *for all* $\mathfrak{p}\in\operatorname{Spec}A$.

(c) $\delta_{\mathfrak{m}}\in\operatorname{Ider}_k(A_{\mathfrak{m}};m)$ *for all* $\mathfrak{m}\in\operatorname{Specmax}A$.

Proof. The implication (a) $\Rightarrow$ (b) is a consequence of Proposition (2.3.1). The implication (b) $\Rightarrow$ (c) is obvious. For the remaining implication (c) $\Rightarrow$ (a), assume that property (c) holds. Then, by Corollary (2.3.5), for each $\mathfrak{m} \in \operatorname{Specmax} A$ there is a $f^{\mathfrak{m}} \in A - \mathfrak{m}$ and a $\zeta^{\mathfrak{m}} \in \operatorname{Ider}_k(A;m)$ such that $f^{\mathfrak{m}}\delta_{\mathfrak{m}} = (\zeta^{\mathfrak{m}})_{\mathfrak{m}}$, and so there is a $g^{\mathfrak{m}} \in A - \mathfrak{m}$ such that $g^{\mathfrak{m}} f^{\mathfrak{m}}\delta = g^{\mathfrak{m}}\zeta^{\mathfrak{m}}$. Since the ideal generated by the $g^{\mathfrak{m}} f^{\mathfrak{m}}$, $\mathfrak{m} \in \operatorname{Specmax} A$, must be the total ideal, we deduce the existence of a finite number of $\mathfrak{m}_i \in \operatorname{Specmax} A$ and $a_i \in A$, $1 \leq i \leq n$, such that $1 = a_1 g_1 f_1 + \cdots + a_n g_n f_n$, with $f_i = f^{\mathfrak{m}_i}$, $g_i = g^{\mathfrak{m}_i}$, and so

$$\delta = \sum_{i=1}^{n} a_i g_i f_i \delta = \sum_{i=1}^{n} a_i g_i \zeta^{\mathfrak{m}_i}$$

is m-integrable. Q.E.D.

(2.3.7) Corollary. *Let $f : X \to S$ be a locally finitely presented morphism of schemes. For each integer $n \geq 1$ there is a quasi-coherent sub-sheaf $\mathit{Ider}_S(\mathcal{O}_X;n) \subset \mathit{Der}_S(\mathcal{O}_X)$ such that, for any affine open sets $U = \operatorname{Spec} A \subset X$ and $V = \operatorname{Spec} k \subset S$, with $f(U) \subset V$, we have $\Gamma(U, \mathit{Ider}_S(\mathcal{O}_X;n)) = \operatorname{Ider}_k(A;n)$ and $\mathit{Ider}_S(\mathcal{O}_X;n)_p = \operatorname{Ider}_{\mathcal{O}_{S,f(p)}}(\mathcal{O}_{X,p};n)$ for each $p \in X$. Moreover, if S is locally noetherian, then $\mathit{Ider}_S(\mathcal{O}_X;n)$ is a coherent sheaf.*

Proof. For each open set $U \subset X$, we define

$$\Gamma(U, \mathit{Ider}_S(\mathcal{O}_X;n)) = \{\delta \in \Gamma(U, \mathit{Der}_S(\mathcal{O}_X)) \mid \delta_p \in \operatorname{Ider}_{\mathcal{O}_{S,f(p)}}(\mathcal{O}_{X,p};n)\ \forall p \in U\}.$$

The behaviour of $\mathit{Ider}_S(\mathcal{O}_X;n)$ on affine open sets and its quasi-coherence is a straightforward consequence of Theorem (2.3.6). Q.E.D.

2.4 Testing the integrability of derivations

In this section k will be an arbitrary commutative ring and A an arbitrary k-algebra.

(2.4.1) Definition. *Let $n \geq m > 1$ be integers and $D \in \operatorname{HS}_k(A;n)$. We say that D is* m-sparse *if $D_i = 0$ whenever $i \notin \mathbb{N}m$. We say that D is* weakly m-sparse *if $\tau_{n,qm} D$ is m-sparse, where $q = \left\lfloor \frac{n}{m} \right\rfloor$. The set of m-sparse (resp. weakly m-sparse) Hasse–Schmidt derivations in $\operatorname{HS}_k(A;n)$ will be denoted by $\operatorname{HS}_k^{m-sp}(A;n)$ (res. $\operatorname{HS}_k^{m-wsp}(A;n)$).*

The proof of the following proposition is easy and its proof is left up to the reader.

(2.4.2) Proposition. *Let $n \geq m > 1$ be integers, $q = \left\lfloor \frac{n}{m} \right\rfloor$ and $r = n - qm$. The following properties hold:*

1) *$\operatorname{HS}_k^{m-sp}(A;n)$ and $\operatorname{HS}_k^{m-wsp}(A;n)$ are subgroups of $\operatorname{HS}_k(A;n)$.*
2) *For any $D \in \operatorname{HS}_k(A;q)$ and any $\underline{\delta} = (\delta_1, \ldots, \delta_r) \in \operatorname{Der}_k(A)^r$, the sequence*

$$\Theta(D,\underline{\delta}) = (\operatorname{Id}_A, 0, \ldots, 0, \overbrace{D_1}^{m}, 0, \ldots, 0, \overbrace{D_2}^{2m}, 0, \ldots, 0, \overbrace{D_q}^{qm}, \overbrace{\ \delta_1\ }^{qm+1}, \ldots, \overbrace{\delta_r}^{n})$$

is a weakly m-sparse Hasse–Schmidt derivation of A (over k) of length n and the map $\Theta : \operatorname{HS}_k(A;q) \times \operatorname{Der}_k(A)^r \to \operatorname{HS}_k^{m-wsp}(A;n)$ is an isomorphism of groups.

(2.4.3) THEOREM. *Let $n \geq 1$ be an integer. The following assertions hold:*

1) *If n is odd and $\operatorname{Ider}_k(A;q) = \operatorname{Der}_k(A)$, with $q = \frac{n+1}{2}$, then any $D \in \operatorname{HS}_k(A;n)$ with $D_1 = 0$ is $(n+1)$-integrable.*

2) *If n is even and $\operatorname{Ider}_k(A;p) = \operatorname{Der}_k(A)$, with $p = \left\lfloor \frac{n+1}{3} \right\rfloor$, then any $D \in \operatorname{HS}_k(A;n)$ with $D_1 = 0$ is $(n+1)$-integrable.*

PROOF. 1) Since $D_1 = 0$ we have $1 \leq \ell(D) \leq n$. If $n = 1$, then D is the identity and the result is clear. Assume $n \geq 3$ and so $q \geq 2$. Let us proceed by decreasing induction on $\ell(D)$. If $\ell(D) = n$ then D is the identity and the result is clear. Let m be an integer with $1 \leq m < n$ and suppose that any $D' \in \operatorname{HS}_k(A;n)$ with $m+1 \leq \ell(D')$ is $(n+1)$-integrable. Let $D \in \operatorname{HS}_k(A;n)$ be a Hasse–Schmidt derivation with $\ell(D) = m$, i.e.

$$D = (\operatorname{Id}_A, 0, \ldots, 0, D_{m+1}, \ldots, D_n) \quad \text{with } D_{m+1} \neq 0.$$

Since $\tau_{n,m+1} D$ is $(m+1)$-sparse, we can apply Proposition (2.4.2), 2) and deduce that D_{m+1} is a derivation and so, by hypothesis, it must be q-integrable. Let $E \in \operatorname{HS}_k(A;q)$ be a q-integral of D_{m+1}. We have that $q(m+1) \geq 2q = n+1$ and so $F = \tau_{q(m+1),n}(E[m+1])$ is $(n+1)$-integrable, an $(n+1)$-integral being $\tau_{q(m+1),n+1}(E[m+1])$, and has the form

$$F = (\operatorname{Id}_A, 0, \ldots, 0, \overbrace{D_{m+1}}^{m+1}, 0, \ldots, F_n).$$

It is clear that for $D' = F^{-1} \circ D$ we have $D'_1 = \cdots = D'_{m+1} = 0$, and so $\ell(D') \geq m+1$. The induction hypothesis implies that D' is $(n+1)$-integrable and we conclude that $D = F \circ D'$ is also $(n+1)$-integrable.

2) If $n = 2$, then $D = (\operatorname{Id}_A, 0, D_2)$ and obviously $(\operatorname{Id}_A, 0, D_2, 0)$ is a 3-integral of D. Assume that n is even ≥ 4, and let us write $n = 2q$, $q \geq 2$, and $n+1 = 3p+r$ with $0 \leq r < 3$, $p \geq 1$. Since $\tau_{n3} D$ is weakly 2-sparse, we deduce that D_3 must be a derivation (see Proposition (2.4.2)) and so, by hypothesis, it is p-integrable. Let $E^3 \in \operatorname{HS}_k(A;p)$ be a p-integral of D_3. It is clear that (see Proposition (2.4.2))

$$F^3 = (\operatorname{Id}_A, 0, 0, \overbrace{E^3_1}^{3}, 0, 0, \overbrace{E^3_2}^{6}, 0, \ldots, 0, \overbrace{E^3_p}^{3p}, 0, 0)$$

is a $(3p+2)$-integral of $E^3[3]$, and since $3p+2 \geq n+1$, $G^3 = \tau_{3p+2,n} F^3$ is $(n+1)$-integrable and $(G^3)^{-1} \circ D$ has the form $(\operatorname{Id}_A, 0, D_2, 0, \ldots)$.

Assume that we have found $G^3, G^5, \ldots, G^{2s-1} \in \operatorname{HS}_k(A;n)$, all of them $(n+1)$-integrable, with $3 \leq 2s-1 < n$, such that $(G^{2s-1})^{-1} \circ \cdots \circ (G^3)^{-1} \circ D$ has the form

$$D' = (\operatorname{Id}_A, 0, \overbrace{D'_2}^{2}, 0, \overbrace{D'_4}^{4}, 0, \ldots, 0, \overbrace{D'_{2s}}^{2s}, D'_{2s+1}, \ldots, D'_n).$$

If $2s = n$, we already have what we are looking for. If $2s < n$, then D'_{2s+1} is a derivation (see Proposition (2.4.2)) and so, by hypothesis, it is p-integrable. Let $E^{2s+1} \in \operatorname{HS}_k(A;p)$ be a p-integral of D'_{2s+1}. Let us consider $F^{2s+1} =$

$E^{2s+1}[2s+1] \in \mathrm{HS}_k(A; p(2s+1))$. Since $p(2s+1) \geq 5p \geq 3p+2 \geq n+1$, $G^{2s+1} := \tau_{p(2s+1),n} F^{2s+1}$ is $(n+1)$-integrable and $(G^{2s+1})^{-1} \circ D'$ has the form

$$D'' = (\mathrm{Id}_A, 0, \overbrace{D''_2}^{2}, 0, \overbrace{D''_4}^{4}, 0, \ldots, 0, \overbrace{D''_{2s}}^{2s}, 0, \overbrace{D''_{2s+2}}^{2s+2}, \ldots, D''_n).$$

We conclude with the existence of $G^3, G^5, \ldots, G^{n-1} \in \mathrm{HS}_k(A; n)$, all of them $(n+1)$-integrable, such that $H = (G^{n-1})^{-1} \circ G^{n-3} \cdots \circ (G^3)^{-1} \circ D \in \mathrm{HS}_k(A; n)$ ($n = 2q$) is 2-sparse. From Proposition (2.4.2) again we deduce that H is $(n+1)$-integrable, and so D is also $(n+1)$-integrable. Q.E.D.

(2.4.4) Definition. *For each integer $n \geq 1$, let us define*

$$\rho(n) = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd} \\ \lfloor \frac{n+1}{3} \rfloor & \text{if } n \text{ is even.} \end{cases}$$

Notice that $\rho(n) < n$ for all $n \geq 2$.

(2.4.5) Corollary. *Let $n \geq 1$ be an integer, and assume that $\mathrm{Ider}_k(A; \rho(n)) = \mathrm{Der}_k(A)$. Then, for any n-integrable derivation $\delta \in \mathrm{Ider}_k(A; n)$, the following properties are equivalent:*

(a) Any n-integral of δ is $(n+1)$-integrable.

(b) There is an n-integral of δ which is $(n+1)$-integrable.

Proof. Assume that $E \in \mathrm{HS}_k(A; n+1)$ is an $(n+1)$-integral of δ and let $D \in \mathrm{HS}_k(A; n)$ be any n-integral of δ. The 1-component of $F = D \circ (\tau_{n+1,n} E)^{-1}$ vanishes and so, by Theorem (2.4.3), F is $(n+1)$-integrable. We deduce that $D = F \circ \tau_{n+1,n} E$ is also $(n+1)$-integrable. Q.E.D.

2.5 Algorithms

Let k be a "computable" base ring k (for instance, any finitely generated extension of $\mathbb{Z}, \mathbb{Q}$ or of any finite field), $f_1, \ldots, f_p \in A = k[x_1, \ldots, x_d]$ and $I = (f_1, \ldots, f_p)$. The starting point is the computation of a system of generators $\{\delta^1, \ldots, \delta^q\}$ of $\mathrm{Der}_k(\log I)$.

The following algorithm decides whether the equality

$$\mathrm{Der}_k(\log I) \stackrel{?}{=} \mathrm{Ider}_k(\log I; 2) \quad \left(\Leftrightarrow \mathrm{Der}_k(A/I) \stackrel{?}{=} \mathrm{Ider}_k(A/I; 2)\right)$$

is true or not, and if yes, returns a 2-integral for each generator of $\mathrm{Der}_k(\log I)$.

ALGORITHM–1:

Step 1: For each $j = 1, \ldots, q$, apply Corollary (2.1.13) as explained in remark (2.1.14), (1) to decide whether δ^j is I-logaritmically 2-integrable or not, and if yes to compute a I-logarithmic 2-integral $D^{j,2}$ of δ^j.

Step 2: (Y) If the answer in Step 1 is YES for all $j = 1, \dots, q$, then save the I-logarithmic 2-integrals $D^{1,2}, \dots, D^{q,2}$ and answer "THE EQUALITY $\mathrm{Der}_k(\log I) = \mathrm{Ider}_k(\log I; 2)$ IS TRUE".
(N) If the answer in step 1 is NOT for some $j = 1, \dots, q$, then answer "THE EQUALITY $\mathrm{Der}_k(\log I) = \mathrm{Ider}_k(\log I; 2)$ IS FALSE".

Assume that we have an ALGORITHM–(N-1) to decide whether the equality

$$\mathrm{Der}_k(\log I) \stackrel{?}{=} \mathrm{Ider}_k(\log I; N) \quad \left(\Leftrightarrow \mathrm{Der}_k(A/I) \stackrel{?}{=} \mathrm{Ider}_k)A/I; N)\right)$$

is true or not, and if yes, to compute an N-integral for each generator of $\mathrm{Der}_k(\log I)$.

ALGORITHM–N:

Step 1: Apply ALGORITHM–(N-1), and if the answer is NOT, then STOP and answer "THE EQUALITY $\mathrm{Der}_k(\log I) = \mathrm{Ider}_k(\log I; N+1)$ IS FALSE". If the answer to ALGORITHM–(N-1) is YES, keep the computed I-logarithmic N-integrals $D^{1,N}, \dots, D^{q,N}$ of $\delta^1, \dots, \delta^q$ and go to step 2.

Step 2: For each $j = 1, \dots, q$, apply Corollary (2.1.13) as explained in remark (2.1.14), (1) to decide whether $D^{j,N}$ is I-logaritmically $(N+1)$-integrable or not, and if yes to compute a I-logarithmic $(N+1)$-integral $D^{j,N+1}$ of $D^{j,N}$.

Step 3: (Y) If the answer in Step 2 is YES for all $j = 1, \dots, q$, then save the I-logarithmic $(N+1)$-integrals $D^{1,N+1}, \dots, D^{q,N+1}$ and answer "THE EQUALITY $\mathrm{Der}_k(\log I) = \mathrm{Ider}_k(\log I; N+1)$ IS TRUE".
(N) If the answer in Step 2 is NOT for some $j = 1, \dots, q$, then answer "THE EQUALITY $\mathrm{Der}_k(\log I) = \mathrm{Ider}_k(\log I; N+1)$ IS FALSE".

Corollary (2.4.5) is the key point for the correctness of Step 3, (N).

3 Examples and questions

We have used Macaulay 2 [4] for the preliminary computations needed in the following examples.

3.1 The cusp $x^2 + y^3$ in characteristic 2 or 3

Let k be a base ring containing the field $\mathbb{F}_p$, $p > 0$, and $f = x^2 + y^3 \in R = k[x, y]$. Let $I = (f)$ and $A = k[x, y]/I$. The computation of $\mathrm{Ider}_k(A; \infty)$ has been treated in [9], example 5. Here we are interested in the computation of $\mathrm{Ider}_k(A; m)$, $m \geq 2$.

Let start with $p = 2$. Then the Jacobian ideal of f is $J = (y^2, f) = (x^2, y^2)$.

The module $\mathrm{Der}_k(\log I)$ is free with basis $\{\partial_x, f\partial_y\}$. It is clear that $f\partial_y$ is I-logarithmically $(\infty$-)integrable. Let $g \in R$ be a polynomial. From Corollary (2.1.13), we have that $g\partial_x$ is I-logarithmically 2-integrable if and only if $g^2 \in J$.

Since $\{g \in R \mid g^2 \in J\} = (x, y)$, we deduce that $\{x\partial_x, y\partial_x, f\partial_y\}$ is a system of generators of $\mathrm{Ider}_k(\log I; 2)$.

The derivation $x\partial_x$ is the Euler vector field for the weights $w(x) = 3, w(y) = 2$. From (2.2.5) we know that $x\partial_x$ is I-logarithmically (∞-)integrable.

Let $c \in R$ be an arbitrary polynomial and $\delta = cy\partial_x$. A I-logarithmic 2-integral of δ is determined by the k-algebra map

$$p(x, y) \in R \mapsto p(x + cyt, y + c^2t^2) + (t^3) \in R_3 = R[[t]]/(t^3).$$

Since the coefficient of t^3 in $f(x + cyt, y + c^2t^2)$ is 0, we deduce that δ is I-logarithmically 3-integrable and so $\mathrm{Ider}_k(\log I; 3) = \mathrm{Ider}_k(\log I; 2)$. A generic I-logarithmic 2-integral of δ is determined by the k-algebra map

$$p(x, y) \in R \mapsto p(x + cyt + dt^2, y - c^2t^2) + (t^3) \in R_3,$$

with $d \in R$, and a generic I-logarithmic 3-integral of δ is determined by the k-algebra map

$$p(x, y) \in R \mapsto p(x + cyt + dt^2 + et^3, y + c^2t^2) + (t^4) \in R_4,$$

with $d, e \in R$. The coefficient of t^4 in $f(x + cyt + dt^2 + et^3, y + c^2t^2)$ is $d^2 + yc^4$, and so, the following conditions are equivalent:

(a) δ is I-logarithmically 4-integrable.

(b) There is a $d \in R$ such that $d^2 + yc^4 \in J$.

The proof of the following lemma is easy:

(3.1.1) Lemma. *The set* $\Gamma := \{c \in R \mid \exists d \in R,\ d^2 + yc^4 \in J\}$ *is the ideal generated by* x, y.

As a consequence of the lemma we deduce that $\{x\partial_x, y^2\partial_x, f\partial_y\}$ is a system of generators of $\mathrm{Ider}_k(\log I; 4)$. But $y^2\partial_x$ is I-logarithmically (∞-)integrable after Proposition (2.2.1), and so

$$\begin{gathered}\mathrm{Der}_k(A) = \langle\overline{\partial_x}\rangle \supsetneq \mathrm{Ider}_k(A; 2) = \langle\overline{x\partial_x}, \overline{y\partial_x}\rangle = \mathrm{Ider}_k(A; 3) \supsetneq \\ \mathrm{Ider}_k(A; 4) = \langle\overline{x\partial_x}, \overline{y^2\partial_x}\rangle = \mathrm{Ider}_k(A; 5) = \cdots = \mathrm{Ider}_k(A; \infty).\end{gathered}$$

In particular, we have

$$\begin{gathered}\mathrm{ann}_A\left(\mathrm{Der}_k(A)/\,\mathrm{Ider}_k(A; 2)\right) = (\overline{x}, \overline{y}) = \sqrt{\overline{J}} \supsetneq \\ \mathrm{ann}_A\left(\mathrm{Der}_k(A)/\,\mathrm{Ider}_k(A; \infty)\right) = (\overline{x}, \overline{y}^2) \supsetneq \overline{J} = (\overline{x}^2, \overline{y}^2).\end{gathered}$$

Let us now compute the case $p = 3$. The Jacobian ideal of f is $J = (x, f) = (x, y^3)$. In a similar way to the preceding case, we obtain that:

-) $\mathrm{Der}_k(\log I) = \langle f\partial_x, \partial_y\rangle$.

-) Since 2 is invertible in k we have $\mathrm{Der}_k(\log I) = \mathrm{Ider}_k(\log I; 2)$.

-) $\mathrm{Ider}_k(\log I; 3) = \langle x\partial_y, y\partial_y, f\partial_x\rangle$.

-) $\operatorname{Ider}_k(\log I;3) = \operatorname{Ider}_k(\log I;\infty)$.

-) $\operatorname{Der}_k(A) = \langle\overline{\partial_y}\rangle = \operatorname{Ider}_k(A;2) \supsetneq \operatorname{Ider}_k(A;3) = \langle\overline{x\partial_y}, \overline{y\partial_y}\rangle = \operatorname{Ider}_k(A;4) = \cdots = \operatorname{Ider}_k(A;\infty)$ and $\operatorname{ann}_A(\operatorname{Der}_k(A)/\operatorname{Ider}_k(A;\infty)) = (\overline{x},\overline{y}) = \sqrt{J_{A/k}}$.

Let us notice that for the cusp in characteristics $\neq 2,3$ we can apply Proposition (2.2.6) and obtain that any derivation is integrable.

3.2 The cusp $x^2 + y^3$ over the integers

Assume that $k = \mathbb{Z}$ and $f = x^2 + y^3 \in R = \mathbb{Z}[x,y]$. Let $I = (f)$ and $A = \mathbb{Z}[x,y]/I$. The Jacobian ideal of f is $J = (2x, 3y^2, f) = (2x, 3y^2, x^2, y^3)$. The I-logarithmic derivations of R are generated by $\delta_1 = 3x\partial_x + 2y\partial_y$, $\delta_2 = 3y^2\partial_x - 2x\partial_y$, $f\partial_x$ and $f\partial_y$. The first derivation δ_1 is the Euler vector field for the weights $w(x) = 3, w(y) = 2$. As in 3.1, δ_1 is I-logarithmically integrable. For the second derivation δ_2, we apply Proposition (2.2.1) and we deduce that it is also I-logarithmically integrable. So this is an example of a non-smooth $\mathbb{Z}$-algebra A for which any derivation is integrable.

3.3 The cusp $3x^2 + 2y^3$ over the integers

Assume that $k = \mathbb{Z}$ and $f = 3x^2 + 2y^3 \in R = \mathbb{Z}[x,y]$. Let $I = (f)$ and $A = \mathbb{Z}[x,y]/I$. The Jacobian ideal of f is $J = (6x, 6y^2, f) = (6x, 6y^2, 3x^2, 2y^3)$. The I-logarithmic derivations of R are generated by $\delta_1 = 3x\partial_x + 2y\partial_y$ and $\delta_2 = -y^2\partial_x + x\partial_y$, which in fact form a basis (we can say that "f is a free divisor" of R). As in 3.1, δ_1 is the Euler vector field for the weights $w(x) = 3, w(y) = 2$ and so it is I-logarithmically integrable.

Let us study the integrability of $a\delta_2$, $a \in R$. The coefficient of t^2 in $f(x - ay^2t, y + axt)$ is $a^2(3y^4 + 6x^2y)$. Since $6x^2 \in J$, this coefficient belongs to J if and only if $3a^2y^4 \in J$, i.e. $a^2 \in J : 3y^4$.

(3.3.1) Lemma.

(a) $J : 3y^4 = (2, x^2)$.

(b) $\{a \in R \mid a^2 \in (2, x^2)\} = (2, x)$.

(3.3.2) Corollary. *The R-module* $\operatorname{Ider}_{\mathbb{Z}}(\log I;2)$ *is generated by* $\{\delta_1, 2\delta_2, x\delta_2\}$ *and so* $\operatorname{ann}_A(\operatorname{Der}_{\mathbb{Z}}(A)/\operatorname{Ider}_{\mathbb{Z}}(A;2)) = (2,x)$.

Let us study the 3-integrability of

$$(2b + cx)\delta_2 = -y^2(2b + cx)\partial_x + (2b + cx)x\partial_y, \quad b, c \in R.$$

Let us write $a = 2b + cx$. The coefficient of t^2 in $f(x - y^2(2b+cx)t, y + (2b+cx)xt)$ is $A(2y^3) + B(3x^2)$ with $A = 6b(b+cx)y, B = c^2y^4 + 2a^2y$, which can be expressed as

$$(A - B)xf'_x + (A - B)yf'_y + (3B - 2A)f.$$

Hence, the coefficient of t^2 in

$$f(x - y^2(2b + cx)t + (B - A)xt^2, y + (2b + cx)xt + (B - A)yt^2)$$

is $(3B-2A)f$ and the reduction $\mod t^3$ of the $\mathbb{Z}$-algebra map

$$\Psi^{(2)}: p(x,y) \in R \mapsto p(x-y^2(2b+cx)t+(B-A)xt^2, y+(2b+cx)xt+(B-A)yt^2) \in R[[t]]$$

is I-logarithmic and gives rise to a I-logarithmic 2-integral of $a\delta_2$. So, the reduction $\mod t^3$ of the $\mathbb{Z}$-algebra map $\Psi_g^{(2)}: R \to R[[t]]$ given by

$$\begin{array}{rcl} x & \mapsto & x-y^2(2b+cx)t+[(B-A)x+3dx-ey^2]t^2, \\ y & \mapsto & y+(2b+cx)xt+[(B-A)y+2dy+ex]t^2 \end{array}$$

is the associated map to a generic I-logarithmic 2-integral of $a\delta_2$. Moreover, the coefficient of t^2 in $\Psi_g^{(2)}(f)$ is $(3B-2A+6d)f$.

The coefficient of t^3 in $\Psi_g^{(2)}(f)$ is $6x^2y^6c^3+12xy^6bc^2+12x^4y^3c^3+36x^3y^3bc^2+2x^6c^3+36x^2y^3b^2c+12x^5bc^2+24xy^3b^3+24x^4b^2c+6xy^4ce+16x^3b^3+6x^2y^2cd+12y^4be+12x^3yce+12xy^2bd+24x^2ybe$, and it belongs to J if and only if

$$2x^6c^3+16x^3b^3 \in J \Leftrightarrow x^3c^3+8b^3 \in (J:2x^3).$$

(3.3.3) LEMMA. *With the above notations, the following assertions hold:*

(a) $J:2x^3=(3,y^3)$.

(b) $x^3c^3+8b^3 \in (J:2x^3) \Leftrightarrow a^3 \in (J:2x^3) \Leftrightarrow a \in (3,y)$.

(3.3.4) COROLLARY. *The I-logarithmic derivation $a\delta_2$ is I-logarithmically* 3-*integrable if and only if $a \in (2,x)\cap(3,y)=(6,3x,2y,xy)$, and so the R-module* $\operatorname{Ider}_{\mathbb{Z}}(\log I;3)$ *is generated by* $\{\delta_1, 6\delta_2, 3x\delta_2, 2y\delta_2, xy\delta_2\}$ *and*

$$\operatorname{ann}_A(\operatorname{Der}_{\mathbb{Z}}(A)/\operatorname{Ider}_{\mathbb{Z}}(A;3)) = (2,\overline{x})\cap(3,\overline{y}),$$
$$\operatorname{ann}_A(\operatorname{Der}_{\mathbb{Z}}(A;2)/\operatorname{Ider}_{\mathbb{Z}}(A;3)) = (3,\overline{y}).$$

The following lemma cannot be deduced directly from Proposition (2.2.1). Its proof proceeds by induction and it is left up to the reader.

(3.3.5) LEMMA. *Let $a \in (2,x)\cap(3,y)$. There are sequences $a_i, b_i \in R$, $i \geq 2$, such that the $\mathbb{Z}$-algebra map*

$$\Psi: p(x,y) \in R \mapsto p\left(x-ay^2t+\sum_{i=2}^{\infty} a_i t^i, y+axt+\sum_{i=2}^{\infty} b_i t^i\right) \in R[[t]]$$

is I-logarithmic, i.e. $\Psi(f) \in R[[t]]f$.

(3.3.6) COROLLARY. *We have*

$$\operatorname{Ider}_{\mathbb{Z}}(A;3) = \operatorname{Ider}_{\mathbb{Z}}(A;4) = \cdots = \operatorname{Ider}_{\mathbb{Z}}(A),$$

and so

$$\operatorname{ann}_A(\operatorname{Der}_{\mathbb{Z}}(A)/\operatorname{Ider}_{\mathbb{Z}}(A)) = (2,\overline{x})\cap(3,\overline{y}) \supsetneq \sqrt{J_{A/\mathbb{Z}}} = (3\overline{x}, 2\overline{y}).$$

The following two examples have been proposed by Herwig Hauser.

3.4 The surface $x_3^2 + x_1(x_1+x_2)^2 = 0$ in characteristic 2

Let k be a field of characteristic 2, $f = x_3^2 + x_1(x_1+x_2)^2 \in R = k[x_1,x_2,x_3]$, $I = (f)$ and $A = R/I$. The Jacobian ideal is $J = (\ell^2, f) = (\ell^2, x_3^2)$ with $\ell = x_1 + x_2$, and $\sqrt{J} = (\ell, x_3)$. A system of generators of $\operatorname{Der}_k(\log I)$ mod. $f \operatorname{Der}_k(R)$ is $\{\partial_2, \partial_3\}$.

(3.4.1) Lemma. *Let $\alpha, \beta \in R$ and $\delta = \alpha\partial_2 + \beta\partial_3$. The following conditions are equivalent:*

(a) δ is I-logarithmically 2-integrable.

(b) $x_1\alpha^2 + \beta^2 \in J$.

(3.4.2) Lemma. *The module $\{(\alpha,\beta) \in R^2 \mid x_1\alpha^2 + \beta^2 \in J\}$ is generated by $(x_3, 0), (\ell, 0), (0, x_3), (0, \ell)$.*

(3.4.3) Corollary. *A system of generators of* $\operatorname{Ider}_k(\log I; 2) \mod f \operatorname{Der}_k(R)$ *is* $\{x_3\partial_2, \ell\partial_2, x_3\partial_3, \ell\partial_3\}$.

(3.4.4) Proposition. $\operatorname{Ider}_k(A;2) = \operatorname{Ider}_k(A)$.

Proof. We need to prove that $x_3\partial_2, \ell\partial_2, x_3\partial_3, \ell\partial_3$ are I-logarithmically integrable.

The derivation $x_3\partial_3$ is the Euler vector field for the weights $w(x_1) = w(x_2) = 2, w(x_3) = 3$. From (2.2.5) we deduce that $x_3\partial_3$ is I-logarithmically integrable.

The derivation $\ell\partial_3$ is I-logarithmically integrable since $f(x_1+t^2, x_2+t^2, x_3+\ell t) = \cdots = f \in R[t] \subset R[[t]]$ and so a I-logarithmic integral of $\ell\partial_3$ is given by the k-algebra map $R \to R[[t]]$ determined by

$$x_1 \mapsto x_1 + t^2, \quad x_2 \mapsto x_2 + t^2, \quad x_3 \mapsto x_3 + \ell t.$$

For the derivation $x_3\partial_2$ let us write $W(t) = \frac{x_1^2 t^2}{1 - x_1 t^2} \in (t^2)R[[t]]$ and consider the homomorphism of k-algebras $\Psi : R \to R[[t]]$ given by:

$$x_1 \mapsto x_1 + W(t), \quad x_2 \mapsto x_2 + x_3 t + W(t), \quad x_3 \mapsto x_3.$$

We have $\Psi(f) = f(x_1 + W, x_2 + x_3 t + W, x_3) = \cdots = \left(\frac{1}{1 - x_1 t^2}\right) f$ and so Ψ gives rise to a I-logarithmic integral of $x_3\partial_2$.

For the derivation $\ell\partial_2$ let us write $V(t) = \frac{x_1 t^2}{1 - t^2} \in (t^2)R[[t]]$ and consider the homomorphism of k-algebras $\Psi : R \to R[[t]]$ given by:

$$x_1 \mapsto x_1 + V(t), \quad x_2 \mapsto x_2 + \ell t + V(t), \quad x_3 \mapsto x_3.$$

We have $\Psi(f) = f(x_1 + V, x_2 + \ell t + V, x_3) = \cdots = f$ and so Ψ gives rise to a I-logarithmic integral of $\ell\partial_2$. Q.E.D.

In this example the descending chain of modules of integrable derivations stabilizes from $N = 2$:

$$\operatorname{Der}_k(A) = \operatorname{Ider}_k(A;1) \supset \operatorname{Ider}_k(A;2) = \operatorname{Ider}_k(A;3) = \cdots = \operatorname{Ider}_k(A;\infty)$$

and

$$\operatorname{ann}_A\left(\operatorname{Der}_k(A)/\operatorname{Ider}_k(A;\infty)\right) = (\ell, x_3) = \sqrt{J}/I.$$

3.5 The surface $x_3^2 + x_1x_2(x_1+x_2)^2 = 0$ in characteristic 2

Let k be a field of characteristic 2, $f = x_3^2 + x_1x_2(x_1+x_2)^2 \in R = k[x_1,x_2,x_3]$, $I = (f)$ and $A = R/I$. The Jacobian ideal is $J = (x_2\ell^2, x_1\ell^2, f) = (x_2\ell^2, x_1\ell^2, x_3^2)$ with $\ell = x_1 + x_2$. It is clear that $\sqrt{J} = (\ell, x_3)$. The module $\operatorname{Der}_k(\log I)$ is generated mod. $f\operatorname{Der}_k(R)$ by ∂_3, $\varepsilon = x_1\partial_1 + x_2\partial_2$ and $\eta = x_1^2\ell^2\partial_1 + x_3^2\partial_2$ ($\partial_3(f) = \varepsilon(f) = 0, \eta(f) = x_1\ell^2 f$). Since ε is the Euler vector field for the weights $w(x_1) = w(x_2) = 1, w(x_3) = 2$, we deduce from (2.2.5) that ε is I-logarithmically integrable. From Proposition (2.2.1) we also deduce that η is I-logarithmically integrable.

To find a system of generators of $\operatorname{Ider}_k(\log I; 2)$ we need the conditions on $a \in R$ which guarantee that $a\partial_3$ is I-logarithmically 2-integrable. The coefficient of t^2 in $f(x_1,x_2,x_3+at) = f + a^2t^2$ is a^2, and so $a\partial_3$ is I-logarithmically 2-integrable if and only if $a^2 \in J$.

(3.5.1) Lemma. $\{a \in R | a^2 \in J\} = (x_3, x_1\ell, x_2\ell)$.

(3.5.2) Corollary. *A system of generators of* $\operatorname{Ider}_k(\log I; 2)$ *mod.* $f\operatorname{Der}_k(R)$ *is* $\{x_3\partial_3, x_1\ell\partial_3, x_2\ell\partial_3, \varepsilon, \eta\}$. *In particular we have*

$$\operatorname{ann}_A\left(\operatorname{Der}_k(A)/\operatorname{Ider}_k(A;2)\right) = (\overline{x_3}, \overline{x_2}\overline{\ell}, \overline{x_1}\overline{\ell}).$$

The following lemma is a very particular case of a general result.

(3.5.3) Lemma. *Any Hasse–Schmidt derivation* $E \in \operatorname{HS}_k(A;2)$ *is* 3*-integrable.*

Proof. Since 3 is invertible in k, we can consider the differential operator $E_3 = E_1E_2 - \frac{1}{3}E_1^3$ and check that $(\mathrm{Id}_A, E_1, E_2, E_3)$ is a Hasse–Schmidt derivation. Q.E.D.

As a consequence of the above lemma we have $\operatorname{Ider}_k(A;2) = \operatorname{Ider}_k(A;3)$.

Let us see the conditions for $a\partial_3$, with $a = \alpha x_3 + \beta x_1\ell + \gamma x_2\ell$, $\alpha,\beta,\gamma \in R$, to be I-logarithmically 4-integrable. The algebra map associated with a general I-logarithmic 3-integral of $a\partial_3$ is $\Psi^{(3)} : R \to R_3$ given by:

$$\begin{aligned}
x_1 &\mapsto x_1 + (\alpha^2x_1 + \gamma^2x_2 + B_1x_1 + C_1x_1^2\ell^2)t^2 + (B_2x_1 + C_2x_1^2\ell^2)t^3,\\
x_2 &\mapsto x_2 + (\beta^2x_1 + B_1x_2 + C_1x_3^2)t^2 + (B_2x_2 + C_2x_3^2)t^3,\\
x_3 &\mapsto x_3 + (\alpha x_3 + \beta x_1\ell + \gamma x_2\ell)t + A_1t^2 + A_2t^3
\end{aligned}$$

with $A_2, B_2, C_2 \in R$, and let $\Psi_0^{(4)} : R \to R_4$ be the obvious lifting of $\Psi^{(3)}$. The coefficient mod J of t^4 in the expression of $\Psi_0^{(4)}(f)$, is $x_1x_2^3(\alpha+\beta+\gamma)^4 + A_1^2$. So, we have proved the following lemma.

(3.5.4) Lemma. *With the above notations, the following assertions are equivalent:*

- *(a) The logarithmic derivation* $a\partial_3$*, with* $a = \alpha x_3 + \beta x_1\ell + \gamma x_2\ell$*, is* I*-logarithmically* 4*-integrable.*
- *(b) There is* $A_1 \in R$ *such that* $x_1x_2^3(\alpha+\beta+\gamma)^4 + A_1^2 \in J$*, or, equivalently,* $x_1x_2^3(\alpha+\beta+\gamma)^4 \in J + R^2$.

(3.5.5) LEMMA. *We have* $\{\varphi \in R \mid x_1 x_2^3 \varphi^4 \in J + R^2\} = (x_3, \ell)$.

PROOF. Let us write $\mathfrak{A} = \{\varphi \in R \mid x_1 x_2^3 \varphi^4 \in J + R^2\}$. It is clear that $x_3, \ell \in \mathfrak{A}$, since $x_3^4 \in J$ and $x_1 x_2^3 \ell^4 \in J$. Let φ be an element in $\mathfrak{A}$ and let us write $\varphi = q x_3 + \varphi_1(x_1, x_2)$, with $q \in R$ and $\varphi_1(x_1, x_2) \in \mathfrak{A}$. We have

$$x_1 x_2^3 \varphi_1^4 = U(x_1, x_2) x_1 \ell^2 + V(x_1, x_2) x_2 \ell^2 + P(x_1, x_2)^2.$$

By taking derivatives with respect to x_1 we obtain $x_2^3 \varphi_1^4 = U'_{x_1} x_1 \ell^2 + U \ell^2 + V'_{x_1} x_2 \ell^2$ and so ℓ divides φ_1. We conclude that $\mathfrak{A} = (x_3, \ell)$. Q.E.D.

As a consequence of the above lemma and the fact that (x_3, ℓ) is a prime ideal, the condition $x_1 x_2^3 (\alpha + \beta + \gamma)^4 \in J + R^2$ is equivalent to $\alpha + \beta + \gamma \in (x_3, \ell)$, i.e. to $\alpha = \alpha_1 x_3 + \alpha_2 \ell + \beta + \gamma$ and so $a = \cdots = \alpha_1 x_3^2 + \alpha_2 x_3 \ell + \beta(x_3 + x_1 \ell) + \gamma(x_3 + x_2 \ell)$. We conclude with the following corollary.

(3.5.6) COROLLARY. *A system of generators of* $\operatorname{Ider}_k(\log I; 4) \mod f \operatorname{Der}_k(R)$ *is* $\{x_3^2 \partial_3, x_3 \ell \partial_3, (x_3 + x_1 \ell)\partial_3, (x_3 + x_2 \ell)\partial_3, \varepsilon, \eta\}$. *In particular we have*

$$\begin{gathered}
\operatorname{ann}_A (\operatorname{Der}_k(A)/\operatorname{Ider}_k(A; 2)) = (\overline{x_3}, \overline{x_2}\overline{\ell}, \overline{x_1}\overline{\ell}), \\
\operatorname{ann}_A (\operatorname{Der}_k(A)/\operatorname{Ider}_k(A; 4)) = (\overline{x_3}^2, \overline{x_3}\overline{\ell}, \overline{x_3} + \overline{x_2}\overline{\ell}, \overline{x_3} + \overline{x_1}\overline{\ell}), \\
\operatorname{ann}_A (\operatorname{Ider}_k(A; 2)/\operatorname{Ider}_k(A; 4)) = (\overline{x_3}, \overline{\ell})
\end{gathered}$$

and all the inclusions

$$J_{A/k} \subset (\overline{x_3}^2, \overline{x_3}\overline{\ell}, \overline{x_3} + \overline{x_2}\overline{\ell}, \overline{x_3} + \overline{x_1}\overline{\ell}) \subset (\overline{x_3}, \overline{x_2}\overline{\ell}, \overline{x_1}\overline{\ell}) \subset (\overline{x_3}, \overline{\ell}) = \sqrt{J_{A/k}}$$

are strict.

From Proposition (2.2.1) we deduce that $x_3^2 \partial_3$ is I-logarithmically integrable.

(3.5.7) LEMMA. *The derivation* $x_3 \ell \partial_3$ *is* I*-logarithmically integrable.*

PROOF. Let us write $\delta = x_3 \ell \partial_3$ and $D = (x_3 \ell) \bullet \Delta^{(3)}$. We have $\Phi_D(f) = f + (x_3 \ell)^2 t^2$ and $(x_3 \ell)^2 = f'_{x_1} f'_{x_2} + \ell^2 f = x_1 x_2 \ell^4 + \ell^2 f$. Let us also write $S = k[x_1, x_2]$ and $\mathfrak{b} = (f'_{x_1}, f'_{x_2}) = (x_2 \ell^2, x_1 \ell^2) \subset S$.

We are going to construct inductively a sequence of differential operators $E_m^m \in \mathfrak{b} \operatorname{Diff}_{S/k}$, $m \geq 1$, with $E_1^1 = 0$, $E_2^2(f) = x_1 x_2 \ell^4$, $E_m^m(f) = 0$ for all $m \geq 3$ and such that $(\mathrm{Id}, E_1^1, E_2^2, E_3^3, \ldots)$ is a Hasse-Schmidt derivation of length ∞.

For $m = 2$, let us take $E_2^2 = f'_{x_2} \partial_1$.

Assume that we have already found a Hasse–Schmidt derivation $E^m = (\mathrm{Id}, E_1^1, \ldots, E_m^m) \in \operatorname{HS}_k(S; m)$ with the required properties. Let us consider $F^m = \varepsilon(E^m) \in \operatorname{HS}_k(S; \infty)$. From Proposition (1.3.1), 2) we deduce that $F_{m+1}^m \in \mathfrak{b}^2 \operatorname{Diff}_{S/k}$ and so $F_{m+1}^m(f) \in \mathfrak{b}^2$. Hence, there are $\alpha, \beta \in \mathfrak{b}$ such that $F_{m+1}^m(f) = \alpha f'_{x_1} + \beta f'_{x_2}$ and consequently we can take $E_{m+1}^{m+1} = F_{m+1}^m - (\alpha \partial_1 + \beta \partial_2)$

Once the Hasse–Schmidt derivation $E = (\mathrm{Id}, 0, E_2^2, E_3^3, \ldots) \in \operatorname{HS}_k(S; \infty)$ has been constructed, we extend it in the obvious way to the ring R (we keep the same name E for the extension). We have $\Phi_{D \circ E}(f) = \widetilde{\Phi}_D(\Phi_E(f)) = \widetilde{\Phi}_D\left(f + x_1 x_2 \ell^4 t^2\right) = \Phi_D(f) + \Phi_D(x_1 x_2 \ell^4) t^2 = f + (x_3 \ell)^2 t^2 + x_1 x_2 \ell^4 t^2 = (1 + \ell^2 t^2) f$ and so $D \circ E$ is a I-logarithmic integral of δ. Q.E.D.

The proof of the following lemma is due to M. Mérida.

(3.5.8) Lemma. *The derivations $(x_3+x_1\ell)\partial_3$ and $(x_3+x_2\ell)\partial_3$ are I-logarithmically integrable.*

Proof. By symmetry, it is enough to consider the case $(x_3 + x_1\ell)\partial_3$, for which the logarithmic integrability is a consequence of the fact that the map $\Psi : R \to R[[t]]$ given by:

$$\begin{array}{rcl} x_1 & \mapsto & x_1 + x_1 V, \\ x_2 & \mapsto & x_2 + x_1 V, \\ x_3 & \mapsto & x_3 + (x_3 + x_1\ell)t + x_3 V, \end{array}$$

with $V = \sum_{i=1}^{\infty} t^{2^i}$, is I-logarithmic. Namely, since $t^2 = V^2 + V$, we have

$$\begin{gathered} f(x_1 + x_1V, x_2 + x_1V, x_3 + (x_3 + x_1\ell)t + x_3V) = \\ (x_3 + (x_3 + x_1\ell)t + x_3V)^2 + (x_1 + x_1V)(x_2 + x_1V)\ell^2 = \\ x_3^2 + (x_3^2 + x_1^2\ell^2)t^2 + x_3^2V^2 + (x_1x_2 + x_1^2V + x_1x_2V + x_1^2V^2)\ell^2 = \\ x_3^2 + (x_3^2 + x_1^2\ell^2)t^2 + x_3^2V^2 + (x_1x_2 + x_1^2t^2 + x_1x_2V)\ell^2 = \\ x_3^2 + x_3^2t^2 + x_3^2V^2 + (x_1x_2 + x_1x_2V)\ell^2 = f + x_3^2t^2 + x_3^2V^2 + x_1x_2V\ell^2 = \\ f + x_3^2V + x_1x_2V\ell^2 = (1+V)f. \end{gathered}$$

Q.E.D.

(3.5.9) Corollary. $\mathrm{Ider}_k(A;4) = \mathrm{Ider}_k(A)$.

3.6 Some questions

(3.6.1) Question. Assume that $R = k[x_1, \ldots, x_d]$, $S \subset R$ is a multiplicative set and $A = S^{-1}R$ or $A = k[[x_1, \ldots, x_d]]$. Let $I \subset A$ be an ideal, $m \geq 1$ an integer, $D \in \mathrm{HS}_k(\log I; m)$ and $E = \overline{D} \in \mathrm{HS}_k(A/I; m)$. Let us consider the following properties:

(a) D is I-logarithmically n-integrable for all integers $n \geq m$ (or equivalently, E is n-integrable for all integers $n \geq m$).

(b) D is I-logarithmically ∞-integrable (or equivalently E is ∞-integrable).

Under which hypotheses on k and on I are properties (a) and (b) equivalent for any $D \in \mathrm{HS}_k(\log I; m)$? Are they equivalent if k is a field or the ring of integers and I is arbitrary?

Notice that this question is the same as asking whether the inclusion in 3 (or in 1 for $m = 1$) is an equality or not.

(3.6.2) Question. The proofs of propositions (2.3.2) and (2.3.3) do not work for $m = \infty$ and, presumably, these propositions are not true for $m = \infty$ without additional finiteness hypotheses on k. Let us notice that if the maps in Proposition (2.3.3) are surjective for $m = \infty$, then the localization conjecture for the Hasse–Schmidt algebra stated in [16] is true.

(3.6.3) Question. For any finitely presented k-algebra A, find an algorithm for deciding whether a given $\delta \in \mathrm{Der}_k(A)$ is m-integrable or not.

(3.6.4) Question. For any finitely presented k-algebra A, find an algorithm to obtain a system of generators of $\mathrm{Ider}_k(A; m)$, $m \geq 2$.

(3.6.5) Question. Assume that the base ring k is a field of positive characteristic or $\mathbb{Z}$, or perhaps a more general noetherian ring, and A a finitely generated k-algebra. Is there an integer $n \geq 1$ such that $\mathrm{Ider}_k(A; n) = \mathrm{Ider}_k(A; \infty)$? Or at least, is the descending chain of A-modules $\mathrm{Ider}_k(A; 1) \supset \mathrm{Ider}_k(A; 2) \supset \mathrm{Ider}_k(A; 3) \supset \cdots$ stationary?

(3.6.6) Question. Assume that the base ring k is a field of positive characteristic or $\mathbb{Z}$, or perhaps a more general noetherian ring. Is there an integer $m \gg 1$, possibly depending on d and e or other numerical invariants, such that

$$\mathrm{Ider}_k(A; m) = \mathrm{Der}_k(A) \quad \Rightarrow \quad \mathrm{Ider}_k(A) = \mathrm{Der}_k(A)$$

for every quotient ring $A = k[x_1, \ldots, x_d]/I$ with $\dim A = e$?

(3.6.7) Question. Assume that the base ring k is a field of positive characteristic or $\mathbb{Z}$, or perhaps a more general noetherian ring, A a local noetherian k-algebra and $\delta : A \to A$ a k-derivation. Under which hypotheses the m-integrability of $\widehat{\delta} : \widehat{A} \to \widehat{A}$ implies the m-integrability of δ?

References

[1] W.C. Brown. On the embedding of derivations of finite rank into derivations of infinite rank. *Osaka J. Math.* 15 (1978), 381–389.

[2] W. Bruns and U. Vetter. Determinantal Rings. Lecture Notes in Math. 1327. Springer-Verlag, Berlin, Heidelberg, 1988.

[3] M. Fernández-Lebrón and L. Narváez-Macarro. Hasse-Schmidt derivations and coefficient fields in positive characteristics. *J. Algebra* 265 (1) (2003), 200–210.

[4] D. R. Grayson and M. E. Stillman. Macaulay 2. Available at http://www.math.uiuc.edu/Macaulay2/.

[5] A. Grothendieck and J. Dieudonné. Eléments de Géométrie Algébrique IV: Etude locale des schémas et de morphismes de schémas (Quatrième Partie), volume 32 of Inst. Hautes Études Sci. Publ. Math., Press Univ. de France, Paris, 1967.

[6] H. Hasse and F. K. Schmidt. Noch eine Begründung der Theorie der höheren Differrentialquotienten in einem algebraischen Funktionenkörper einer Unbestimmten. *J. Reine U. Angew. Math.* 177 (1937), 223-239.

[7] J. Lipman. On the Jacobian ideal of the module of differentials. *Proc. Amer. Math. Soc.* 21 (1969), 422–426.

[8] G. Másson. Differential Operators and Étale Extensions. Ph.D. Thesis, Massachusetts Institute of Technology, 1991.

[9] H. Matsumura. Integrable derivations. *Nagoya Math. J.* 87 (1982), 227–245.

[10] H. Matsumura. Commutative Ring Theory. Vol. 8 of Cambridge studies in advanced mathematics, Cambridge Univ. Press, Cambridge, 1986.

[11] S. Molinelli. Sul modulo delle derivazioni integrabili in caractteristica positiva. *Ann. Mat. Pura Appl.* 121 (1979), 25–38.

[12] L. Narváez Macarro. Hasse–Schmidt derivations, divided powers and differential smoothness. *Ann. Inst. Fourier (Grenoble)* 59 (7) (2009), 2979–3014.

[13] A. Seidenberg. Derivations and integral closure. *Pacific J. Math.* 16 (1) (1966), 167–173.

[14] W.N. Traves. Differential Operators and Nakai's Conjecture. PhD Thesis, Department of Mathematics, University of Toronto, 1998.

[15] W.N. Traves: Tight Closure and Differential Simplicity. *J. of Algebra* 228 (2000), 457–476.

[16] W.N. Traves: Localization of the Hasse-Schmidt Algebra *Canad. Math. Bull.* 46 (2) (2003), 304–309.

[17] P. Vojta. Jets via Hasse–Schmidt derivations. In "Diophantine geometry", CRM Series, vol. 4, Ed. Norm., Pisa, 2007, 335–361.

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