Existence of positive solution of a nonlocal logistic population model

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Abstract. In this paper, we study the existence of positive solutions for a class of nonlocal problem arising in population dynamic. Basically, we prove our results via bifurcation theory.

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1. Introduction

In this paper, we study the existence of positive solution for the following class of nonlocal problem

$$\begin{cases} -\Delta u = u\left(\lambda - \displaystyle\int_\Omega K(x,y)u^p(y)dy\right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^N, N \geq 1$, is a smooth bounded domain, $p > 0$ and $K : \Omega \times \Omega \to \mathbb{R}$, is a non-negative function with $K \in L^\infty(\Omega \times \Omega)$ and verifying other hypotheses that will be detailed below.

Our motivation to study the above problem begins with the most used equation to model the behaviour of a species inhabiting in a domain Ω, that is the classical logistic equation

$$\begin{cases} -\Delta u = u(\lambda - b(x)u^p) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here, $u(x)$ is the population density at location $x \in \Omega$, $\lambda \in \mathbb{R}$ is the growth rate of the species, and b is a positive function denoting the carrying capacity, that is, $b(x)$ describes the limiting effect of crowding of the population. In (1), we are assuming that Ω is surrounded by inhospitable areas, due to the homogeneous Dirichlet boundary conditions. Equation (1) is a local equation, and so the crowding effect of the population u at x only depends on the value of the population in the same point x. It seems more realistic (see for instance [3]) to consider that this crowding effect depends also on the value of the population around of x, that is, the crowding effect depends on the value of u in a neighborhood of x, $B_r(x)$, the centered ball at x of radius $r > 0$. So, we consider the equation

$$\begin{cases} -\Delta u = u\left(\lambda - \displaystyle\int_{\Omega\cap B_r(x)} b(y)u^p(y)dy\right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where b is a nonnegative and nontrivial continuous function. In fact, we are going to study a more general problem, that is, problem (P).

We would like to mention that the nonlocal term has been also used to model the selection process of a population structured by phenotypical trait, see [7].

Before to statement our main results, we will recall some known results with respect to (P), involving different conditions on K. Hereafter, λ_1 denotes the first eigenvalue of the Laplacian under homogeneous Dirichlet boundary conditions.

When K is separable variable, i.e.,

$$K(x,y) = g(x)h(y), h \geq 0, h \neq 0 \text{ and } g(x) > 0 \text{ in } \Omega, \tag{K_1}$$

it is proved in [4] that (P) possesses a unique positive solution for $\lambda > \lambda_1$. Moreover, in [5] and [6], assuming $g \equiv 1$, $p > 1$ and under homogeneous Neumann boundary conditions, it is proved that the positive solution of (P) attracts all the possible solutions of the corresponding parabolic associated to (P). When $g \geq 0$, $g \neq 0$, $g \equiv 0$ in $\Omega_0 \subset \Omega$, then (P) possesses a unique positive solution for $\lambda \in (\lambda_1, \lambda_0)$ where λ_0 is the principal eigenvalue of the Laplacian in Ω_0, see [4].

In [1], a similar result is proved when $K(x,y) = K_\delta(|x-y|)$ is a mollifier in $\mathbb{R}^N$, i. e., $K_\delta(|x-y|) \in C_0^\infty$, $\int_{\mathbb{R}^N} K_\delta(|x-y|)dy = 1$ for any x with

$$K_\delta(|x-y|) = 0 \text{ if } |x-y| \geq \delta \tag{K_2}$$

and

$$K_\delta(|x-y|) \text{ bounded away from zero is } |x-y| < \mu < \delta. \tag{K_3}$$

Observe that in this case, K vanishes away from the diagonal of $\Omega \times \Omega$.

For kernel functions $K(x,y)$ verifying that

$$K(x,y) \geq K_0 > 0 \text{ for all } (x,y) \in \Omega \times \Omega, \tag{K_4}$$

in [6] it was proved that there exists a positive solution of (P) if, and only if, $\lambda > \lambda_1$, see also [5].

In [2], when $p = 1$ and $K \in C(\overline{\Omega} \times \overline{\Omega})$ is a nonnegative function such that for all $\phi > 0$ it holds that

$$\int_\Omega K(x,y)\phi(y)dy > 0 \tag{K_5}$$

then it is shown the existence of $\lambda^* > \lambda_1$ such that (P) possesses at least a positive solution for $\lambda \in (\lambda_1, \lambda^*]$; for that the authors use the implicit function theorem.

Finally, (P) has been studied also for the case $N = 1$. Indeed, in [6] the existence of positive solution it proved if

$$K(x,x) \geq K_0 > 0 \text{ for all } x \in \Omega \tag{K_6}$$

and also in [9] for $K(x,y) = K_1(|x-y|)$ and $\Omega = (-1,1)$, where $K_1 : [0,2] \mapsto (0,\infty)$ is a nondecreasing and piecewise continuous map with

$$\int_0^2 K_1(y)dy > 0. \tag{K_7}$$

In this paper we are interested in giving new conditions on K to assure the existence of positive solution for all $\lambda > \lambda_1$ or the non-existence of positive solution for λ large. To this end, we introduce the class $\mathcal{K}$, which is formed by functions $K : \Omega \times \Omega \to \mathbb{R}$ verifying:

i) $K \in L^\infty(\Omega \times \Omega)$ *and* $K(x,y) \geq 0$ *for all* $x, y \in \Omega$.

ii) *If* w *is measurable and* $\displaystyle\int_{\Omega\times\Omega} K(x,y)|w(y)|^p w(x)^2 dxdy = 0$, *then* $w = 0$ *a.e in* Ω.

Theorem 1. *Suppose that* $K \in \mathcal{K}$. *Then problem* (P) *has a positive solution if, and only if,* $\lambda > \lambda_1$.

Here, we would like to point out that Theorem 1 implies that (2) has a positive solution if, and only if, $\lambda > \lambda_1$, because the function $K(x,y) = b(y)\chi_{B_r(x)}(y)$ belongs to $\mathcal{K}$, once that b is a positive function on Ω. Moreover, we observe that Theorem 1 also improves the above results, allowing that K vanishes in some part of $\Omega \times \Omega$ in a general way, not only in a symmetric as in $(K_2) - (K_3)$.

If K does not belong to $(\mathcal{K})$, then K vanishes in some neighborhood of the diagonal of $\Omega \times \Omega$, see Section 3. Here, we are also able to prove results of existence and non-existence of positive solution for some values of λ if K belongs to a class $\mathcal{K}'$, which is formed by functions $K : \Omega \times \Omega \to \mathbb{R}$ verifying the following condition:

There are $r > 0$ and m connected open sets $\Omega_1, \Omega_2, \Omega_3, ..., \Omega_m \subset \Omega$ such that $\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset$, $i \neq j$, and

$$K(x,y) > 0, \text{ for all } (x,y) \in \Omega \times \Omega \text{ such that } x \notin U := \bigcup_{j=1}^{m} \Omega_j \text{ and } |x-y| < r.$$

Our main results involving this class of function is

Theorem 2. *Suppose that $K \in \mathcal{K}'$. Then, for any $\lambda_1 < \lambda < \min\{\lambda_1(\Omega_1), , ..., \lambda_1(\Omega_m)\}$, problem (P) has a positive solution.*

In the above result, $\lambda_1(\Omega_i)$ denotes the principal eigenvalue of the Laplacian in Ω_i under homogeneous Dirichlet boundary conditions. As a by product, we have the following corollary

Corollary 3. *Suppose that $K \in \mathcal{K}'$ with $m = 1$ and $K(x,y) = 0$ in U for any $y \in \Omega$. Suppose that ∂U is C^1. For any $\lambda_1 < \lambda < \lambda_1(U)$, there exists a positive solution u for the problem (P). Moreover, (P) does not have any positive solution for $\lambda \geq \lambda_1(U)$.*

An outline of the paper is as follows: in Section 2 we show, using bifurcation arguments, the existence of positive solution under class $\mathcal{K}$. Section 3 is devoted to the case when K belongs to $\mathcal{K}'$.

2. Proof of Theorem 1

In whole this section, we are assuming that $K \in \mathcal{K}$. Moreover, for any $w \in L^\infty(\Omega)$, we will consider the function $\phi_w : \Omega \to \mathbb{R}$ given by

$$\phi_w(x) := \int_\Omega K(x,y)|w(y)|^p dy.$$

Once that K and w are bounded, we have that ϕ_w is well defined. Furthermore, the ensuing properties will be useful along the paper:

$$t^p \phi_w = \phi_{tw}; \text{ for all } w \in L^\infty(\Omega), t > 0; \tag{ϕ_1}$$

$$||\phi_w||_\infty \leq ||K||_\infty |\Omega| \, ||w||_\infty^p, \text{ for all } w \in L^\infty(\Omega); \tag{ϕ_2}$$

$$||\phi_w - \phi_v||_\infty \leq ||K||_\infty |\Omega| \, || \, |w|^p - |v|^p ||_\infty, \text{ for all } w, v \in L^\infty(\Omega); \tag{ϕ_3}$$

and

$$\phi : L^\infty(\Omega) \to L^\infty(\Omega), \ \phi(u) = \phi_u \text{ is uniformly continuous in } L^\infty(\Omega). \tag{ϕ_4}$$

Using the above notation, it is easy to observe (P) can be rewritten by

$$\begin{cases} -\Delta u + \phi_u u = \lambda u & \text{in } \Omega, \\ u(x) > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_1}$$

Here, we recall that u satisfies the above problem in weak sense, if $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and

$$\int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega \phi_u u v dx = \lambda \int_\Omega u v dx \ \forall v \in H_0^1(\Omega). \tag{3}$$

Hereafter, we intend to solve problem (P_1) by using the classical bifurcation result of Rabinowitz, see [8]. To this end, we recall that there exists $c_\infty = c_\infty(\Omega) > 0$ such that: for each $f \in L^\infty(\Omega)$, there exists a unique $\omega \in C^1(\overline{\Omega})$ satisfying

$$\begin{cases} -\Delta \omega = f(x) & \text{in } \Omega, \\ \omega = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$||\omega||_{C^1(\overline{\Omega})} \leq c_\infty ||f||_\infty.$$

Hence, the solution operator $S : C^0(\overline{\Omega}) \to C^1(\overline{\Omega})$ given by

$$Sv = \omega_1 \iff \begin{cases} -\Delta\omega_1 = v & \text{in } \Omega, \\ \omega_1 = 0 & \text{on } \partial\Omega, \end{cases}$$

is well-defined, it is linear and verifies

$$||Sv||_{C^1(\overline{\Omega})} \leq c_\infty ||v||_{C^0(\overline{\Omega})}, \ \forall v \in C^0(\overline{\Omega}).$$

Moreover, using the Schuart imbeeding, $S : C^0(\overline{\Omega}) \to C^0(\overline{\Omega})$ is a compact operator. Related to spectrum of S, it easy to see that

$$\sigma(S) = \{\lambda_j^{-1} \ : \ \lambda_j \text{ is a eigenvalue of the Laplacian}\},$$

On the other hand, define the nonlinear operator $G : C^0(\overline{\Omega}) \to C^1(\overline{\Omega})$ given by

$$G(v) = \omega_2 \iff \begin{cases} -\Delta\omega_2 + \phi_v v = 0 & \text{in } \Omega, \\ \omega_2 = 0 & \text{on } \partial\Omega, \end{cases}$$

which is continuous and satisfies

$$||G(v)||_{C^1(\overline{\Omega})} \leq c_\infty ||\phi_v||_\infty ||v||_{C^0(\overline{\Omega})}, \forall v \in C^0(\overline{\Omega}).$$

Using again the Schuart imbedding, we have that $G : C^0(\overline{\Omega}) \to C^0(\overline{\Omega})$ is compact. Furthermore, since

$$||G(v)||_{C^0(\overline{\Omega})} \leq ||G(v)||_{C^1(\overline{\Omega})},$$

we have

$$\left\| \frac{G(v)}{||v||_{C^0(\overline{\Omega})}} \right\|_{C^0(\overline{\Omega})} \leq \frac{||G(v)||_{C^1(\overline{\Omega})}}{||v||_{C^0(\overline{\Omega})}} \leq c_\infty ||\phi_v||_\infty,$$

from where it follows that

$$\lim_{v \to 0} \frac{G(v)}{||v||_{C^0(\overline{\Omega})}} = 0, \qquad (\mathcal{G})$$

i.e.,

$$G(v) = o(||v||_{C^0(\overline{\Omega})}).$$

Of course, under these new notations: (λ, u) solves (P) if, and only if,

$$u = F(\lambda, u) := \lambda S u + G(u).$$

Now, as a direct consequence of [8], we have the following result

Theorem 4. *(***Global bifurcation***) Let E be a Banach space. Suppose that S is a compact linear operator and $\lambda^{-1} \in \sigma(S)$ and its multiplicity is odd. If G satisfies condition $(\mathcal{G})$, then set*

$$\Sigma = \overline{\{(\lambda, u) \in \mathbb{R} \times E : u = \lambda S u + G(u), u \neq 0\}}$$

has a closed connected component $\mathcal{C} = \mathcal{C}_\lambda$ such that $(\lambda, 0) \in \mathcal{C}$ and

(i) *$\mathcal{C}$ is unbounded in $\mathbb{R} \times E$, or*
(ii) *there exists $\hat{\lambda} \neq \lambda$ such that $(\hat{\lambda}, 0) \in \mathcal{C}$ and $\hat{\lambda}^{-1} \in \sigma(S)$.*

It is known that the first eigenfunction φ_1 associated to λ_1 can be chosen positive. Moreover, λ_1^{-1} is an eigenvalue with odd multiplicity for S.

From global bifurcation theorem, there exists a closed connected component $\mathcal{C} = \mathcal{C}_{\lambda_1}$ of solutions for (P), which satisfies (i) or (ii).

Lemma 5. *There exists $\delta > 0$ such that if $(\lambda, u) \in \mathcal{C}$ with $|\lambda - \lambda_1| + ||u||_{C^0(\overline{\Omega})} < \delta$ and $u \neq 0$, then u has defined signal, i.e.,*

$$u(x) > 0 \ \forall x \in \Omega \quad or \quad u(x) < 0 \ \forall x \in \Omega.$$

Proof: Take (u_n) in $C^0(\overline{\Omega})$ and $\lambda_n \to \lambda_1$ such that,

$$u_n \neq 0,\ ||u_n||_{C^0(\overline{\Omega})} \to 0 \text{ and } u_n = F(\lambda_n, u_n).$$

Consider $w_n = u_n/||u_n||_{C^0(\overline{\Omega})}$ and observe that

$$\begin{cases} -\Delta w_n + \phi_{u_n} w_n = \lambda_n w_n & \text{in } \Omega, \\ w_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

It is easy to check that

$$||w_n||_{C^1(\overline{\Omega})} \leq c_\infty(\lambda_n + ||\phi_{u_n}||_\infty)||w_n||_{C^0(\overline{\Omega})} \leq c_\infty(\lambda_n + ||\phi_{u_n}||_\infty)\ \ \forall n \in \mathbb{N}$$

Once that (u_n) is bounded in $C^0(\overline{\Omega})$, it follows from (ϕ_2) that $(||\phi_{u_n}||_\infty)$ is bounded, therefore (w_n) is bounded in $C^1(\overline{\Omega})$. By using Arzelá-Áscoli theorem, (w_n) converges to some $w \in C^1(\overline{\Omega})$, uniformly in $\overline{\Omega}$, under a convenient subsequence. Of course $||w||_{C^0(\overline{\Omega})} = 1$, showing that $w \neq 0$ in Ω.

Now, by (ϕ_3), we know that (ϕ_{u_n}) is a Cauchy sequence in $C^0(\overline{\Omega})$. Then, this fact combined with the below inequality

$$\begin{aligned} ||w_n - w_m||_{C^1(\overline{\Omega})} \leq c_\infty[||\lambda_n u_n - \lambda_m u_m||_{C^0(\overline{\Omega})} + ||\phi_{u_n} - \phi_{u_m}||_\infty \\ + ||\phi_{u_n}||_\infty ||w_n - w_m||_{C^0(\overline{\Omega})}], \end{aligned}$$

give that (w_n) converges to w in $C^1(\overline{\Omega})$, and so, passing to the limit in

$$\int_\Omega \nabla w_n \cdot \nabla v dx + \int_\Omega \phi_{u_n} w_n v dx = \lambda_n \int_\Omega w_n v dx,$$

and recalling that by (ϕ_2), $\phi_{u_n} w_n \to 0$ in $C^0(\overline{\Omega})$, we get

$$\begin{cases} -\Delta w = \lambda_1 w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $w \neq 0$, by spectral theory, we must have

$$w(x) > 0\ \forall x \in \Omega \text{ or } w(x) < 0\ \forall x \in \Omega.$$

Without loss of generality, we can suppose that $w(x) > 0$ for all $x \in \Omega$. As w is the $C^1(\overline{\Omega})$-limit of (w_n), we must have $w_n(x) > 0$ for all $x \in \Omega$ for n large enough. Thereby, the sign of u_n is the same of w_n for n large enough finishing the proof.

□

It is easy to check that: if $(\lambda, u) \in \Sigma$, the pair $(\lambda, -u)$ also is in Σ. In what follows, we decompose $\mathcal{C}$ into $\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^-$ where

$$\mathcal{C}^+ = \{(\lambda, u) \in \mathcal{C} : u(x) \geq 0, \forall x \in \Omega\}$$

and

$$\mathcal{C}^- = \{(\lambda, u) \in \mathcal{C} : u(x) \leq 0, \forall x \in \Omega\}.$$

A simple computation gives that $\mathcal{C}^- = \{(\lambda, u) \in \mathcal{C} : (\lambda, -u) \in \mathcal{C}^+\}$, $\mathcal{C}^+ \cap \mathcal{C}^- = \{(\lambda_1, 0)\}$ and $\mathcal{C}^+$ is unbounded if, and only if, $\mathcal{C}^-$ is also unbounded.

Lemma 6. *$\mathcal{C}^+$ is unbounded.*

Proof: Suppose that $\mathcal{C}^+$ is bounded. Then $\mathcal{C}$ is also bounded. From global bifurcation theorem, $\mathcal{C}$ contains $(\hat{\lambda}, 0)$, where $\hat{\lambda} \neq \lambda_1$ and $\hat{\lambda}^{-1} \in \sigma(S)$.

In this way, we can take (u_n) in $C^0(\overline{\Omega})$ and $\lambda_n \to \hat{\lambda}$ such that,

$$u_n \neq 0,\ ||u_n||_{C^0(\overline{\Omega})} \to 0 \text{ and } u_n = F(\lambda_n, u_n).$$

Considering $w_n = u_n/||u_n||_{C^0(\overline{\Omega})}$, we know that it satisfies problem (4). Moreover, as in the proof of previous lemma, under an adequate subsequence, (w_n) converges to w in $C^1(\overline{\Omega})$, which is a nonzero solution of the eigenvalue problem

$$\begin{cases} -\Delta w = \hat{\lambda} w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

showing that w is a eigenfunction related to $\hat{\lambda}$. Since $\hat{\lambda} \neq \lambda_1$, w must change sign. Then, for n large, each w_n must change sign, and the same should be hold for $u_n = ||u_n||_{C^0(\overline{\Omega})} w_n$. But this is not possible, because $(\lambda_n, u_n) \in \mathcal{C}^+$ or $(\lambda_n, u_n) \in \mathcal{C}^-$.

□

2.1. A priori estimate

From Lemma 6, the connected component $\mathcal{C}^+$ is unbounded. Now, our goal it is to show that this component intersects any set of the form $\{\lambda\} \times H_0^1(\Omega)$, for $\lambda > \lambda_1$.

Lemma 7. *Suppose that $K \in \mathcal{K}$. For any $\Lambda > 0$, there exists $r > 0$ such that: if $(\lambda, u) \in \mathcal{C}^+$ and $\lambda \leq \Lambda$, we must have $||u||_{C^0(\overline{\Omega})} \leq r$.*

Proof: From now on, we denote by $||\ ||$ the usual norm in $H_0^1(\Omega)$, i.e.,

$$||u||^2 = ||u||^2_{H_0^1(\Omega)} = \int_\Omega |\nabla u|^2 dx.$$

We start with the following claim:

Claim. *For any $\Lambda > 0$, there exists $r > 0$ such that: if $(\lambda, u) \in \mathcal{C}^+$ and $\lambda \leq \Lambda$, we should have $||u|| \leq r$.*

Indeed, arguing by contradiction, if it is not true, there are $(u_n) \subset H_0^1(\Omega)$ and $(\lambda_n) \subset [0, \Lambda]$ such that,

$$||u_n|| \to \infty \text{ and } u_n = F(\lambda_n, u_n).$$

Considering $w_n = u_n/||u_n||$, it follows that

$$\int_\Omega \nabla w_n \cdot \nabla v dx + \int_\Omega \phi_{u_n} w_n v dx = \lambda_n \int_\Omega w_n v dx, \quad \forall v \in H_0^1(\Omega).$$

Once that (w_n) is bounded in $H_0^1(\Omega)$, without loss of generality, we can suppose that there is $w \in H_0^1(\Omega)$ verifying

$$w_n \rightharpoonup w \text{ in } H_0^1(\Omega), w_n \to w \text{ in } L^2(\Omega) \text{ and } w_n(x) \to w(x) \text{ a.e. in } \Omega.$$

Taking $v = \dfrac{u_n}{||u_n||^{p+1}}$ as a test function, and recalling that $t^p \phi_{u_n} = \phi_{tu_n}$ for all $t > 0$, we obtain

$$\frac{1}{||u_n||^p} + \int_\Omega \phi_{w_n} w_n^2 dx = \frac{\lambda_n}{||u_n||^p} \int_\Omega w_n^2 dx, \ \forall n.$$

Passing to the limit in the above equality, we derive

$$\lim_n \int_\Omega \phi_{w_n} w_n^2 dx = 0.$$

From Fatou Lemma

$$\int_\Omega \phi_w w^2 dx \leq \lim_n \int_\Omega \phi_{w_n} w_n^2 dx = 0,$$

and so,

$$\int_{\Omega\times\Omega} K(x,y)|w(y)|^p |w(x)|^2 \ dxdy = 0.$$

Since $K \in \mathcal{K}$, we should have $w \equiv 0$. Thereby, (w_n) converges to 0 in $L^2(\Omega)$. Taking $v = w_n$ as test function, we see that

$$\int_\Omega |\nabla w_n|^2 dx + \int_\Omega \phi_{u_n} w_n^2 dx = \lambda_n \int_\Omega w_n^2 dx.$$

Since (λ_n) is bounded from above by Λ and $\int_\Omega \phi_{u_n} w_n^2 dx \geq 0$, we have

$$\int_\Omega |\nabla w_n|^2 dx \leq \Lambda \int_\Omega w_n^2 dx.$$

Taking the limit, we conclude that $||w_n|| \to 0$, which is an absurd, because $||w_n|| = 1$ for all n, proving the claim.

Since (u_n) is bounded in $H_0^1(\Omega)$, iteration arguments imply that (u_n) is bounded in $L^\infty(\Omega)$, and the proof is done.

□

Next, we will show the non-existence of solution for $\lambda \leq \lambda_1$, proving that $\mathcal{C}^+$ does not intersect $[0, \lambda_1] \times H_0^1(\Omega)$. In fact, suppose that

$$(\lambda, u) \in \mathcal{C}^+ \cap ([0, \lambda_1] \times H_0^1(\Omega)).$$

Using $v = \varphi_1$ as the test function in (3), we get

$$\lambda_1 \int_\Omega u\varphi_1 dx < \lambda_1 \int_\Omega u\varphi_1 dx + \int_\Omega \phi_u u\varphi_1 dx = \lambda \int_\Omega u\varphi_1 dx.$$

Since $\int_\Omega u\varphi_1 dx > 0$, the above inequality leads to $\lambda_1 < \lambda$.

Corollary 8. *Consider problem (2). There exists at least a positive solution of (2) if and only if $\lambda > \lambda_1$.*

Proof: It is clear that

$$K(x, y) = \chi_{\Omega \cap B_r(x)}(y) b(y) = \begin{cases} b(y) & y \in \Omega \cap B_r(x), \\ 0 & y \notin \Omega \cap B_r(x), \end{cases}$$

belongs to class $\mathcal{K}$. □

3. Proof of Theorem 2

We begin this section observing that if kernel K does not belong to $\mathcal{K}$, then there exists a measurable function $w : \Omega \to \mathbb{R}$ such that

$$\int_{\Omega\times\Omega} K(x, y)|w(y)|^p w(x)^2 dxdy = 0 \quad \text{but} \quad w \neq 0.$$

Thus, there exists $a > 0$ such that $A = \{x \in \Omega : |w(x)| \geq a\}$ has positive measure.

Observe that

$$a^{p+2} \int_{A\times A} K(x, y) dxdy \leq \int_{\Omega\times\Omega} K(x, y)|w(y)|^p w(x)^2 dxdy = 0,$$

i.e, $K = 0$ a.e. in $A \times A$.

Lemma 9. *If $K \in \mathcal{K}'$ and*

$$\int_\Omega \phi_w w^2 dx = 0,$$

then $w = 0$, a.e. in $\Omega \setminus U$.

Proof: It is easy to verify that $\phi_w(x)w(x)^2 = 0$, for all $x \in \Omega$. The, fixing $\varepsilon > 0$ and $A_\varepsilon = \{x \in \Omega \setminus U : |w(x)| \geq \varepsilon\}$, it follows that $\phi_w(x) = 0$ for all $x \in A_\varepsilon$ and

$$0 = \phi_w(x) = \int_\Omega K(x, y)|w(y)|^p dy \geq \int_{B_r(x)\cap A_\varepsilon} K(x, y)|w(y)|^p dy$$
$$\geq \varepsilon^p \int_{B_r(x)\cap A_\varepsilon} K(x, y) dy.$$

Once that $K \in \mathcal{K}'$, we can deduce that $|A_\varepsilon \cap B_r(x)| = 0$, for all $x \in A_\varepsilon$. Hence, $|A_\varepsilon| = 0$ for all $\varepsilon > 0$, from where it follows that $w = 0$ a.e. in $\Omega \setminus U$.

□

In what follows, for $D \subset \Omega$, $\lambda_1(D)$ denotes the first eigenvalue for the problem

$$\begin{cases} -\Delta w = \mu w & \text{in } D, \\ w = 0 & \text{on } \partial D, \end{cases}$$

which has a positive associated eigenfunction.

Now, we are ready to prove Theorem 2:

Proof of Theorem 2: It is enough to obtain an a priori estimate for $(\lambda, u) \in \mathcal{C}^+$ such that $\lambda \in [\lambda_1, \lambda^*]$ with $\lambda^* < \Lambda = \min\{\lambda_1(\Omega_1), \lambda_1(\Omega_2), ..., \lambda_1(\Omega_m)\}$.

Hereafter, we proceed as in the proof of Lemma 7. Let $(\lambda_n, u_n) \in \mathcal{C}^+$ such that

$$\lambda_n \in [\lambda_1, \lambda^*],\ u_n \neq 0,\ ||u_n|| \to \infty \text{ and } F(\lambda_n, u_n) = 0.$$

Then, $w_n = u_n/||u_n||$ satisfies

$$\int_\Omega \nabla w_n \cdot \nabla v dx + \int_\Omega \phi_{u_n} w_n v dx = \lambda_n \int_\Omega w_n v dx, \quad \forall v \in H_0^1(\Omega)$$

and under a subsequence, there exists $w \in H_0^1(\Omega)$ verifying

$$w_n \rightharpoonup w \text{ in } H_0^1(\Omega), w_n \to w \text{ in } L^2(\Omega), w_n(x) \to w(x) \text{ a.e. in } \Omega$$
$$\text{and } \int_\Omega \phi_w |w|^2 dx = 0.$$

We claim that $w \neq 0$. In fact, if (w_n) converges to 0 in $L^2(\Omega)$, we will get a contradiction as in the proof of Lemma 7.

Since $w \neq 0$, from Lemma 9, there exists some $j \in \{1, 2, ..., m\}$ such that $w \neq 0$, a. e. in Ω_j. Of course $w = w|_{\Omega_j} \in H_0^1(\Omega_j)$.

For any $v \in H_0^1(\Omega_j)$ with $v \geq 0$, we have

$$\int_\Omega \nabla w_n \cdot \nabla v dx \leq \int_\Omega \nabla w_n \cdot \nabla v dx + \int_\Omega \phi_{u_n} w_n v dx = \lambda_n \int_\Omega w_n v dx \leq \lambda^* \int_\Omega w_n v dx.$$

Passing to the limit in this last inequality, we derive

$$\int_{\Omega_j} \nabla w \cdot \nabla v dx \leq \lambda^* \int_{\Omega_j} w v dx, \text{ for any } v \in H_0^1(\Omega_j), v \geq 0.$$

Taking $v = \varphi_0$, where φ_0 is a positive eigenfunction associated to eigenvalue $\lambda_1(\Omega_j)$, we find

$$\lambda_1(\Omega_j) \int_{\Omega_j} w\varphi_0 dx = \int_{\Omega_j} \nabla w \cdot \nabla \varphi_0 dx \leq \lambda^* \int_{\Omega_j} w\varphi_0 dx,$$

which is a contradiction, because $\int_{\Omega_j} w\varphi_0 dx > 0$. □

In the following result, we prove the non-existence of positive solution for λ large when K vanishes in a sub-domain.

Lemma 10. *Assume that $D \subset \Omega$ is a sub-domain and suppose that $K(x, y) = 0$ in $D \subset \Omega$ for any $y \in \Omega$. Then, (P) does not possess positive solution for $\lambda \geq \lambda_1(D)$.*

Proof. Take φ_0, a positive eigenfunction associated to eigenvalue $\lambda_1(D)$, and v the prolongation by zero of φ_0 as test function in (P). Then,

$$\int_D \nabla u \cdot \nabla \varphi_0 dx + \int_\Omega \phi_u u\varphi_0 dx = \lambda \int_D u\varphi_0 dx.$$

We can check that

$$\int_\Omega \phi_u u\varphi_0 dx = \int_D \left(\int_\Omega K(x, y)|u(y)|^p dy \right) u\varphi_0 dx$$
$$= \int_\Omega \left(\int_D K(x, y)\varphi_0 dx \right) |u(y)|^p u dy = 0,$$

because $K(x, y) = 0$ in D for any $y \in \Omega$.

Using the Green identity,

$$\int_D \nabla u \cdot \nabla \varphi_0 dx = -\int_D u\Delta\varphi_0 dx + \int_{\partial D} u\frac{\partial \varphi_0}{\partial \nu} d\sigma < \lambda_1(D)\int_D u\varphi_0 dx,$$

because $\dfrac{\partial \varphi_0}{\partial \nu} < 0$ on ∂D, where ν is the outward unit normal vector.

Combining everything we have

$$\lambda \int_D u\varphi_0 dx = \int_D \nabla u \cdot \nabla \varphi_0 dx < \lambda_1(D)\int_D u\varphi_0 dx$$

which implies $\lambda < \lambda_1(D)$. □

As a direct consequence of Lemma 10 and Theorem 2 we have the Corollary 3.

Remark 11. *In the hypothesis of the Corollary* 3 *we see that* $(\lambda_1(U), \infty)$ *is a bifurcation at infinity for equation* $u = F(\lambda, u)$.

We finish with two examples in $N = 1$, $\Omega = (0, \pi)$ and so $\lambda_1((0, \pi)) = 1$. In the first one, K vanishes in the diagonal and there exists positive solution for all $\lambda > 1$. Consider

$$K(x,y) = \begin{cases} 1 & \text{if } (x,y) \in (0,\pi/2)\times(\pi/2,\pi) \cup (\pi/2,\pi)\times(0,\pi/2), \\ 0 & \text{in other cases.} \end{cases}$$

Observe that

$$A = \int_{\pi/2}^{\pi} \sin^p(y)dy = \int_0^{\pi/2} \sin^p(y)dy,$$

and then a solution for (P) is

$$u(x) = \left(\frac{\lambda-1}{A}\right)^{1/p} \sin(x).$$

In the second example, K is positive only in a horizontal band and there exists positive solution for all $\lambda > 1$. Consider $a_0 \in (\pi/2, \pi)$ and

$$K(x,y) = \begin{cases} 1 & \text{if } y > a_0, \\ 0 & \text{in other cases.} \end{cases}$$

Denote

$$A_0 = \int_{a_0}^{\pi} \sin^p(y)dy,$$

then a solution for (P) is

$$u(x) = \left(\frac{\lambda-1}{A_0}\right)^{1/p} \sin(x).$$

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