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A REDUCED DISCRETE INF-SUP CONDITION IN L^p FOR INCOMPRESSIBLE FLOWS AND APPLICATION

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Abstract. In this work, we introduce a discrete specific inf-sup condition to estimate the L^p norm, $1 < p < +\infty$, of the pressure in a number of fluid flows. It applies to projection-based stabilized finite element discretizations of incompressible flows, typically when the velocity field has a low regularity. We derive two versions of this inf-sup condition: The first one holds on shape-regular meshes and the second one on quasi-uniform meshes. As an application, we derive reduced inf-sup conditions for the linearized Primitive equations of the Ocean that apply to the surface pressure in weighted L^p norm. This allows to prove the stability and convergence of quite general stabilized discretizations of these equations: SUPG, Least Squares, Adjoint-stabilized and OSS discretizations.

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1. INTRODUCTION

Stabilized methods are designed to provide stable discretizations with reduced computational complexity of several sources of spurious instabilities that may arise in the discretization of incompressible flows (that may be due either to incompressibility or to large convection, Coriolis or reaction terms, among others). In this paper, we focus on the treatment of the incompressibility constraint. Concretely, we deal with projection-stabilized methods, introduced by Blasco and Codina in [3] by means of a local L^2 projection, that produce discretizations with high-order accuracy. This method was extended to the local projection-stabilization methods that use element-wise L^2 projections instead of the global L^2 projection, while satisfying some orthogonality properties. Among the many references on different versions of local projection-stabilization, let us quote Braack and Burman [4], Braack *et al.* [5], Ganesan *et al.* [17], Knobloch [19], Matthies *et al.* [21], Roos *et al.* [23].

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A special class of projection-stabilized methods is the interior penalty method, in which stabilization is achieved by introducing inter-element jumps of the terms to be stabilized. This method is equivalent to a projection-stabilized method, where the L^2 projection operator is replaced by the Oswald (cf. [22]) quasi-interpolant operator on the discrete velocity space (cf. [7–9]).

A further simplification is introduced in [15] where the local projection operator is replaced by a quasi-local approximation operator that does not need to satisfy any orthogonality property. This method has a more compact stencil while retaining the same optimal accuracy as all projection-stabilization methods, in the sense that its convergence order is optimal with respect to the degree of the finite element spaces.

In the present work, we extend the method introduced in [15] to stabilize the discretization of the pressure in flows where the velocity has a low accuracy, typically in some space $W^{1,s}(\Omega)$ with $1 < s < 2$. Then the pressure has only $L^p(\Omega)$ regularity, where p is the conjugate exponent of s . This is the case, for instance, of the weak solutions of the Primitive equations of the Ocean, that we consider as an application of our general setting.

We introduce a specific inf-sup condition in L^p norms, which is the main technical contribution of this paper. As in [15], the derivation of this condition faces the difficulty of the reduced number of degrees of freedom of the buffer space. This difficulty is solved by a finite-dimensional argument of equivalence of norms (cf. Lem. 2.4). We derive two versions of this inf-sup condition: the first one holds on shape-regular meshes and the second one on quasi-uniform meshes. As an application, we derive reduced inf-sup conditions for the linearized Primitive equations of the Ocean that apply to the surface pressure in weighted L^p norms. This allows to prove the stability and convergence of quite general stabilized discretizations of these equations: SUPG, Least Squares, Adjoint-stabilized and OSS discretizations. This condition generalizes a similar one for L^2 weighted norms introduced in [14].

The paper is structured as follows: in Section 2 we introduce the inf-sup condition in L^p norms for shape-regular meshes, as well as a simplified condition for quasi-uniform meshes. These conditions are applied to stabilized discretizations of a linearized version of the Primitive equations in Section 3. In this section, reduced inf-sup conditions for the surface pressure are deduced, for both shape-regular and quasi-uniform meshes. Also, the stability and convergence of the stabilized discretizations are proved by means of these conditions.

2. INF-SUP CONDITION FOR SHAPE-REGULAR MESHES

Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) be a bounded domain. We assume that Ω is a polygon if $d = 2$ or a polyhedron if $d = 3$. Let $\{\mathcal{T}_h\}_{h>0}$ be a family of conforming triangulations of $\bar{\Omega}$ formed by simplicial elements, where the parameter h denotes the largest diameter of the elements of \mathcal{T}_h . We assume the following.

Hypothesis 2.1. *The family $\{\mathcal{T}_h\}_{h>0}$ is shape-regular in the sense of Ciarlet [16] and no element of \mathcal{T}_h has 3 nodes (when $d = 2$) or 4 nodes (when $d = 3$) on the boundary of Ω .*

We decompose $\bar{\Omega}$ into a finite union of macro-elements:

$$\bar{\Omega} = \bigcup_{i=1}^R \mathcal{O}_i,$$

such that each \mathcal{O}_i is the support of the piecewise affine basis function associated to the node i . Note that any element $K \in \mathcal{T}_h$ belongs to at most M macro-elements, where M is independent of h .

For all i , $1 \leq i \leq R$, we set

$$h_i = \max_{K \subset \mathcal{O}_i} \{h_K\} \quad \text{and} \quad \rho_i = \min_{K \subset \mathcal{O}_i} \{\rho_K\},$$

where h_K denotes the diameter of the element K of \mathcal{T}_h and ρ_K the diameter of the ball inscribed in K .

As the mesh is regular, then it is locally uniformly regular (or locally quasi-uniform) and this implies that there exist positive constants C_1 and C_2 independent of h , such that for all $K \in \mathcal{T}_h$ and for all i for which $K \subset \mathcal{O}_i$,

$$C_1 \rho_i \leq h_K \leq h_i, \quad \frac{h_i}{\rho_i} \leq C_2. \quad (2.1)$$

This implies immediately that

$$\frac{C_1}{C_2} h_i \leq h_K \leq h_i. \tag{2.2}$$

For a domain $\mathcal{O} \subset \mathbb{R}^d$, we denote by $\|\cdot\|_{k,p,\mathcal{O}}$ and $|\cdot|_{k,p,\mathcal{O}}$ the norm and, respectively, the seminorm in $W^{k,p}(\mathcal{O})$. In $L^p(\Omega)/\mathbb{R}$, we also denote by $\|\cdot\|_{0,p,\Omega}$ the quotient norm in order to simplify the notation.

For all $p \in [1, +\infty)$ and for all $\mathbf{v} \in L^p(\Omega)^d$, we define

$$\|\mathbf{v}\|_{h,p} = \left(\sum_{i=1}^R h_i^p \|\mathbf{v}\|_{0,p,\mathcal{O}_i}^p \right)^{\frac{1}{p}}. \tag{2.3}$$

Given an integer $l \geq 1$, we denote by $\mathbb{P}_l(K)$ the space of polynomials of degree smaller than, or equal to, l defined on an element $K \in \mathcal{T}_h$ and define the following finite element spaces:

$$V_h^l = \{v_h \in C^0(\bar{\Omega}) \text{ such that } v_h|_K \in \mathbb{P}_l(K), \forall K \in \mathcal{T}_h\},$$

$$X_h = (V_h^l \cap H_0^1(\Omega))^d,$$

$$M_h = V_h^m / \mathbb{R},$$

$$V_h^l(\mathcal{O}_i) = \{v_h \in C^0(\bar{\mathcal{O}}_i) \text{ such that } v_h|_K \in \mathbb{P}_l(K), \forall K \in \mathcal{T}_h \text{ such that } K \subset \mathcal{O}_i\},$$

$$X_h(\mathcal{O}_i) = (V_h^l(\mathcal{O}_i) \cap H_0^1(\mathcal{O}_i))^d.$$

We recall now the inverse inequalities in finite element spaces, that we use frequently in the sequel.

Lemma 2.2. *Let p_1 and p_2 be numbers in $[1, \infty)$, and l_1 and l_2 two non-negative integers such that $l_1 \geq l_2$ and $l_1 - \frac{d}{p_1} \geq l_2 - \frac{d}{p_2}$.*

[Local inverse inequalities] ([2], Prop. 4.2) *For all $K \in \mathcal{T}_h$,*

$$\forall v \in \mathbb{P}_l(K), \quad |v|_{W^{l_1,p_1}(K)} \leq C \rho_K^{l_2-l_1-\frac{d}{p_2}+\frac{d}{p_1}} h_K^{\frac{d}{p_1}} |v|_{W^{l_2,p_2}(K)}, \tag{2.4}$$

where C is a constant independent of K .

[Global inverse inequalities] ([6], Thm 4.5.11) *If $p_1 \geq p_2$ and Z_h is a finite element space of polynomials in each K , then*

$$\forall w_h \in Z_h, \quad \left(\sum_{K \in \mathcal{T}_h} |w_h|_{W^{l_1,p_1}(K)}^{p_1} \right)^{\frac{1}{p_1}} \leq C \rho_{min}^{l_2-l_1-\frac{d}{p_2}+\frac{d}{p_1}} \left(\sum_{K \in \mathcal{T}_h} |w_h|_{W^{l_2,p_2}(K)}^{p_2} \right)^{\frac{1}{p_2}}, \tag{2.5}$$

where C is a constant independent of h and $\rho_{min} = \inf_{K \in \mathcal{T}_h} \{\rho_K\}$.

Throughout this work, C represents a constant that is always independent of h but may vary from an inequality to another.

Let us now consider an interpolation or projection operator

$$J_h : L^2(\Omega)^d \rightarrow (V_h^{l-1})^d, \tag{2.6}$$

and set $J_h^* = Id - J_h$. Our main result is the following.

Theorem 2.3. *Assume that Hypothesis 2.1 holds. Then for any $p \in (1, +\infty)$ there exists a constant $\gamma_p > 0$ independent of h such that for all $q_h \in M_h$,*

$$\gamma_p \|q_h\|_{0,p,\Omega} \leq \sup_{\mathbf{v}_h \in X_h} \frac{(\nabla \cdot \mathbf{v}_h, q_h)}{|\mathbf{v}_h|_{1,s,\Omega}} + \left(\sum_{i=1}^R \left(\sup_{\mathbf{v}_h \in X_h(\mathcal{O}_i)} \frac{|(\nabla \cdot \mathbf{v}_h, q_h)_{\mathcal{O}_i}|^p}{|\mathbf{v}_h|_{1,s,\mathcal{O}_i}^p} \right) \right)^{\frac{1}{p}} + \|J_h^*(\nabla q_h)\|_{h,p} \tag{2.7}$$

where s is the conjugate exponent of p ($\frac{1}{p} + \frac{1}{s} = 1$).

To prove this theorem we need the following auxiliary result.

Lemma 2.4. *Under Hypothesis 2.1, there exists a positive constant C independent of h , such that for all i , $1 \leq i \leq R$,*

$$\forall g_h \in V_h^{l-1}(\mathcal{O}_i), \quad \|g_h\|_{0,p,\mathcal{O}_i} \leq C \sup_{v_h \in V_h^l(\mathcal{O}_i) \cap H_0^1(\mathcal{O}_i)} \frac{(g_h, v_h)_{\mathcal{O}_i}}{\|v_h\|_{0,s,\mathcal{O}_i}}, \tag{2.8}$$

where $p \in (1, +\infty)$ and s is the conjugate exponent of p .

Proof. For $p = s = 2$ this result was proved in [15]. We extend it here to any conjugate pair of exponents p and s .

We denote by $\{a_j\}$ the nodes of \mathcal{T}_h that belong to $\overline{\mathcal{O}_i}$ and we define the function $w_h \in V_h^1(\mathcal{O}_i) \cap H_0^1(\mathcal{O}_i)$ such that:

- $w_h|_K \in \mathbb{P}_1(K)$, for all $K \subset \mathcal{O}_i$;
- $w_h(a_j) = 1$ if a_j is an interior node of \mathcal{O}_i ;
- $w_h(a_j) = 0$ if a_j is a node belonging to $\partial\mathcal{O}_i$.

Because each macro-element \mathcal{O}_i is the support of one piecewise affine basis function and no element of \mathcal{T}_h has 3 nodes (when $d = 2$) or 4 nodes (when $d = 3$) on the boundary of Ω , there exists at least one interior node in \mathcal{O}_i . So, this function w_h is well-defined and it is positive in the interior of \mathcal{O}_i .

Let $g_h \in V_h^{l-1}(\mathcal{O}_i)$ and $v_h = g_h w_h$. Then, $v_h \in V_h^l(\mathcal{O}_i) \cap H_0^1(\mathcal{O}_i)$ and it satisfies

$$|(g_h, v_h)_{\mathcal{O}_i}| \geq C \|g_h\|_{0,\mathcal{O}_i}^2, \tag{2.9}$$

with C a positive constant independent of h (cf. [15], Lem. 3.2).

Given $K \subset \mathcal{O}_i$, let $\hat{g}_K = g_h|_K \circ F_K$, where F_K is the affine mapping that transforms the reference element \hat{K} onto K . The shape regularity of the mesh implies that

$$\|g_h\|_{0,\mathcal{O}_i}^2 = \sum_{K \subset \mathcal{O}_i} \int_K |g_h|^2 = C \sum_{K \subset \mathcal{O}_i} |K| \int_{\hat{K}} |\hat{g}_K|^2 \geq C |\mathcal{O}_i| \sum_{K \subset \mathcal{O}_i} \int_{\hat{K}} |\hat{g}_K|^2. \tag{2.10}$$

Denote by N_i the number of elements of $\mathcal{T}_h|_{\mathcal{O}_i}$ and consider the norm in \mathbb{R}^{N_i} :

$$\|(x_1, \dots, x_{N_i})\|_{p,N_i} = \left(\sum_{j=1}^{N_i} |x_j|^p \right)^{\frac{1}{p}}.$$

As the grids are regular, $N_i \leq N$ (independent of h) and

$$\|(x_1, \dots, x_{N_i})\|_{p,N_i} = \|(x_1, \dots, x_{N_i}, 0, \dots, 0)\|_{p,N}.$$

By the equivalence of norms in \mathbb{R}^N , there exists a constant C_p (independent of h) such that

$$\|(x_1, x_2, \dots, x_{N_i}, 0, \dots, 0)\|_{2,N} \geq C_p \|(x_1, x_2, \dots, x_{N_i}, 0, \dots, 0)\|_{p,N}. \tag{2.11}$$

Then, applying (2.11) with

$$x_j = \left(\int_{\hat{K}} |\hat{g}_{K_j}|^p \right)^{\frac{1}{p}} \quad \forall j = 1, \dots, N_i,$$

we have

$$\left(\sum_{K \subset \mathcal{O}_i} \int_{\hat{K}} |\hat{g}_K|^2 \right)^{\frac{1}{2}} \geq C_p \left(\sum_{K \subset \mathcal{O}_i} \int_{\hat{K}} |\hat{g}_K|^p \right)^{\frac{1}{p}}.$$

Thus, from (2.10) and the shape regularity of the mesh

$$\begin{aligned} \|g_h\|_{0,\mathcal{O}_i}^2 &\geq C |\mathcal{O}_i| C_p C_s \left(\sum_{K \subset \mathcal{O}_i} \int_{\hat{K}} |\hat{g}_K|^p \right)^{\frac{1}{p}} \left(\sum_{K \subset \mathcal{O}_i} \int_{\hat{K}} |\hat{g}_K|^s \right)^{\frac{1}{s}} \\ &\geq C \left(\sum_{K \subset \mathcal{O}_i} |K| \int_{\hat{K}} |\hat{g}_K|^p \right)^{\frac{1}{p}} \left(\sum_{K \subset \mathcal{O}_i} |K| \int_{\hat{K}} |\hat{g}_K|^s \right)^{\frac{1}{s}}. \end{aligned}$$

That is,

$$\|g_h\|_{0,\mathcal{O}_i}^2 \geq C \|g_h\|_{0,p,\mathcal{O}_i} \|g_h\|_{0,s,\mathcal{O}_i}. \tag{2.12}$$

Moreover,

$$\|v_h\|_{0,s,\mathcal{O}_i} \leq \|g_h\|_{0,s,\mathcal{O}_i} \|w_h\|_{0,\infty,\mathcal{O}_i} \leq \|g_h\|_{0,s,\mathcal{O}_i}. \tag{2.13}$$

Then, from (2.9) and taking into account (2.12) and (2.13) we obtain

$$C \|g_h\|_{0,p,\mathcal{O}_i} \leq \frac{(g_h, v_h)_{\mathcal{O}_i}}{\|v_h\|_{0,s,\mathcal{O}_i}},$$

whence we deduce (2.8). □

Proof of Theorem 2.3. We adapt Verfürth’s technique (cf. [24]). Given $q_h \in M_h$, by Amrouche and Girault (cf. [1]), there exists a constant $C > 0$ independent of h such that

$$C \|q_h\|_{0,p,\Omega} \leq \sup_{\mathbf{v} \in [W_0^{1,s}(\Omega)]^d - \{0\}} \frac{(\nabla \cdot \mathbf{v}, q_h)}{|\mathbf{v}|_{1,s,\Omega}}.$$

So there exists $\mathbf{v} \in [W_0^{1,s}(\Omega)]^d$ such that

$$\frac{1}{2} C \|q_h\|_{0,p,\Omega} \leq \frac{(\nabla \cdot \mathbf{v}, q_h)}{|\mathbf{v}|_{1,s,\Omega}}. \tag{2.14}$$

Since the family of grids is regular, following the standard finite element interpolation theory (cf. [2] or [6], Sect. 4.8), there exists an interpolate of \mathbf{v} , $\mathbf{v}_h \in X_h$, such that

$$|\mathbf{v}_h|_{1,s,\Omega} \leq C |\mathbf{v}|_{1,s,\Omega}, \tag{2.15}$$

$$\|\mathbf{v} - \mathbf{v}_h\|_{0,s,K} \leq C h_K |\mathbf{v}|_{1,s,\omega_K}, \tag{2.16}$$

where ω_K denotes the union of all elements of \mathcal{T}_h that intersect K .

Let us rewrite the r.h.s. of (2.14) as

$$\frac{(\nabla \cdot \mathbf{v}, q_h)}{|\mathbf{v}|_{1,s,\Omega}} = \frac{(\nabla \cdot \mathbf{v}_h, q_h)}{|\mathbf{v}|_{1,s,\Omega}} + \frac{(\nabla \cdot (\mathbf{v} - \mathbf{v}_h), q_h)}{|\mathbf{v}|_{1,s,\Omega}}. \tag{2.17}$$

Using (2.15),

$$\frac{(\nabla \cdot \mathbf{v}_h, q_h)}{|\mathbf{v}|_{1,s,\Omega}} \leq C \frac{(\nabla \cdot \mathbf{v}_h, q_h)}{|\mathbf{v}_h|_{1,s,\Omega}}. \tag{2.18}$$

Also, because q_h belongs to $H^1(\Omega)$ and $(\mathbf{v} - \mathbf{v}_h) \cdot \mathbf{n} = 0$ on $\partial\Omega$,

$$\begin{aligned} (\nabla \cdot (\mathbf{v} - \mathbf{v}_h), q_h) &= -(\mathbf{v} - \mathbf{v}_h, \nabla q_h) \\ &\leq \sum_{K \in \mathcal{T}_h} \|\mathbf{v} - \mathbf{v}_h\|_{0,s,K} \|\nabla q_h\|_{0,p,K} \\ &\leq \sum_{K \in \mathcal{T}_h} C h_K |\mathbf{v}|_{1,s,\omega_K} \|\nabla q_h\|_{0,p,K} \quad (\text{using (2.16)}) \\ &\leq C \left(\sum_{K \in \mathcal{T}_h} |\mathbf{v}|_{1,s,\omega_K}^s \right)^{\frac{1}{s}} \left(\sum_{K \in \mathcal{T}_h} h_K^p \|\nabla q_h\|_{0,p,K}^p \right)^{\frac{1}{p}} \\ &\leq C |\mathbf{v}|_{1,s,\Omega} \left(\sum_{K \in \mathcal{T}_h} h_K^p \|\nabla q_h\|_{0,p,K}^p \right)^{\frac{1}{p}}. \end{aligned} \tag{2.19}$$

Then from (2.14), combining (2.17)–(2.19), we have

$$C \|q_h\|_{0,p,\Omega} \leq \sup_{\mathbf{v}_h \in X_h} \frac{(\nabla \cdot \mathbf{v}_h, q_h)}{|\mathbf{v}_h|_{1,s,\Omega}} + \left(\sum_{K \in \mathcal{T}_h} h_K^p \|\nabla q_h\|_{0,p,K}^p \right)^{\frac{1}{p}}. \tag{2.20}$$

As each element $K \in \mathcal{T}_h$ belongs to some macro-element \mathcal{O}_i ,

$$\sum_{K \in \mathcal{T}_h} h_K^p \|\nabla q_h\|_{0,p,K}^p \leq C \sum_{i=1}^R h_i^p \|\nabla q_h\|_{0,p,\mathcal{O}_i}^p.$$

So, from (2.20)

$$C \|q_h\|_{0,p,\Omega} \leq \sup_{\mathbf{v}_h \in X_h} \frac{(\nabla \cdot \mathbf{v}_h, q_h)}{|\mathbf{v}_h|_{1,s,\Omega}} + \|\nabla q_h\|_{h,p}. \tag{2.21}$$

To estimate the last term in (2.21), we use the relation $J_h + J_h^* = Id$ and write

$$\|\nabla q_h\|_{0,p,\mathcal{O}_i} \leq \|J_h(\nabla q_h)\|_{0,p,\mathcal{O}_i} + \|J_h^*(\nabla q_h)\|_{0,p,\mathcal{O}_i}. \tag{2.22}$$

Since $J_h(\nabla q_h)|_{\mathcal{O}_i} \in (V_h^{l-1}(\mathcal{O}_i))^d$ we can apply the inf-sup condition (2.8) to each of its components,

$$\|J_h(\nabla q_h)\|_{0,p,\mathcal{O}_i} \leq C \sup_{\mathbf{v}_h \in X_h(\mathcal{O}_i)} \frac{(J_h(\nabla q_h), \mathbf{v}_h)_{\mathcal{O}_i}}{\|\mathbf{v}_h\|_{0,s,\mathcal{O}_i}}.$$

Using again $J_h + J_h^* = Id$,

$$\begin{aligned} |(J_h(\nabla q_h), \mathbf{v}_h)_{\mathcal{O}_i}| &\leq |(\nabla q_h, \mathbf{v}_h)_{\mathcal{O}_i}| + |(J_h^*(\nabla q_h), \mathbf{v}_h)_{\mathcal{O}_i}| \\ &\leq |(\nabla \cdot \mathbf{v}_h, q_h)_{\mathcal{O}_i}| + \|J_h^*(\nabla q_h)\|_{0,p,\mathcal{O}_i} \|\mathbf{v}_h\|_{0,s,\mathcal{O}_i}, \end{aligned}$$

as $(\nabla q_h, \mathbf{v}_h)_{\mathcal{O}_i} = -(\nabla \cdot \mathbf{v}_h, q_h)_{\mathcal{O}_i}$ because $\mathbf{v}_h = \mathbf{0}$ on $\partial\mathcal{O}_i$. So,

$$\|J_h(\nabla q_h)\|_{0,p,\mathcal{O}_i} \leq C \left(\sup_{\mathbf{v}_h \in X_h(\mathcal{O}_i)} \frac{|(\nabla \cdot \mathbf{v}_h, q_h)_{\mathcal{O}_i}|}{\|\mathbf{v}_h\|_{0,s,\mathcal{O}_i}} + \|J_h^*(\nabla q_h)\|_{0,p,\mathcal{O}_i} \right). \tag{2.23}$$

Thus, from (2.22) and (2.23)

$$\|\nabla q_h\|_{h,p}^p \leq C \left(\sum_{i=1}^R \left(\sup_{\mathbf{v}_h \in X_h(\mathcal{O}_i)} h_i^p \frac{|(\nabla \cdot \mathbf{v}_h, q_h)_{\mathcal{O}_i}|^p}{\|\mathbf{v}_h\|_{0,s,\mathcal{O}_i}^p} \right) + \|J_h^*(\nabla q_h)\|_{h,p}^p \right). \tag{2.24}$$

The local inverse inequality (2.4) between $W^{1,s}(K)$ and $L^s(K)$ on each $K \subset \mathcal{O}_i$ yields

$$|\mathbf{v}_h|_{1,s,\mathcal{O}_i} \leq C \rho_i^{-1} \|\mathbf{v}_h\|_{0,s,\mathcal{O}_i}.$$

This inequality and (2.1) imply that

$$\frac{h_i^p}{\|\mathbf{v}_h\|_{0,s,\mathcal{O}_i}^p} \leq C \frac{1}{|\mathbf{v}_h|_{1,s,\mathcal{O}_i}^p},$$

and we obtain

$$\|\nabla q_h\|_{h,p}^p \leq C \left(\sum_{i=1}^R \left(\sup_{\mathbf{v}_h \in X_h(\mathcal{O}_i)} \frac{|(\nabla \cdot \mathbf{v}_h, q_h)_{\mathcal{O}_i}|^p}{|\mathbf{v}_h|_{1,s,\mathcal{O}_i}^p} \right) + \|J_h^*(\nabla q_h)\|_{h,p}^p \right). \tag{2.25}$$

Finally, (2.7) follows from (2.21) and (2.25). □

2.1. The case of uniformly regular meshes

The inf-sup condition (2.7) may be simplified if the grids are uniformly regular. We assume the following.

Hypothesis 2.5. *The family $\{\mathcal{T}_h\}_{h>0}$ is uniformly regular (also called quasi-uniform): there exist positive constants α and β independent of h such that*

$$\beta h \leq h_K \leq \alpha \rho_K, \quad \forall K \in \mathcal{T}_h. \tag{2.26}$$

We define the space

$$Y = \{ \mathbf{v} \in H^1(\Omega)^d \text{ such that } \mathbf{v} \cdot \mathbf{n} = 0 \text{ a.e. on } \partial\Omega \},$$

and consider an interpolation or projection operator

$$I_h : L^2(\Omega)^d \rightarrow Y_h, \quad \text{where } X_h \subseteq Y_h \subseteq (V_h^1)^d \cap Y. \tag{2.27}$$

We shall denote $I_h^* = Id - I_h$.

In this case the inf-sup condition (2.7) reduces to a simpler one. This is stated as follows.

Theorem 2.6. *Assume that Hypothesis 2.5 holds. Then for any $p \in (1, +\infty)$ there exists a constant $\lambda_p > 0$ independent of h such that for all $q_h \in M_h$,*

$$\lambda_p \|q_h\|_{0,p,\Omega} \leq \sup_{\mathbf{v}_h \in Y_h} \frac{(\nabla \cdot \mathbf{v}_h, q_h)}{|\mathbf{v}_h|_{1,s,\Omega}} + h \|I_h^*(\nabla q_h)\|_{0,p,\Omega}, \tag{2.28}$$

where s is the conjugate exponent of p .

Proof. As in Theorem 2.3 we obtain (2.20) because this estimate only requires that the grids be regular. Moreover, as $X_h \subseteq Y_h$,

$$C \|q_h\|_{0,p,\Omega} \leq \sup_{\mathbf{v}_h \in Y_h} \frac{(\nabla \cdot \mathbf{v}_h, q_h)}{|\mathbf{v}_h|_{1,s,\Omega}} + h \|\nabla q_h\|_{0,p,\Omega}. \tag{2.29}$$

In order to bound the last term in (2.29), we argue as in the proof of Theorem 2.3 but now we do not need to follow a local argument. Using the relation $I_h + I_h^* = Id$,

$$\|\nabla q_h\|_{0,p,\Omega} \leq \|I_h(\nabla q_h)\|_{0,p,\Omega} + \|I_h^*(\nabla q_h)\|_{0,p,\Omega}. \tag{2.30}$$

As $I_h(\nabla q_h) \in L^p(\Omega)^d$, there exists $\mathbf{v} \in L^s(\Omega)^d$ such that

$$(I_h(\nabla q_h), \mathbf{v}) = \|I_h(\nabla q_h)\|_{0,p,\Omega} \|\mathbf{v}\|_{0,s,\Omega}.$$

Now, let \mathbf{v}_h be the L^2 orthogonal projection of \mathbf{v} on Y_h . As the mesh is uniformly regular, the L^2 projection is stable in the L^s norm (cf. [25]):

$$\|\mathbf{v}_h\|_{0,s,\Omega} \leq C \|\mathbf{v}\|_{0,s,\Omega}.$$

Then, as $I_h(\nabla q_h) \in Y_h$,

$$(I_h(\nabla q_h), \mathbf{v}_h) = (I_h(\nabla q_h), \mathbf{v}) \geq C \|I_h(\nabla q_h)\|_{0,p,\Omega} \|\mathbf{v}_h\|_{0,s,\Omega}.$$

Thus,

$$\|I_h(\nabla q_h)\|_{0,p,\Omega} \leq C \sup_{\mathbf{v}_h \in Y_h} \frac{(I_h(\nabla q_h), \mathbf{v}_h)}{\|\mathbf{v}_h\|_{0,s,\Omega}}.$$

Using again $I_h + I_h^* = Id$ and $(\nabla q_h, \mathbf{v}_h) = -(\nabla \cdot \mathbf{v}_h, q_h)$ because $\mathbf{v}_h \in Y$, we have

$$\|I_h(\nabla q_h)\|_{0,p,\Omega} \leq C \left(\sup_{\mathbf{v}_h \in Y_h} \frac{|(\nabla \cdot \mathbf{v}_h, q_h)|}{\|\mathbf{v}_h\|_{0,s,\Omega}} + \|I_h^*(\nabla q_h)\|_{0,p,\Omega} \right). \tag{2.31}$$

Therefore, from (2.30) and (2.31),

$$h \|\nabla q_h\|_{0,p,\Omega} \leq C \left(\sup_{\mathbf{v}_h \in Y_h} h \frac{|(\nabla \cdot \mathbf{v}_h, q_h)|}{\|\mathbf{v}_h\|_{0,s,\Omega}} + h \|I_h^*(\nabla q_h)\|_{0,p,\Omega} \right). \tag{2.32}$$

The global inverse inequality (2.5) between $W^{1,s}(\Omega)$ and $L^s(\Omega)$ and the quasi-uniformity of the mesh implies

$$|\mathbf{v}_h|_{1,s,\Omega} \leq C h^{-1} \|\mathbf{v}_h\|_{0,s,\Omega}.$$

Using this estimate in (2.32), we obtain

$$h \|\nabla q_h\|_{0,p,\Omega} \leq C \left(\sup_{\mathbf{v}_h \in Y_h} \frac{|(\nabla \cdot \mathbf{v}_h, q_h)|}{|\mathbf{v}_h|_{1,s,\Omega}} + h \|I_h^*(\nabla q_h)\|_{0,p,\Omega} \right). \tag{2.33}$$

Finally, (2.28) follows from (2.29) and (2.33). □

Remark 2.7. At the angular corner points of the boundary, the condition $\mathbf{v} \cdot \mathbf{n} = 0$ implies that $\mathbf{v} = \mathbf{0}$. So, when the domain Ω is approximating a curved domain the space Y_h in (2.27) coincides with X_h . In the application to the Primitive equations of the Ocean studied in the next section, the boundary of the domain has a flat component (the surface) and the space X_h is strictly contained into Y_h .

3. APPLICATION TO THE PRIMITIVE EQUATIONS OF THE OCEAN

We study the fluid in a domain

$$\Omega = \{(\mathbf{x}, z) \in \mathbb{R}^d \text{ such that } \mathbf{x} \in \omega, -D(\mathbf{x}) \leq z \leq 0\},$$

where ω is a bounded domain in \mathbb{R}^{d-1} and $D : \bar{\omega} \rightarrow \mathbb{R}$ is a Lipschitz-continuous non-negative function that represents the depth. The boundary is split as $\partial\Omega = \Gamma_s \cup \Gamma_b$, where $\Gamma_s = \{(\mathbf{x}, 0) \in \mathbb{R}^d \text{ such that } \mathbf{x} \in \omega\}$ represents the ocean surface, and Γ_b represents the ocean bottom and, eventually, sidewalls.

We consider a linearized version of the steady reduced Primitive equations model (cf. [20]). The problem consists in finding a horizontal velocity field $\mathbf{u} : \overline{\Omega} \mapsto \mathbb{R}^{d-1}$ and a surface pressure $p : \omega \mapsto \mathbb{R}$ such that

$$\begin{cases} \mathbf{W} \cdot \nabla \mathbf{u} - \mu \Delta \mathbf{u} + \nabla_{\mathbf{x}} p + \varphi \mathbf{u}^\perp = \mathbf{f} & \text{in } \Omega, \\ \nabla_{\mathbf{x}} \cdot \langle \mathbf{u} \rangle = 0 & \text{in } \omega, \\ \mathbf{u}|_{\Gamma_b} = 0, \quad \mu_z \partial_z \mathbf{u}|_{\Gamma_s} = \mathbf{g}. \end{cases} \tag{3.1}$$

Here $\mathbf{W} : \overline{\Omega} \mapsto \mathbb{R}^d$ is a given convection velocity and $\langle u \rangle : \omega \mapsto \mathbb{R}^{d-1}$ is defined by

$$\langle \mathbf{u} \rangle(\mathbf{x}) = \int_{-D(\mathbf{x})}^0 \mathbf{u}(\mathbf{x}, s) ds. \tag{3.2}$$

Also μ is the viscosity coefficient, that we assume to be isotropic for simplicity. The term $\varphi \mathbf{u}^\perp$ is due the Coriolis acceleration and it appears only when $d = 3$. In this case, $\mathbf{u} = (u_1, u_2)$ and $\mathbf{u}^\perp = (-u_2, u_1)$. The function φ is defined by $\varphi = 2\theta \sin \phi$, where θ is the angular rotation rate of the earth and ϕ is the latitude. The source term \mathbf{f} takes into account variable density effects, due to variations of temperature and salinity, and \mathbf{g} represents the wind tension at the surface.

This model includes the rigid-lid assumption, stating that the free surface is flat ($z = 0$ in our case), and that the vertical velocity vanishes at the surface (cf. [13]). This last condition and the incompressibility of the velocity $\mathbf{U} = (\mathbf{u}, u_z)$ are used to define the vertical velocity $u_z : \overline{\Omega} \mapsto \mathbb{R}$ from the horizontal velocity:

$$u_z(\mathbf{x}, z) = \int_z^0 \nabla_{\mathbf{x}} \cdot \mathbf{u}(\mathbf{x}, s) ds. \tag{3.3}$$

To define weak solutions of problem (3.1), we consider the spaces

$$W_b^{1,s}(\Omega) = \{v \in W^{1,s}(\Omega) \text{ such that } v|_{\Gamma_b} = 0\}, \quad \text{for an integer } s \geq 1.$$

In particular $H_b^1(\Omega) = W_b^{1,2}(\Omega)$. We denote $\mathbf{H}_b^{-1}(\Omega)$ the dual space of $H_b^1(\Omega)^{d-1}$ and $\mathbf{H}^{-\frac{1}{2}}(\Gamma_s)$ the dual space of $H^{\frac{1}{2}}(\Gamma_s)^{d-1}$. We also consider the spaces

$$L_D^p(\omega) = \left\{ q : \omega \mapsto \mathbb{R} \text{ such that } \int_{\omega} D(\mathbf{x}) |q(\mathbf{x})|^p dx < \infty \right\}$$

and

$$L_{D,0}^p(\omega) = L_D^p(\omega) / \mathbb{R},$$

for all $p \in (1, +\infty)$.

Problem (3.1) is a linearized version of the non-linear Primitive equations that in fact may be used as an intermediate step to prove its well posedness. We thus assume that the velocity field is $\mathbf{W} = (\mathbf{w}, w_z)$ with

$$\mathbf{w} \in H_b^1(\Omega)^{d-1} \quad \text{and} \quad w_z(\mathbf{x}, z) = \int_z^0 \nabla_{\mathbf{x}} \cdot \mathbf{w}(\mathbf{x}, s) ds. \tag{3.4}$$

The weak formulation of problem (3.1) that we consider is: Given $\mathbf{f} \in \mathbf{H}_b^{-1}(\Omega)$ and $\mathbf{g} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_s)$,

$$\begin{cases} \text{Find } (\mathbf{u}, p) \in H_b^1(\Omega)^{d-1} \times L_{D,0}^{\frac{3}{2}}(\omega) \text{ such that} \\ B((\mathbf{u}, p), (\mathbf{v}, q)) = L(\mathbf{v}), \quad \forall (\mathbf{v}, q) \in W_b^{1,3}(\Omega)^{d-1} \times L_{D,0}^2(\omega), \end{cases} \tag{3.5}$$

where the forms B and L are defined as follows:

$$\begin{aligned} B((\mathbf{u}, p), (\mathbf{v}, q)) &= -(\mathbf{W} \cdot \nabla \mathbf{v}, \mathbf{u}) + \mu (\nabla \mathbf{u}, \nabla \mathbf{v}) + (\varphi \mathbf{u}^\perp, \mathbf{v}) \\ &\quad - (p, \nabla_{\mathbf{x}} \cdot \mathbf{v}) + (\nabla_{\mathbf{x}} \cdot \mathbf{u}, q), \\ L(\mathbf{v}) &= \langle \mathbf{f}, \mathbf{v} \rangle_\Omega + \langle \mathbf{g}, \mathbf{v} \rangle_{\Gamma_s}. \end{aligned} \quad (3.6)$$

Here $\langle \cdot, \cdot \rangle_\Omega$ stands for the duality between $\mathbf{H}_b^{-1}(\Omega)$ and $H_b^1(\Omega)^{d-1}$ and $\langle \cdot, \cdot \rangle_{\Gamma_s}$ stands for the duality between $\mathbf{H}^{-\frac{1}{2}}(\Gamma_s)$ and $H^{\frac{1}{2}}(\Gamma_s)^{d-1}$.

In this formulation the convection vertical velocity w_z has only L^2 regularity, as it is obtained by vertical integration of the divergence of the horizontal velocity in (3.4). As a consequence, the convection operator has not H^{-1} regularity and a Petrov–Galerkin formulation is needed, with test functions smoother than the velocity: $\mathbf{v} \in W_b^{1,3}(\Omega)^{d-1}$. This allows to define the convection term by duality as

$$\langle \mathbf{W} \cdot \nabla \mathbf{u}, \mathbf{v} \rangle = - \int_\Omega (\mathbf{W} \cdot \nabla \mathbf{v}) \mathbf{u},$$

and justifies the expression of the convection term in (3.6). The existence of solutions of problem (3.5) is proved in [12].

Now we define a discretization of problem (3.5). To do this, we assume that ω is polygonal and D is a piecewise affine function on $\{\mathcal{C}_h\}_{h>0}$, a family of conforming triangulations of $\bar{\omega}$. We introduce the prisms

$$P_T = \{(\mathbf{x}, z) \in \mathbb{R}^d, \text{ such that } \mathbf{x} \in T, -D(\mathbf{x}) \leq z \leq 0\} \quad \forall T \in \mathcal{C}_h,$$

and consider a family of triangulations of $\bar{\Omega}$ constructed by subdividing each prism P_T into triangles ($d = 2$) or tetrahedra ($d = 3$).

We consider the following finite element spaces:

$$\mathcal{U}_h = (V_h^l \cap H_b^1(\Omega))^{d-1}, \quad (3.7)$$

$$Q_h = \{q_h \in C^0(\bar{\omega}) : q_h|_T \in \mathbb{P}_m(T), \forall T \in \mathcal{C}_h\}, \quad \mathcal{P}_h = Q_h/\mathbb{R}, \quad (3.8)$$

for integers $l, m \geq 1$.

In order to fully discretize problem (3.5), we use an interpolate of \mathbf{w} , $\mathbf{w}_h \in \mathcal{U}_h$, such that

$$\|\mathbf{w}_h\|_{1,\Omega} \leq C \|\mathbf{w}\|_{1,\Omega}, \quad \lim_{h \rightarrow 0} \|\mathbf{w}_h - \mathbf{w}\|_{1,\Omega} = 0, \quad (3.9)$$

and we define

$$w_{hz}(\mathbf{x}, z) = \int_z^0 \nabla_{\mathbf{x}} \cdot \mathbf{w}_h(\mathbf{x}, s) \, ds. \quad (3.10)$$

Note that the field $\mathbf{W}_h = (\mathbf{w}_h, w_{hz})$ verifies that $\nabla \cdot \mathbf{W}_h = 0$ a.e. in Ω , $w_{hz} = 0$ on Γ_s , and

$$\|\mathbf{W}_h\|_{0,\Omega} \leq C \|\mathbf{w}\|_{1,\Omega}, \quad \lim_{h \rightarrow 0} \|\mathbf{W}_h - \mathbf{W}\|_{0,\Omega} = 0. \quad (3.11)$$

We also consider an interpolate of \mathbf{f} , $\mathbf{f}_h \in (V_h^l)^{d-1}$, such that

$$\|\mathbf{f}_h\|_{\mathbf{H}_b^{-1}(\Omega)} \leq C \|\mathbf{f}\|_{\mathbf{H}_b^{-1}(\Omega)}, \quad \lim_{h \rightarrow 0} \|\mathbf{f} - \mathbf{f}_h\|_{\mathbf{H}_b^{-1}(\Omega)} = 0. \quad (3.12)$$

We define the following scalar product:

$$\forall \mathbf{u}, \mathbf{v} \in L^2(\Omega)^{d-1}, \quad (\mathbf{u}, \mathbf{v})_\tau = \sum_{K \in \mathcal{T}_h} \tau_K(\mathbf{u}, \mathbf{v})_K,$$

where τ_K are stabilization coefficients, and denote by $\|\cdot\|_\tau$ the associated norm:

$$\|\mathbf{v}\|_\tau = \left(\sum_{K \in \mathcal{T}_h} \tau_K \|\mathbf{v}\|_{0,K}^2 \right)^{\frac{1}{2}}.$$

We propose the following projection-stabilized discretization of problem (3.5):

$$\begin{cases} \text{Find } (\mathbf{u}_h, p_h) \in \mathcal{U}_h \times \mathcal{P}_h \text{ such that} \\ B_h((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) = L_h(\mathbf{v}_h, q_h), \quad \forall (\mathbf{v}_h, q_h) \in \mathcal{U}_h \times \mathcal{P}_h, \end{cases} \tag{3.13}$$

where

$$\begin{aligned} B_h((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) &= B((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) \\ &+ (\Pi_h^*(\mathbf{W}_h \cdot \nabla \mathbf{v}_h - \varepsilon \mu \Delta \mathbf{v}_h + \nabla_{\mathbf{x}} q_h + \varphi \mathbf{v}_h^\perp), \Pi_h^*(\mathbf{W}_h \cdot \nabla \mathbf{u}_h - \mu \Delta \mathbf{u}_h + \nabla_{\mathbf{x}} p_h + \varphi \mathbf{u}_h^\perp))_\tau, \\ L_h(\mathbf{v}_h, q_h) &= L(\mathbf{v}_h) + (\Pi_h^*(\mathbf{W}_h \cdot \nabla \mathbf{v}_h - \varepsilon \mu \Delta \mathbf{v}_h + \nabla_{\mathbf{x}} q_h + \varphi \mathbf{v}_h^\perp), \Pi_h^*(\mathbf{f}_h))_\tau. \end{aligned} \tag{3.14}$$

Here Π_h is an interpolation or projection operator from $L^2(\Omega)^{d-1}$ into a finite element space and $\Pi_h^* = Id - \Pi_h$. Also ε is a real parameter that determines the method. In particular,

- If $\Pi_h \equiv 0$,
 - For $l = m = 1$ in (3.7) and (3.8), (3.13) is known as Streamline Upwinding Petrov–Galerkin (SUPG) method.
 - For others values of $l \geq 1$ and $m \geq 1$, (3.13) is known as Adjoint Stabilized/Variational Multi-Scale (AdS/VMS) method when $\varepsilon = -1$, generalized SUPG method when $\varepsilon = 0$ and Galerkin Least Squares (GLS) method when $\varepsilon = 1$.
- If Π_h is the orthogonal projection onto \mathcal{U}_h with respect to the inner product $(\cdot, \cdot)_\tau$ and $\varepsilon = -1$, (3.13) is the Orthogonal Sub-Scales (OSS) method proposed in [14].

3.1. Stability and convergence analysis

We make the following hypotheses on the stabilization coefficients:

Hypothesis 3.1. *There exist positive constants α_1, α_2 such that*

$$\alpha_1 h_K^2 \leq \tau_K \leq \alpha_2 h_K^2, \quad \forall K \in \mathcal{T}_h. \tag{3.15}$$

This hypothesis is verified by the usual stabilization coefficients and, in particular, by those given by Codina (cf. [10]) or Chacón (cf. [11]). This assumption holds even in the convection-dominated regime. Indeed, in this case, typically

$$\tau_K = C \frac{h_K}{\|\mathbf{u}\|_{0,p,K}},$$

if the local Péclet number verifies $Pe_K > P$ with $Pe_K = \frac{\|\mathbf{u}\|_{0,p,K} h_K}{\mu}$ and $P > 0$ some preset threshold. Then,

$$\tau_K \leq \frac{C}{P \mu} h_K^2.$$

Note that as a consequence of Hypothesis 3.1, when the grids are uniformly regular, there holds

$$\forall \mathbf{v} \in L^2(\Omega)^{d-1}, \quad \beta \sqrt{\alpha_1} h \|\mathbf{v}\|_{0,\Omega} \leq \|\mathbf{v}\|_\tau \leq \sqrt{\alpha_2} h \|\mathbf{v}\|_{0,\Omega}. \tag{3.16}$$

In reference [14] a stability and convergence analysis of the OSS method was realized when $\mathbf{W} \in L^d(\Omega)^d$. However, if we only assume the natural regularity for the convection velocity (3.4), it is not possible to bound the pressure in the L^2 norm when $d = 3$. We next state a weaker inf-sup condition that allows to estimate the pressure in $L^{\frac{3}{2}}$ when $d = 3$.

3.1.1. Case of uniformly regular meshes

We assume that the operator in (3.14) is an interpolation or projection operator from $L^2(\Omega)^{d-1}$ into \mathcal{U}_h , denoted by $\Pi_{\mathcal{U}_h}$ and $\Pi_{\mathcal{U}_h}^* = Id - \Pi_{\mathcal{U}_h}$.

Lemma 3.2. *Assume that Hypothesis 2.5 holds. Then for any $p \in (1, +\infty)$ there exists a constant $\lambda_p > 0$ independent of h such that for all $q_h \in \mathcal{P}_h$,*

$$\lambda_p \|q_h\|_{L^p_{D,0}(\omega)} \leq \sup_{\mathbf{v}_h \in \mathcal{U}_h} \frac{(\nabla_{\mathbf{x}} \cdot \mathbf{v}_h, q_h)}{|\mathbf{v}_h|_{1,s,\Omega}} + h \|\Pi_{\mathcal{U}_h}^*(\nabla_{\mathbf{x}} q_h)\|_{0,p,\Omega}, \quad (3.17)$$

where s is the conjugate exponent of p .

Proof. Given $q_h \in \mathcal{P}_h$, denote by \tilde{q}_h its extension to Ω as the function defined by

$$\tilde{q}_h(\mathbf{x}, z) = q_h(\mathbf{x}), \quad \forall \mathbf{x} \in \omega, \forall z \in (-D(\mathbf{x}), 0). \quad (3.18)$$

Then $\tilde{q}_h \in M_h = V_h^m / \mathbb{R}$, because \mathcal{T}_h is a prismatic grid. Moreover, $\|\tilde{q}_h\|_{0,p,\Omega} = \|q_h\|_{L^p_{D,0}(\omega)}$ and $\partial_z \tilde{q}_h = 0$ in Ω .

Let $Y_h = \mathcal{U}_h \times (V_h^l \cap H_0^1(\Omega))$ and let $I_h : L^2(\Omega)^d \rightarrow Y_h$ be an interpolation or projection operator such that $I_h|_{L^2(\Omega)^{d-1}} = \Pi_{\mathcal{U}_h}$. Note that all \mathbf{V}_h in Y_h satisfy $\mathbf{V}_h \cdot \mathbf{n} = 0$ on $\partial\Omega$. We may thus apply the inf-sup condition (2.28) with this choice of I_h :

$$\lambda_p \|\tilde{q}_h\|_{0,p,\Omega} \leq \sup_{\mathbf{V}_h \in Y_h} \frac{(\nabla \cdot \mathbf{V}_h, \tilde{q}_h)}{|\mathbf{V}_h|_{1,s,\Omega}} + h \|I_h^*(\nabla \tilde{q}_h)\|_{0,p,\Omega}. \quad (3.19)$$

We set $\mathbf{V}_h = (\mathbf{v}_h, v_{hz}) \in Y_h$ and observe that $(\nabla \cdot \mathbf{V}_h, \tilde{q}_h) = (\nabla_{\mathbf{x}} \cdot \mathbf{v}_h, q_h)$ because $\partial_z \tilde{q}_h = 0$ in Ω and $v_{hz} = 0$ on $\partial\Omega$. Thus,

$$\sup_{\mathbf{V}_h \in Y_h} \frac{(\nabla \cdot \mathbf{V}_h, \tilde{q}_h)}{|\mathbf{V}_h|_{1,s,\Omega}} \leq \sup_{\mathbf{v}_h \in \mathcal{U}_h} \frac{(\nabla_{\mathbf{x}} \cdot \mathbf{v}_h, q_h)}{|\mathbf{v}_h|_{1,s,\Omega}}. \quad (3.20)$$

As $\nabla \tilde{q}_h = (\nabla_{\mathbf{x}} q_h, 0)$,

$$\|I_h^*(\nabla \tilde{q}_h)\|_{0,p,\Omega} = \|\Pi_{\mathcal{U}_h}^*(\nabla_{\mathbf{x}} q_h)\|_{0,p,\Omega}. \quad (3.21)$$

Then, (3.17) follows by substituting (3.20) and (3.21) into (3.19). \square

We assume the following global stability properties of the operator $\Pi_{\mathcal{U}_h}$:

Hypothesis 3.3. *All $\mathbf{v} \in L^2(\Omega)^{d-1}$ satisfy*

$$\|\Pi_{\mathcal{U}_h}(\mathbf{v})\|_{0,\Omega} \leq C \|\mathbf{v}\|_{0,\Omega}, \quad (3.22)$$

$$\|\Pi_{\mathcal{U}_h}(\mathbf{v})\|_{0,\frac{3}{2},\Omega} \leq C \|\mathbf{v}\|_{0,\frac{3}{2},\Omega}. \quad (3.23)$$

A large class of interpolation or local L^2 projection operators satisfies these properties, in particular, the Lagrange finite element interpolation operators, based upon averaged nodal values such as the variant of the Scott–Zhang ([6], Sect. 4.8) operator in ([18], Appendix) or the Clément operator in [2]. The global $L^2(\Omega)$ orthogonal projection also satisfies these assumptions when the grids are uniformly regular (cf. [25]).

We next state the stability and convergence of discretization (3.13) and (3.14) with first-degree finite elements for velocities (for the sake of simplicity).

Theorem 3.4. *Assume that Hypotheses 2.5, 3.1 and 3.3 hold. Then, the discrete problem (3.13)–(3.14) with $l = 1$ in (3.7) admits a unique solution $(\mathbf{u}_h, p_h) \in \mathcal{U}_h \times \mathcal{P}_h$ which is bounded in $H_b^1(\Omega)^{d-1} \times L_{D,0}^{\frac{3}{2}}(\omega)$, satisfying the estimates*

$$|\mathbf{u}_h|_{1,\Omega} \leq \frac{C}{\mu} \left(\|\mathbf{f}\|_{\mathbf{H}_b^{-1}(\Omega)} + \|\mathbf{g}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_s)} \right), \tag{3.24}$$

$$\|p_h\|_{L_{D,0}^{\frac{3}{2}}(\omega)} \leq C \left(1 + \frac{1}{\mu} + \frac{1}{\sqrt{\mu}} \right) (1 + |\mathbf{w}|_{1,\Omega}) \left(\|\mathbf{f}\|_{\mathbf{H}_b^{-1}(\Omega)} + \|\mathbf{g}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_s)} \right), \tag{3.25}$$

$$\|II_{\mathcal{U}_h}^* (\mathbf{W}_h \cdot \nabla \mathbf{u}_h + \nabla_{\mathbf{x}} p_h + \varphi \mathbf{u}_h^\perp)\|_\tau \leq \frac{C}{\sqrt{\mu}} \left(\|\mathbf{f}\|_{\mathbf{H}_b^{-1}(\Omega)} + \|\mathbf{g}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_s)} \right), \tag{3.26}$$

where C is a constant independent of h .

Moreover, the sequence $\{(\mathbf{u}_h, p_h)\}_{h>0}$ contains a subsequence which is weakly convergent in $H_b^1(\Omega)^{d-1} \times L_{D,0}^{\frac{3}{2}}(\omega)$ to a solution of the continuous problem (3.5). If this solution belongs to $W_b^{1,3}(\Omega)^{d-1}$, then the convergence is strong.

Proof. Part of this proof can be found in reference [14] (Thms. 1 and 2) when $\mathbf{W} \in L^d(\Omega)^d$. Here we only detail substantial differences when $d = 3$ that are due to the low regularity of the vertical velocity.

Taking $\mathbf{v}_h = \mathbf{u}_h$ and $q_h = p_h$ in the variational formulation (3.13) and (3.14) we have:

$$\mu |\mathbf{u}_h|_{1,\Omega}^2 + \|II_{\mathcal{U}_h}^*(\mathbf{c}_h)\|_\tau^2 = L(\mathbf{u}_h) + (II_{\mathcal{U}_h}^*(\mathbf{c}_h), II_{\mathcal{U}_h}^*(\mathbf{f}_h))_\tau, \tag{3.27}$$

where $\mathbf{c}_h = \mathbf{W}_h \cdot \nabla \mathbf{u}_h + \nabla_{\mathbf{x}} p_h + \varphi \mathbf{u}_h^\perp$. Using (3.16) and (3.22),

$$\|II_{\mathcal{U}_h}^*(\mathbf{f}_h)\|_\tau \leq C h \|II_{\mathcal{U}_h}^*(\mathbf{f}_h)\|_{0,\Omega} \leq C h \|\mathbf{f}_h\|_{0,\Omega} \leq C \|\mathbf{f}_h\|_\tau. \tag{3.28}$$

Moreover,

$$\|\mathbf{f}_h\|_\tau \leq C \|\mathbf{f}_h\|_{\mathbf{H}_b^{-1}(\Omega)} \leq C \|\mathbf{f}\|_{\mathbf{H}_b^{-1}(\Omega)}. \tag{3.29}$$

This bound is obtained by using a representation of the term $\|\mathbf{f}_h\|_\tau$ on spaces of bubble functions by means of static condensation operators and (3.12). (See [14] for details). Combining (3.28) and (3.29), we obtain

$$\|II_{\mathcal{U}_h}^*(\mathbf{f}_h)\|_\tau \leq C \|\mathbf{f}\|_{\mathbf{H}_b^{-1}(\Omega)}. \tag{3.30}$$

Then, from (3.27), we derive

$$\sqrt{\mu} |\mathbf{u}_h|_{1,\Omega} + \|II_{\mathcal{U}_h}^*(\mathbf{c}_h)\|_\tau \leq \frac{C}{\sqrt{\mu}} \left(\|\mathbf{f}\|_{\mathbf{H}_b^{-1}(\Omega)} + \|\mathbf{g}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_s)} \right), \tag{3.31}$$

whence (3.24) and (3.26).

To estimate the pressure we use the inf-sup condition (3.17) with $p = \frac{3}{2}$ and $s = 3$:

$$C \|p_h\|_{L^{\frac{3}{2},0}(\omega)} \leq \sup_{\mathbf{v}_h \in \mathcal{U}_h} \frac{(\nabla_{\mathbf{x}} \cdot \mathbf{v}_h, p_h)}{|\mathbf{v}_h|_{1,3,\Omega}} + h \|II_{\mathcal{U}_h}^*(\nabla_{\mathbf{x}} p_h)\|_{0,\frac{3}{2},\Omega}. \tag{3.32}$$

To estimate the first summand, we take $q_h = 0$ in the discrete problem (3.13) and (3.14) and obtain:

$$\begin{aligned} (\nabla_{\mathbf{x}} \cdot \mathbf{v}_h, p_h) &= -(\mathbf{W}_h \cdot \nabla \mathbf{v}_h, \mathbf{u}_h) + \mu (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) + (\varphi \mathbf{u}_h^\perp, \mathbf{v}_h) \\ &\quad + (II_{\mathcal{U}_h}^*(\mathbf{W}_h \cdot \nabla \mathbf{v}_h + \varphi \mathbf{v}_h^\perp), II_{\mathcal{U}_h}^*(\mathbf{c}_h))_\tau \\ &\quad - (II_{\mathcal{U}_h}^*(\mathbf{W}_h \cdot \nabla \mathbf{v}_h + \varphi \mathbf{v}_h^\perp), II_{\mathcal{U}_h}^*(\mathbf{f}_h))_\tau - L(\mathbf{v}_h). \end{aligned} \tag{3.33}$$

Next, to estimate the terms in the right-hand side, we first observe that

$$|(\mathbf{W}_h \cdot \nabla \mathbf{v}_h, \mathbf{u}_h)| \leq \|\mathbf{W}_h\|_{0,\Omega} \|\nabla \mathbf{v}_h\|_{0,3,\Omega} \|\mathbf{u}_h\|_{0,6,\Omega} \leq C |\mathbf{w}|_{1,\Omega} |\mathbf{u}_h|_{1,\Omega} |\mathbf{v}_h|_{1,3,\Omega},$$

using (3.11) and Sobolev’s imbeddings. Also,

$$\|II_{\mathcal{U}_h}^*(\mathbf{W}_h \cdot \nabla \mathbf{v}_h + \varphi \mathbf{v}_h^\perp)\|_\tau \leq C h \|\mathbf{W}_h \cdot \nabla \mathbf{v}_h + \varphi \mathbf{v}_h^\perp\|_{0,\Omega},$$

using (3.16) and (3.22), the global L^2 stability of the operator $II_{\mathcal{U}_h}$. Next,

$$\|\mathbf{W}_h \cdot \nabla \mathbf{v}_h\|_{0,\Omega} \leq \|\mathbf{W}_h\|_{0,\Omega} \|\nabla \mathbf{v}_h\|_{0,\infty,\Omega} \leq C h^{-1} |\mathbf{w}|_{1,\Omega} |\mathbf{v}_h|_{1,3,\Omega},$$

using (3.11) and the global inverse inequality (2.5) between $W^{1,\infty}(\Omega)$ and $W^{1,3}(\Omega)$. Moreover,

$$\|\varphi \mathbf{v}_h^\perp\|_{0,\Omega} \leq C \|\varphi\|_{0,\infty,\Omega} \|\mathbf{v}_h\|_{0,\Omega} \leq C \|\varphi\|_{0,\infty,\Omega} |\mathbf{v}_h|_{1,3,\Omega}.$$

Thus,

$$\|II_{\mathcal{U}_h}^*(\mathbf{W}_h \cdot \nabla \mathbf{v}_h + \varphi \mathbf{v}_h^\perp)\|_\tau \leq C (|\mathbf{w}|_{1,\Omega} + \|\varphi\|_{0,\infty,\Omega}) |\mathbf{v}_h|_{1,3,\Omega}.$$

Then, from (3.33) we have

$$\begin{aligned} \frac{(\nabla_{\mathbf{x}} \cdot \mathbf{v}_h, p_h)}{|\mathbf{v}_h|_{1,3,\Omega}} &\leq C \left(|\mathbf{w}|_{1,\Omega} |\mathbf{u}_h|_{1,\Omega} + \mu |\mathbf{u}_h|_{1,\Omega} + \|\varphi\|_{0,\infty,\Omega} |\mathbf{u}_h|_{1,\Omega} \right. \\ &\quad \left. + (|\mathbf{w}|_{1,\Omega} + \|\varphi\|_{0,\infty,\Omega}) \|II_{\mathcal{U}_h}^*(\mathbf{c}_h)\|_\tau + (|\mathbf{w}|_{1,\Omega} + \|\varphi\|_{0,\infty,\Omega}) \|\mathbf{f}\|_{\mathbf{H}_b^{-1}(\Omega)} \right. \\ &\quad \left. + \|\mathbf{f}\|_{\mathbf{H}_b^{-1}(\Omega)} + \|\mathbf{g}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_s)} \right), \end{aligned} \tag{3.34}$$

and taking into account the estimates (3.24) and (3.26) we obtain

$$\frac{(\nabla_{\mathbf{x}} \cdot \mathbf{v}_h, p_h)}{|\mathbf{v}_h|_{1,3,\Omega}} \leq C \left(1 + \frac{1}{\mu} + \frac{1}{\sqrt{\mu}} \right) (1 + |\mathbf{w}|_{1,\Omega}) \left(\|\mathbf{f}\|_{\mathbf{H}_b^{-1}(\Omega)} + \|\mathbf{g}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_s)} \right), \quad \forall \mathbf{v}_h \in \mathcal{U}_h. \tag{3.35}$$

To estimate the second summand in (3.32) we split it in the following way:

$$\begin{aligned} h \|II_{\mathcal{U}_h}^*(\nabla_{\mathbf{x}} p_h)\|_{0,\frac{3}{2},\Omega} &\leq h \|II_{\mathcal{U}_h}^*(\mathbf{c}_h)\|_{0,\frac{3}{2},\Omega} \\ &\quad + h \|II_{\mathcal{U}_h}^*(\mathbf{W}_h \cdot \nabla \mathbf{u}_h)\|_{0,\frac{3}{2},\Omega} + h \|II_{\mathcal{U}_h}^*(\varphi \mathbf{u}_h^\perp)\|_{0,\frac{3}{2},\Omega}. \end{aligned} \tag{3.36}$$

We bound the first term by:

$$h \|II_{\mathcal{U}_h}^*(\mathbf{c}_h)\|_{0,\frac{3}{2},\Omega} \leq C h \|II_{\mathcal{U}_h}^*(\mathbf{c}_h)\|_{0,\Omega} \leq C \|II_{\mathcal{U}_h}^*(\mathbf{c}_h)\|_\tau,$$

using (3.16). The second term is bounded using first the stability (3.23) of the operator $\Pi_{\mathcal{U}_h}$,

$$h \|\Pi_{\mathcal{U}_h}^*(\mathbf{W}_h \cdot \nabla \mathbf{u}_h)\|_{0, \frac{3}{2}, \Omega} \leq Ch \|\mathbf{W}_h \cdot \nabla \mathbf{u}_h\|_{0, \frac{3}{2}, \Omega} \leq Ch \|\mathbf{W}_h\|_{0, \Omega} \|\nabla \mathbf{u}_h\|_{0, 6, \Omega}$$

and then the global inverse inequality (2.5) between $W^{1,6}(\Omega)$ and $H^1(\Omega)$,

$$\|\nabla \mathbf{u}_h\|_{0, 6, \Omega} \leq Ch^{-1} |\mathbf{u}_h|_{1, \Omega}.$$

Thus,

$$h \|\Pi_{\mathcal{U}_h}^*(\mathbf{W}_h \cdot \nabla \mathbf{u}_h)\|_{0, \frac{3}{2}, \Omega} \leq C |\mathbf{w}|_{1, \Omega} |\mathbf{u}_h|_{1, \Omega}. \tag{3.37}$$

The last term is bounded using also (3.23):

$$h \|\Pi_{\mathcal{U}_h}^*(\varphi \mathbf{u}_h^\perp)\|_{0, \frac{3}{2}, \Omega} \leq Ch \|\varphi \mathbf{u}_h^\perp\|_{0, \frac{3}{2}, \Omega} \leq Ch \|\varphi \mathbf{u}_h^\perp\|_{0, \Omega} \leq C \|\varphi\|_{0, \infty, \Omega} |\mathbf{u}_h|_{1, \Omega}.$$

Combining these estimates with (3.24) and (3.26), from (3.36) we have

$$h \|\Pi_{\mathcal{U}_h}^*(\nabla_{\mathbf{x}} p_h)\|_{0, \frac{3}{2}, \Omega} \leq C \left(\frac{1}{\mu} (1 + |\mathbf{w}|_{1, \Omega}) + \frac{1}{\sqrt{\mu}} \right) \left(\|\mathbf{f}\|_{\mathbf{H}_b^{-1}(\Omega)} + \|\mathbf{g}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_s)} \right). \tag{3.38}$$

Therefore, we deduce (3.25) from (3.32) with (3.35) and (3.38).

Due the estimates (3.24) and (3.25), the sequence $\{(\mathbf{u}_h, p_h)\}_{h>0}$ is uniform bounded in $H_b^1(\Omega)^{d-1} \times L^{\frac{3}{2}}_{D,0}(\omega)$ that is a reflexive space. Then, it contains a subsequence, that we still denote in the same way, weakly convergent in that space to a pair (\mathbf{u}, p) . This pair is a solution of problem (3.5). The proof of this convergence is similar to that performed in [14]. We skip the details here for brevity. \square

3.1.2. The case of regular meshes

We assume that the operator Π_h in (3.13) is an interpolation or projection operator from $L^2(\Omega)^{d-1}$ into $(V_h^{l-1})^{d-1}$. We define the space $\mathcal{U}_h(\mathcal{O}_i) = (V_h^l(\mathcal{O}_i) \cap H_0^1(\mathcal{O}_i))^{d-1}$, and recall the notation $\|\cdot\|_{h,p}$ for the norm defined in (2.3).

Lemma 3.5. *Assume that Hypothesis 2.1 holds. Then for any $p \in (1, +\infty)$ there exists a constant $\gamma_p > 0$ independent of h such that for all $q_h \in M_h$,*

$$\gamma_p \|q_h\|_{L^p_{D,0}(\omega)} \leq \sup_{\mathbf{v}_h \in \mathcal{U}_h} \frac{(\nabla_{\mathbf{x}} \cdot \mathbf{v}_h, q_h)}{|\mathbf{v}_h|_{1,s,\Omega}} + \left(\sum_{i=1}^R \left(\sup_{\mathbf{v}_h \in \mathcal{U}_h(\mathcal{O}_i)} \frac{|(\nabla_{\mathbf{x}} \cdot \mathbf{v}_h, q_h)_{\mathcal{O}_i}|^p}{|\mathbf{v}_h|_{1,s,\mathcal{O}_i}^p} \right) \right)^{\frac{1}{p}} + \|\Pi_h^*(\nabla_{\mathbf{x}} q_h)\|_{h,p}, \tag{3.39}$$

where s is the conjugate exponent of p .

Proof. Given $q_h \in \mathcal{P}_h$ we define its extension to Ω , \tilde{q}_h , by (3.18). Let J_h be an interpolation or projection operator from $L^2(\Omega)^d$ into $(V_h^{l-1})^d$ such that $J_h|_{L^2(\Omega)^{d-1}} = \Pi_h$. We use the inf-sup condition (2.7),

$$\gamma_p \|\tilde{q}_h\|_{0,p,\Omega} \leq \sup_{\mathbf{V}_h \in X_h} \frac{(\nabla \cdot \mathbf{V}_h, \tilde{q}_h)}{|\mathbf{V}_h|_{1,s,\Omega}} + \left(\sum_{i=1}^R \left(\sup_{\mathbf{V}_h \in X_h(\mathcal{O}_i)} \frac{|(\nabla \cdot \mathbf{V}_h, \tilde{q}_h)_{\mathcal{O}_i}|^p}{|\mathbf{V}_h|_{1,s,\mathcal{O}_i}^p} \right) \right)^{\frac{1}{p}} + \|J_h^*(\nabla \tilde{q}_h)\|_{h,p}. \tag{3.40}$$

On one hand,

$$\|\tilde{q}_h\|_{0,p,\Omega} = \|q_h\|_{L^p_{D,0}(\omega)}.$$

On the other hand, we have $\partial_z \tilde{q}_h = 0$ in Ω . Moreover, if $\mathbf{V} = (\mathbf{v}, v_{hz}) \in X_h$ then $v_{hz} = 0$ on $\partial\Omega$ and also, if $\mathbf{V} = (\mathbf{v}, v_{hz}) \in X_h(\mathcal{O}_i)$ then $v_{hz} = 0$ on $\partial\mathcal{O}_i$. Thus,

$$\sup_{\mathbf{V}_h \in X_h} \frac{(\nabla \cdot \mathbf{V}_h, \tilde{q}_h)}{|\mathbf{V}_h|_{1,s,\Omega}} \leq \sup_{\mathbf{v}_h \in (V_h^l \cap H_0^1(\Omega))^{d-1}} \frac{(\nabla_{\mathbf{x}} \cdot \mathbf{v}_h, q_h)}{|\mathbf{v}_h|_{1,s,\Omega}} \leq \sup_{\mathbf{v}_h \in \mathcal{U}_h} \frac{(\nabla_{\mathbf{x}} \cdot \mathbf{v}_h, q_h)}{|\mathbf{v}_h|_{1,s,\Omega}}, \tag{3.41}$$

and similarly

$$\sup_{\mathbf{v}_h \in X_h(\mathcal{O}_i)} \frac{(\nabla \cdot \mathbf{V}_h, \tilde{q}_h)_{\mathcal{O}_i}}{\|\mathbf{V}_h\|_{1,s,\mathcal{O}_i}} \leq \sup_{\mathbf{v}_h \in \mathcal{U}_h(\mathcal{O}_i)} \frac{(\nabla_{\mathbf{x}} \cdot \mathbf{v}_h, q_h)_{\mathcal{O}_i}}{\|\mathbf{v}_h\|_{1,s,\mathcal{O}_i}}. \tag{3.42}$$

Also, considering that $\nabla \tilde{q}_h = (\nabla_{\mathbf{x}} q_h, 0)$, we have

$$\|J_h^*(\nabla \tilde{q}_h)\|_{0,p,\mathcal{O}_i} = \|\Pi_h^*(\nabla_{\mathbf{x}} q_h)\|_{0,p,\mathcal{O}_i}. \tag{3.43}$$

Then, (3.39) follows from (3.40) by taking into account (3.41)–(3.43). □

We assume the following local stability properties of the operator Π_h :

Hypothesis 3.6. *The operator Π_h satisfies for all $\mathbf{v} \in L^2(\Omega)^{d-1}$,*

$$\|\Pi_h(\mathbf{v})\|_{0,K} \leq C \|\mathbf{v}\|_{0,w_K}, \tag{3.44}$$

$$\|\Pi_h(\mathbf{v})\|_{0,\frac{3}{2},K} \leq C \|\mathbf{v}\|_{0,\frac{3}{2},w_K}, \tag{3.45}$$

where w_K is the union of all elements of \mathcal{T}_h that intersect K .

This hypothesis is stronger than Hypothesis 3.3, but is also verified by local L^2 projection or Lagrange interpolation-based operators. However it is not verified by the global $L^2(\Omega)$ projection operator.

Observe that as a consequence of (3.44), the operator Π_h is also stable with respect to the norm $\|\cdot\|_\tau$:

$$\|\Pi_h(\mathbf{v})\|_\tau \leq C \|\mathbf{v}\|_\tau, \quad \forall \mathbf{v} \in L^2(\Omega)^{d-1}. \tag{3.46}$$

Theorem 3.7. *Assume that Hypotheses 2.1, 3.1 and 3.6 hold. Then, the discrete problem (3.13) and (3.14) with $l = 1$ in (3.7) admits a unique solution $(\mathbf{u}_h, p_h) \in \mathcal{U}_h \times \mathcal{P}_h$ which is bounded in $H_b^1(\Omega)^{d-1} \times L_{D,0}^{\frac{3}{2}}(\omega)$, satisfying the estimates*

$$\|\mathbf{u}_h\|_{1,\Omega} \leq \frac{C}{\mu} \left(\|\mathbf{f}\|_{\mathbf{H}_b^{-1}(\Omega)} + \|\mathbf{g}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_s)} \right), \tag{3.47}$$

$$\|p_h\|_{L_{D,0}^{\frac{3}{2}}(\omega)} \leq C \left(1 + \frac{1}{\mu} + \frac{1}{\sqrt{\mu}} \right) (1 + \|\mathbf{w}\|_{1,\Omega}) \left(\|\mathbf{f}\|_{\mathbf{H}_b^{-1}(\Omega)} + \|\mathbf{g}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_s)} \right), \tag{3.48}$$

$$\|\Pi_h^*(\mathbf{W}_h \cdot \nabla \mathbf{u}_h + \nabla_{\mathbf{x}} p_h + \varphi \mathbf{u}_h^\perp)\|_\tau \leq \frac{C}{\sqrt{\mu}} \left(\|\mathbf{f}\|_{\mathbf{H}_b^{-1}(\Omega)} + \|\mathbf{g}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_s)} \right), \tag{3.49}$$

where C is a constant independent of h .

Moreover, the sequence $\{(\mathbf{u}_h, p_h)\}_{h>0}$ contains a subsequence which is weakly convergent in $H_b^1(\Omega)^{d-1} \times L_{D,0}^{\frac{3}{2}}(\omega)$ to a solution of the continuous problem (3.5). If this solution belongs to $W_b^{1,3}(\Omega)^{d-1}$, then the convergence is strong.

Proof. The estimations (3.47) and (3.49) are obtained in the same way as in Theorem 3.4 but the estimate of the pressure is now based on the inf-sup condition (3.39) with $p = s = 2$, when $d = 2$, or $p = \frac{3}{2}$ and $s = 3$, when $d = 3$. We indicate the estimates in this last case. To estimate the first summand in (3.39), we start from (3.33). We treat all terms in the same way as before but to bound the term $\|\Pi_h^*(\mathbf{W}_h \cdot \nabla \mathbf{v}_h + \varphi \mathbf{v}_h^\perp)\|_\tau$ we have to argue locally because in this case we only can use local inverse inequalities:

From (3.46),

$$\|\Pi_h^*(\mathbf{W}_h \cdot \nabla \mathbf{v}_h + \varphi \mathbf{v}_h^\perp)\|_\tau \leq C \|\mathbf{W}_h \cdot \nabla \mathbf{v}_h + \varphi \mathbf{v}_h^\perp\|_\tau.$$

Moreover,

$$\|\mathbf{W}_h \cdot \nabla \mathbf{v}_h\|_\tau^2 = \sum_{K \in \mathcal{T}_h} \tau_K \|\mathbf{W}_h \cdot \nabla \mathbf{v}_h\|_{0,K}^2 \leq \alpha_2 \sum_{K \in \mathcal{T}_h} h_K^2 \|\mathbf{W}_h\|_{0,K}^2 \|\nabla \mathbf{v}_h\|_{0,\infty,K}^2.$$

Then, using (3.11), the local inverse inequality (2.4) between $W^{1,\infty}(K)$ and $W^{1,3}(K)$, and (2.1), we derive

$$\|\mathbf{W}_h \cdot \nabla \mathbf{v}_h\|_\tau \leq C |\mathbf{w}|_{1,\Omega} |\mathbf{v}_h|_{1,3,\Omega}.$$

Likewise,

$$\|\varphi \mathbf{v}_h^\perp\|_\tau^2 = \sum_{K \in \mathcal{T}_h} \tau_K \|\varphi \mathbf{v}_h^\perp\|_{0,K}^2 \leq \alpha_2 \sum_{K \in \mathcal{T}_h} h_K^2 \|\varphi\|_{0,\infty,\Omega}^2 \|\mathbf{v}_h\|_{0,K}^2 \leq \alpha_2 \|\varphi\|_{0,\infty,\Omega}^2 h \|\mathbf{v}_h\|_{0,\Omega}^2.$$

So,

$$\|\varphi \mathbf{v}_h^\perp\|_\tau \leq C \|\varphi\|_{0,\infty,\Omega} |\mathbf{v}_h|_{1,3,\Omega}.$$

Also from (3.46),

$$\|II_{\mathcal{U}_h}^*(\mathbf{f}_h)\|_\tau \leq C \|\mathbf{f}_h\|_\tau.$$

In this way, we obtain (3.35).

To estimate the second summand in (3.39), we consider the variational formulation (3.13) and (3.14) with $\mathbf{v}_h \in \mathcal{U}_h(\mathcal{O}_i)$ and $q_h = 0$. Then,

$$\begin{aligned} (\nabla_{\mathbf{x}} \cdot \mathbf{v}_h, p_h)_{\mathcal{O}_i} &= (\mathbf{W}_h \cdot \nabla \mathbf{u}_h, \mathbf{v}_h)_{\mathcal{O}_i} + \mu (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h)_{\mathcal{O}_i} + (\varphi \mathbf{u}_h^\perp, \mathbf{v}_h)_{\mathcal{O}_i} \\ &+ (II_h^*(\mathbf{W}_h \cdot \nabla \mathbf{v}_h + \varphi \mathbf{v}_h^\perp), II_{\mathcal{U}_h}^*(\mathbf{c}_h))_{\tau,\mathcal{O}_i} - (II_h^*(\mathbf{W}_h \cdot \nabla \mathbf{v}_h + \varphi \mathbf{v}_h^\perp), II_{\mathcal{U}_h}^*(\mathbf{f}_h))_{\tau,\mathcal{O}_i} - L(\mathbf{v}_h). \end{aligned} \tag{3.50}$$

All terms in the right-hand side are estimated as before but here again we argue locally. For the stabilizing term, we have:

$$|(II_h^*(\mathbf{W}_h \cdot \nabla \mathbf{v}_h + \varphi \mathbf{v}_h^\perp), II_{\mathcal{U}_h}^*(\mathbf{c}_h))_{\tau,\mathcal{O}_i}| \leq \|II_h^*(\mathbf{W}_h \cdot \nabla \mathbf{v}_h + \varphi \mathbf{v}_h^\perp)\|_{\tau,\mathcal{O}_i} \|II_{\mathcal{U}_h}^*(\mathbf{c}_h)\|_{\tau,\mathcal{O}_i}.$$

To estimate the first factor, we first observe that we have the local version of (3.46):

$$\|II_h^*(\mathbf{W}_h \cdot \nabla \mathbf{v}_h + \varphi \mathbf{v}_h^\perp)\|_{\tau,\mathcal{O}_i} \leq C \|\mathbf{W}_h \cdot \nabla \mathbf{v}_h + \varphi \mathbf{v}_h^\perp\|_{\tau,\mathcal{O}_i},$$

because $\mathbf{v}_h \in \mathcal{U}_h(\mathcal{O}_i)$. Now, we expand the terms $\|\mathbf{W}_h \cdot \nabla \mathbf{v}_h\|_{\tau,\mathcal{O}_i}$ and $\|\varphi \mathbf{v}_h^\perp\|_{\tau,\mathcal{O}_i}$:

$$\begin{aligned} \|\mathbf{W}_h \cdot \nabla \mathbf{v}_h\|_{\tau,\mathcal{O}_i}^2 &= \sum_{K \in \mathcal{O}_i} \tau_K \|\mathbf{W}_h \cdot \nabla \mathbf{v}_h\|_{0,K}^2 \\ &\leq \alpha_2 \sum_{K \in \mathcal{O}_i} h_K^2 \|\mathbf{W}_h\|_{0,K}^2 \|\nabla \mathbf{v}_h\|_{0,\infty,K}^2 \leq C |\mathbf{w}|_{1,\Omega}^2 \sum_{K \in \mathcal{O}_i} |\mathbf{v}_h|_{1,3,K}^2, \end{aligned}$$

applying (3.15), (3.11), the local inverse inequality (2.4) between $W^{1,\infty}(K)$ and $W^{1,3}(K)$ and (2.1). Next,

$$\begin{aligned} \|\varphi \mathbf{v}_h^\perp\|_{\tau,\mathcal{O}_i}^2 &= \sum_{K \in \mathcal{O}_i} \tau_K \|\varphi \mathbf{v}_h^\perp\|_{0,K}^2 \\ &\leq \alpha_2 \sum_{K \in \mathcal{O}_i} h_K^2 \|\varphi\|_{0,\infty,\Omega}^2 \|\mathbf{v}_h\|_{0,K}^2 \leq C \|\varphi\|_{0,\infty,\Omega}^2 h^2 \sum_{K \in \mathcal{O}_i} |\mathbf{v}_h|_{1,3,K}^2. \end{aligned}$$

But

$$\left(\sum_{K \in \mathcal{O}_i} |\mathbf{v}_h|_{1,3,K}^2 \right)^{\frac{1}{2}} \leq C |\mathbf{v}_h|_{1,3,\mathcal{O}_i},$$

because the number of elements in \mathcal{O}_i is bounded by a constant independent of h and i .

Then,

$$|(II_h^*(\mathbf{W}_h \cdot \nabla \mathbf{v}_h + \varphi \mathbf{v}_h^\perp), II_{\mathcal{U}_h}^*(\mathbf{c}_h))_{\tau, \mathcal{O}_i}| \leq C(1 + |\mathbf{w}|_{1, \Omega}) |\mathbf{v}_h|_{1,3, \mathcal{O}_i} \|II_{\mathcal{U}_h}^*(\mathbf{c}_h)\|_{\tau, \mathcal{O}_i}.$$

From here, applying Jensen's inequality,

$$\begin{aligned} & \left[\sum_{i=1}^R \sup_{\mathbf{v}_h \in \mathcal{U}_h(\mathcal{O}_i)} \left(\frac{|(II_h^*(\mathbf{W}_h \cdot \nabla \mathbf{v}_h + \varphi \mathbf{v}_h^\perp), II_{\mathcal{U}_h}^*(\mathbf{c}_h))_{\tau, \mathcal{O}_i}|}{|\mathbf{v}_h|_{1,3, \mathcal{O}_i}} \right)^{\frac{2}{3}} \right]^{\frac{3}{2}} \\ & \leq C(1 + |\mathbf{w}|_{1, \Omega}) \left(\sum_{i=1}^R \|II_{\mathcal{U}_h}^*(\mathbf{c}_h)\|_{\tau, \mathcal{O}_i}^2 \right)^{\frac{1}{2}} \leq C(1 + |\mathbf{w}|_{1, \Omega}) \|II_{\mathcal{U}_h}^*(\mathbf{c}_h)\|_{\tau}, \end{aligned}$$

because the number of repetitions of a given element K in all macro-elements is bounded by a fixed constant independent of h . Hence, by (3.49),

$$\begin{aligned} & \left(\sum_{i=1}^R \sup_{\mathbf{v}_h \in \mathcal{U}_h(\mathcal{O}_i)} \left(\frac{|(II_h^*(\mathbf{W}_h \cdot \nabla \mathbf{v}_h + \varphi \mathbf{v}_h^\perp), II_{\mathcal{U}_h}^*(\mathbf{c}_h))_{\tau, \mathcal{O}_i}|}{|\mathbf{v}_h|_{1,3, \mathcal{O}_i}} \right)^{\frac{3}{2}} \right)^{\frac{2}{3}} \\ & \leq C \frac{1}{\sqrt{\mu}} (1 + |\mathbf{w}|_{1, \Omega}) \left(\|\mathbf{f}\|_{\mathbf{H}_b^{-1}(\Omega)} + \|\mathbf{g}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_s)} \right). \end{aligned}$$

We proceed in a similar way for the remaining terms and obtain

$$\begin{aligned} & \left(\sum_{i=1}^R \sup_{\mathbf{v}_h \in \mathcal{U}_h(\mathcal{O}_i)} \left(\frac{|(\nabla_{\mathbf{x}} \cdot \mathbf{v}_h, p_h)_{\mathcal{O}_i}|}{|\mathbf{v}_h|_{1,3, \mathcal{O}_i}} \right)^{\frac{3}{2}} \right)^{\frac{2}{3}} \\ & \leq C \left(1 + \frac{1}{\mu} + \frac{1}{\sqrt{\mu}} \right) (1 + |\mathbf{w}|_{1, \Omega}) \left(\|\mathbf{f}\|_{\mathbf{H}_b^{-1}(\Omega)} + \|\mathbf{g}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_s)} \right). \end{aligned} \tag{3.51}$$

It remains to estimate the third summand in (3.39). First, we write:

$$\|II_h^*(\nabla_{\mathbf{x}} p_h)\|_{h, \frac{3}{2}} \leq \|II_h^*(\mathbf{c}_h)\|_{h, \frac{3}{2}} + \|II_h^*(\mathbf{W}_h \cdot \nabla \mathbf{u}_h)\|_{h, \frac{3}{2}} + \|II_h^*(\varphi \mathbf{u}_h^\perp)\|_{h, \frac{3}{2}}. \tag{3.52}$$

Again we argue locally to bound the first term in the r.h.s. of (3.52). By definition,

$$\|II_h^*(\mathbf{c}_h)\|_{h, \frac{3}{2}} = \left(\sum_{i=1}^R h_i^{\frac{3}{2}} \|II_{\mathcal{U}_h}^*(\mathbf{c}_h)\|_{0, \frac{3}{2}, \mathcal{O}_i}^2 \right)^{\frac{1}{2}}.$$

To bound this term, it is convenient to use the function

$$H(\mathbf{x}) = \sum_{K \in \mathcal{T}_h} h_K \chi_K(\mathbf{x}),$$

where χ_K is the characteristic function of K . Then (2.2) implies that

$$\|II_h^*(\mathbf{c}_h)\|_{h, \frac{3}{2}} \leq C \left(\sum_{i=1}^R \|H II_h^*(\mathbf{c}_h)\|_{0, \frac{3}{2}, \mathcal{O}_i}^2 \right)^{\frac{1}{2}}.$$

Since any element $K \in \mathcal{T}_h$ belongs to at most M macro-elements, this yields

$$\|II_h^*(\mathbf{c}_h)\|_{h, \frac{3}{2}} \leq C \left(\sum_{K \in \mathcal{T}_h} \|H II_h^*(\mathbf{c}_h)\|_{0, \frac{3}{2}, K}^2 \right)^{\frac{1}{2}}.$$

Finally, we associate with τ_k the analogue of H :

$$T(\mathbf{x}) = \sum_{K \in \mathcal{T}_h} \tau_K \chi_K(\mathbf{x}).$$

Then we infer from (3.15) and Cauchy–Schwarz’s inequality:

$$\begin{aligned} \|II_h^*(\mathbf{c}_h)\|_{h, \frac{3}{2}} &\leq C \left(\sum_{K \in \mathcal{T}_h} \|\sqrt{T}II_h^*(\mathbf{c}_h)\|_{0, \frac{3}{2}, K}^{\frac{3}{2}} \right)^{\frac{2}{3}} = C \|\sqrt{T}II_h^*(\mathbf{c}_h)\|_{0, \frac{3}{2}, \Omega} \\ &\leq C \|\sqrt{T}II_h^*(\mathbf{c}_h)\|_{0, 2, \Omega} = C \|II_h^*(\mathbf{c}_h)\|_{\tau}, \end{aligned} \tag{3.53}$$

which is bounded by (3.49).

To estimate the second term in the r.h.s. of (3.52), we use first the local stability of II_h :

$$\|II_h^*(\mathbf{W}_h \cdot \nabla \mathbf{u}_h)\|_{h, \frac{3}{2}}^{\frac{3}{2}} \leq C \sum_{i=1}^R h_i^{\frac{3}{2}} \sum_{K \in \mathcal{O}_i} \|\mathbf{W}_h \cdot \nabla \mathbf{u}_h\|_{0, \frac{3}{2}, w_K}^{\frac{3}{2}}.$$

Then we use the local quasi-uniformity of the mesh and the local inverse inequality (2.4) between $W^{1,6}(K)$ and $W^{1,2}(K)$,

$$\|\nabla \mathbf{u}_h\|_{0, 6, K} \leq C \rho_K^{-1} \|\nabla \mathbf{u}_h\|_{0, K}.$$

Therefore

$$\begin{aligned} \|II_h^*(\mathbf{W}_h \cdot \nabla \mathbf{u}_h)\|_{h, \frac{3}{2}}^{\frac{3}{2}} &\leq C \sum_{K \in \mathcal{T}_h} h_K^{\frac{3}{2}} \|\mathbf{W}_h \cdot \nabla \mathbf{u}_h\|_{0, \frac{3}{2}, w_K}^{\frac{3}{2}} \leq C \sum_{K \in \mathcal{T}_h} h_K^{\frac{3}{2}} \left(\frac{1}{\rho_K} \|\mathbf{W}_h\|_{0, w_K} \|\nabla \mathbf{u}_h\|_{0, w_K} \right)^{\frac{3}{2}} \\ &\leq C \sum_{K \in \mathcal{T}_h} \|\mathbf{W}_h\|_{0, w_K}^{\frac{3}{2}} \|\nabla \mathbf{u}_h\|_{0, w_K}^{\frac{3}{2}}. \end{aligned}$$

Hence,

$$\begin{aligned} \|II_h^*(\mathbf{W}_h \cdot \nabla \mathbf{u}_h)\|_{h, \frac{3}{2}} &\leq C \left(\sum_{K \in \mathcal{T}_h} \|\mathbf{W}_h\|_{0, K}^{\frac{3}{2}} \|\nabla \mathbf{u}_h\|_{0, K}^{\frac{3}{2}} \right)^{\frac{2}{3}} \leq C \left(\sum_{K \in \mathcal{T}_h} \|\mathbf{W}_h\|_{0, K}^6 \right)^{\frac{1}{6}} |\mathbf{u}_h|_{1, \Omega} \\ &\leq C \|\mathbf{W}_h\|_{0, \Omega} |\mathbf{u}_h|_{1, \Omega} \leq C |\mathbf{w}|_{1, \Omega} |\mathbf{u}_h|_{1, \Omega}, \end{aligned} \tag{3.54}$$

by another application of Jensen’s inequality and (3.11). Similarly, we bound the last term in the r.h.s. of (3.52) using again (3.45) and obtain

$$\|II_h^*(\varphi \mathbf{u}_h^\perp)\|_{h, \frac{3}{2}} \leq C h \|\varphi\|_{0, \infty, \Omega} |\mathbf{u}_h|_{1, \Omega}. \tag{3.55}$$

Then, from (3.52) and taking into account (3.53)–(3.55):

$$\|II_h^*(\nabla_{\mathbf{x}} p_h)\|_{h, \frac{3}{2}} \leq \left(\frac{1}{\mu} (1 + |\mathbf{w}|_{1, \Omega}) + \frac{1}{\sqrt{\mu}} \right) \left(\|\mathbf{f}\|_{\mathbf{H}_b^{-1}(\Omega)} + \|\mathbf{g}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_s)} \right). \tag{3.56}$$

Finally, we derive (3.48) from (3.39) by combining (3.35), (3.51) and (3.56). The rest of the proof follows as in Theorem 3.4. \square

Remark 3.8. When $l > 1$ in (3.7), Theorems 3.4 and 3.7 also hold if the constant α_2 in (3.15) is small enough. In this case, we have also to bound the term $\|\Delta \mathbf{u}_h\|_{\tau}$. (See [14] for details).

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