Lagrangian Formalism in Perturbed Nonlinear Klein-Gordon Equations

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Abstract

We develop an alternative approach to study the effect of the generic perturbation (in addition to explicitly considering the loss term) in the nonlinear Klein-Gordon equations. By a change of the variables that cancel the dissipation term we are able to write the Lagrangian density and then, calculate the Lagrangian as a function of collective variables. We use the Lagrangian formalism together with the Rice *Ansatz* to derive the equations of motion of the collective coordinates (CCs) for the perturbed sine-Gordon (sG) and ϕ^4 systems. For the N collective coordinates, regardless of the *Ansatz* used, we show that, for the nonlinear Klein-Gordon equations, this approach is equivalent to the *Generalized Traveling Wave Ansatz* (*GTWA*).

Key words: Collective coordinates, Solitons, Solitary waves, Perturbed Nonlinear Klein-Gordon equations.
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1 Introduction

The solitons and solitary waves under the external perturbations have been extensively studied in the last three decades (see e.g. [1,2,3] and references there in). This has been more than justified since in the physical real systems, modeled by the equations related with these solutions, the dissipation

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and the external forces are unavoidably present [4,5]. In this work we use the Lagrangian formalism to study the effect of damping and external forces in the nonlinear Klein-Gordon systems, and in particular we use as examples sine-Gordon (sG) and ϕ^4 (this method can be extended to other nonlinear Klein-Gordon equations such as, the double sine-Gordon (DsG) [3] or the asymmetric double sine-Gordon (ADSG) [6]). To illustrate the advantage of using this method over previous ones used in this kind of problem, let us revise the situation of the perturbation theory and collective coordinates (CCs in short) on the aforesaid systems. First of all, the authors of [4] developed a technique to investigate the influence of external perturbations, such as damping, dc force and spatial inhomogeneity, on solitary waves. This method is based on the expansion of the solution of the perturbed problem in the complete set of eigenfunctions of the Sturm-Liouville problem associated with the linearized partial differential equation (PDE) around a kink solution. Later, the dynamic of the breather and kink-antikink solution in the perturbed sG equation was also considered using a new perturbative analysis [5,7,8], where the radiation field was also included [5,7]. All these methods involve cumbersome calculations, even more so if we extend them to the ϕ^4 equation where the internal mode is present. In particular, if it is assumed that the perturbations only alter the center of the kink, $X(t)$, the calculations can be simplified. In this case, it was shown that the perturbative method is equivalent to the variation of the energy of the system [5]. However, the presence of the internal mode in ϕ^4 and some phonon's modes in sG are also able to change the kink's width, $l(t)$. Then, the solitary waves under certain perturbations can exhibit resonance's phenomena [9,10,11] and it is not appropriate to only consider one collective coordinate, for example, the center of the kink, in order to describe its dynamics. These kind of resonances due to the action of the ac force and damping were successfully explained by using a more general *Antsatz*, the so called Generalized Traveling Wave Ansatz (*GTWA*) [9,10,12] together with the Rice approximated solution [7,13]. This latter method is easier to apply than the former perturbation theory, however it is only supported in the projection technique. Recently, the equivalence of the *GTWA* and the variation of the momentum and the energy (two constants of motion of the unperturbed problem) of the nonlinear Klein-Gordon systems when two CCs are considered [14], has been shown. When it is necessary to use more that two CCs, this equivalence is unclear due to the absence of (or lack of knowledge of) the constants of motion for the general unperturbed nonlinear Klein-Gordon equation (except for the integrable sG equation).

In order to solve the perturbed nonlinear Klein-Gordon equation, we need to assume how explicit the approximated solution of the problem is, i.e. the *Ansatz*. Usually the *Ansatz* involves a number of unknown variables (collective coordinates) and by using either the perturbation theory or CCs approach we obtain the equations of motion (system of ordinary differential equations) that the CCs satisfy.

The equations of the CCs have been also derived by using the *Lagrangian formalism*. In spite of its simplicity, this method has only been used to study the kink-antikink collisions in the unperturbed problem [15] and the kink-impurity interactions in the sG and ϕ^4 models [16,17]. Furthermore, it has been used to analyze kink-antikink collisions in the damped ϕ^4 and sG models driven by an external force, however the projection technique was used to treat the dissipation separately [18]. Probably, the main difficulty in using this method was the fact that there is not any systematic way to construct the Lagrangian density when a generic perturbation is added into the sG or ϕ^4 equations.

The aim of this work is to extend the Lagrangian formalism to the generic perturbed nonlinear Klein-Gordon equation in order to obtain the evolution equations that obey the CCs. We explicitly consider a damping term as a perturbation. In this case, we are able to write the Lagrangian density by introducing a new time variable and calculate the Lagrangian as a function of the CCs for a given *Ansatz* (see the section 2). Some examples are also considered in section 2, where we analyze perturbed sG and ϕ^4 equations and we show the equivalence between the *GTWA* and the Lagrangian formalism for the nonlinear Klein-Gordon equations. Finally, in section 3 we discuss and summarize the main results of this work.

2 Lagrangian formalism

In this section we study the approximated solution of the following perturbed nonlinear Klein-Gordon equation

$$\phi_{tt} - \phi_{xx} = -\frac{dU}{d\phi} - \beta\phi_t + z(x,t,\phi), \tag{1}$$

where the subindex t and x indicate the partial derivatives with respect to time and space, respectively; $U(\phi)$ is the nonlinear Klein-Gordon potential, β is the damping coefficient and $z(x,t,\phi)$ represents a generic perturbation on the system. Notice that, by making the change of variable in time $\tau = \exp(-\beta t)$ [19], Eq. (1) becomes in dissipationless equation,

$$\phi_{\tau\tau} - \frac{\phi_{xx}}{\beta^2\tau^2} = -\frac{1}{\beta^2\tau^2}\frac{dU}{d\phi} + \frac{\tilde{z}(x,\tau,\phi)}{\beta^2\tau^2}. \tag{2}$$

Let us to remark that the idea to suppress the damping in the nonlinear systems have been already used in [20], where the authors showed that the effect of damping in the Landau-Lifshitz equation is only a rescaling of time by a complex constant.

Now by using the Euler-Lagrange equation [21] it is not difficult to show that Eq. (2) can be obtained from the Lagrangian

$$L = \int_{-\infty}^{+\infty} dx \mathcal{L} = \int_{-\infty}^{+\infty} dx \left\{ \frac{1}{2}\phi_\tau^2 - \frac{1}{2}\frac{\phi_x^2}{\beta^2\tau^2} - \frac{U(\phi)}{\beta^2\tau^2} + \frac{W(x,\tau)\phi_x}{\beta^2\tau^2} \right\}, \tag{3}$$

where $\mathcal{L}$ is the Lagrangian density, and

$$W(x,\tau) = -\int_{x_0}^{x} \tilde{z}(y,\tau,\phi) dy, \tag{4}$$

where ϕ in the last expression represents the approximated solution of Eq. (1), i.e. the *Ansatz*. Notice that, in general the expression (3) is an approximated Lagrangian density corresponding to the Eq. (2). In particular, if $z(x,t,\phi)$ is not a function of the field ϕ, the Eq. (3) represents an exact Lagrangian density corresponding to Eq. (1). To proceed we need to assume an approximated solution for Eq. (2). The *Ansatz* we use involves N CCs ($\vec{Y} = Y_1, Y_2, ..., Y_N$) and so, by inserting it in the Eq. (3) we obtain the Lagrangian as a function of the N CCs and their derivatives with respect to the new time variable τ, $L(\vec{Y}, \vec{Y}')$. From this moment on with the prime and the dot we denote the time derivative in relation to τ and t, respectively. The next step is to derive the equations of motion for a given CC Y_i ($i = 1, 2, ..., N$) by using the Lagrange equation

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial Y_i'}\right) - \frac{\partial L}{\partial Y_i} = 0. \tag{5}$$

In particular, we will analyze two cases, $U(\phi) = 1 - \cos(\phi)$ corresponding to sG equation, and $U(\phi) = (1/4)(1-\phi^2)^2$ to ϕ^4 one. In both examples, we use the Rice's *Ansatz* [7,13]. This approximated solution reads

$$\phi(x,\tau) = 4\,\mathrm{atan}\left(\exp\left[\frac{x - X(\tau)}{l(\tau)}\right]\right), \tag{6}$$

for the sG and

$$\phi(x,\tau) = \tanh\left[\frac{x - X(\tau)}{l(\tau)}\right], \tag{7}$$

for ϕ^4 equation, where $X(\tau)$ and $l(\tau)$ represent the center and the width of the kink, respectively. Substituting Eqs. (6) and (7) in (3) and integrating we obtain

$$L(X,X',l,l') = \frac{M_0 l_0}{2l}(X')^2 + \frac{\alpha M_0 l_0}{2l}(l')^2 - \frac{M_0}{2\beta^2\tau^2}\left(\frac{l_0}{l} + \frac{l}{l_0}\right) + \frac{1}{\beta^2\tau^2}\int_{-\infty}^{+\infty} W(X+\theta l,\tau)\phi_\theta d\theta, \tag{8}$$

where M_0 is the mass of the kink, l_0 represents the width of the static unperturbed kink and α is a coefficient. In particular, $M_0 = 8$, $l_0 = 1$ and $\alpha = \pi^2/12$ for the sG and $M_0 = 4/(3\sqrt{2})$, $l_0 = \sqrt{2}$ and $\alpha = (\pi^2-6)/12$ for ϕ^4. From the Eq. (5) and the Lagrangian (8), the equations of motion for the CCs $X(\tau)$ and $l(\tau)$ are given by

$$\begin{aligned} &\frac{dP}{d\tau} = -\frac{1}{\beta^2\tau^2}\int_{-\infty}^{+\infty} \tilde{z}(X+\theta l,\tau,\phi)\phi_\theta d\theta, \qquad P(\tau) = \frac{M_0 l_0}{l(\tau)}X', \\ &\frac{\alpha M_0 l_0}{l}\left(l'' - \frac{(l')^2}{2l}\right) + \frac{P^2}{2M_0 l_0} + \frac{M_0}{2\beta^2\tau^2 l}\left(\frac{l}{l_0} - \frac{l_0}{l}\right) = \\ &\frac{1}{\beta^2\tau^2}\int_{-\infty}^{+\infty} \frac{\partial W}{\partial l}(X+\theta l,\tau)\phi_\theta d\theta, \end{aligned} \tag{9}$$

or equivalent, in a more compact form, the equations for $X(t)$ and $l(t)$ read

$$\begin{aligned} &\frac{dP}{dt} = -\beta P(t) - \int_{-\infty}^{+\infty} z(X+\theta l,\tau,\phi)\phi_\theta d\theta, \qquad P(t) = \frac{M_0 l_0 \dot{X}}{l(t)}, \\ &\alpha[\dot{l}^2 - 2\beta l\dot{l} - 2l\ddot{l}] = \frac{l^2}{l_0^2}\left(\frac{P^2}{M_0^2} + 1\right) - 1 + \frac{2l^2}{M_0 l_0}\int_{-\infty}^{+\infty} z(X+\theta l,\tau,\phi)\theta\phi_\theta d\theta. \end{aligned} \tag{10}$$

This system of equations has been also obtained by using the *GTWA* or the variation of the energy, $E(t)$, and the momentum, $P(t)$, together with the Rice's *Ansatz* [22]. As we will later show the Lagrangian formalism and the *GTWA* are equivalent in a more general sense, i.e. the equivalence between both methods is regardless of the *Ansatz* that we use as approximated solution of Eq. (1) and also of how many CCs we use. Notice that, for the ac forces $z(x,t,\phi) = \epsilon_1 \sin(\delta t+\delta_0)+\epsilon_2 \sin(m\delta t+\delta_0)$ [9,10,23], for the parametric periodic force $z(x,t,\phi) = \epsilon \sin(\delta t+\delta_0)\phi$ [24], and for the driven $z(x,t,\phi) = -\epsilon$ [25] we recover the equations related with the resonance phenomena and the ratchet effect studied in these references by using either the *GTWA* or the variation of $E(t)$ and $P(t)$.

In order to establish the relation between *GTWA* and Lagrangian formalism, presented here for the nonlinear Klein-Gordon equations, let us assume that the solution of Eq. (2) depends on the N CCs, so $\phi(x,\vec{Y}(\tau))$ with $\vec{Y}=Y_1,Y_2,...,Y_N$. Then, the Lagrangian density in Eq. (3) is a function of $\phi(x,\vec{Y}(\tau))$, so that

$$L=\int_{-\infty}^{+\infty} dx\,\mathcal{L}(\phi(x,\vec{Y}(\tau))). \tag{11}$$

By inserting Eq. (11) in (5), after some straightforward calculations (see e.g. [18]a) we obtain

$$\int_{-\infty}^{+\infty} dx\left\{\frac{\partial}{\partial\tau}\left(\frac{\partial\mathcal{L}}{\partial\phi_\tau}\right)+\frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial\phi_x}\right)-\frac{\partial\mathcal{L}}{\partial\phi}\right\}\frac{\partial\phi}{\partial Y_i}. \tag{12}$$

This equation is just the projection of Eq. (2) into the functions $\partial\phi/\partial Y_i$. By substituting the *Ansatz* $\phi(x,\vec{Y})$ in Eq. (12) we obtain the following set of N equations for the N CCs by using the Lagrangian density defined in (3):

$$\begin{aligned}&\int_{-\infty}^{+\infty} dx\left\{\frac{\partial\phi}{\partial Y_k}\sum_{i=1}^{N}\frac{\partial\tilde{\psi}}{\partial Y_i'}Y_i''+\sum_{i=1(i\neq k)}^{N}\left[\frac{\partial\phi}{\partial Y_k}\frac{\partial\tilde{\psi}}{\partial Y_i}-\frac{\partial\tilde{\psi}}{\partial Y_k}\frac{\partial\phi}{\partial Y_i}\right]Y_i'+\right.\\&\left.\frac{1}{\beta^2\tau^2}\left[\frac{\partial\phi}{\partial Y_k}\frac{\delta H}{\delta\phi}-\beta\tau\frac{\partial\tilde{\psi}}{\partial Y_k}\frac{\delta H}{\delta\psi}\right]-\frac{\tilde{z}}{\beta^2\tau^2}\frac{\partial\phi}{\partial Y_k}\right\}=0,\end{aligned} \tag{13}$$

where $k=1,2,...,N$, $\tilde{\psi}\equiv\phi_\tau=\tilde{\psi}(x,\vec{Y},\vec{Y}')$ and H is the Hamiltonian corresponding to Eq. (1) with $\beta=0$ and $z=0$. In the variable t this system of N equations reads

$$\begin{aligned}&\int_{-\infty}^{+\infty} dx\left\{\frac{\partial\phi}{\partial Y_k}\sum_{i=1}^{N}\frac{\partial\psi}{\partial\dot{Y}_i}\ddot{Y}_i+\sum_{i=1(i\neq k)}^{N}\left[\frac{\partial\phi}{\partial Y_k}\frac{\partial\psi}{\partial Y_i}-\frac{\partial\psi}{\partial Y_k}\frac{\partial\phi}{\partial Y_i}\right]\dot{Y}_i+\right.\\&\left.\frac{\partial\mathcal{H}}{\partial Y_k}-(z-\beta\psi)\frac{\partial\phi}{\partial Y_k}\right\}=0,\end{aligned} \tag{14}$$

where $\mathcal{H}$ is the Hamiltonian density of the unperturbed Eq. (1) [put in (1) β and z equal to zero]. This set of equations for the N CCs can be obtained also by using the *GTWA*. This set of equations is reduced when two CCs, $Y_1\equiv X(t)$ and $Y_2\equiv l(t)$, are considered. In this case we recover the Eqs. (7) and (10) of [14], obtained by using either the *GTWA* or the variation of the energy and the momentum.

3 Conclusions

In this work, using the Lagrangian formalism, we have developed an alternative method in order to obtain the system of ODEs that satisfy the CCs in the generic perturbed nonlinear Klein-Gordon models. The main advantage of this method over previous ones is that it involves less calculations, so that the equations of motion for the CCs can be derived straightforwardly. Indeed, given the generic perturbed nonlinear Klein-Gordon equation (1), we have shown that with a nonlinear change of the time variable in (1) it is possible to write the dissipationless equation (2) and its Lagrangian density (3). (In general $\mathcal{L}$ is an approximated Lagrangian density of (1). In particular, when the perturbation $z(x, t, \phi)$ does not depend on ϕ, we are able to obtain an exact Lagrangian density corresponding to Eq. (1)). Furthermore, given an *Ansatz* for the solution of (1) we can, first, calculate the Lagrangian as a function of the N CCs by using the Lagrangian density defined by Eq. (3). Then, from the Lagrange equation (5) we can obtain the corresponding system of ODEs for the collective variables.

By using the Rice *Ansatz* [7,13] we have obtained the system of ODEs that satisfy the collective variables for the perturbed sine-Gordon and ϕ^4 systems. These equations coincide with those obtained by the $GTWA$ [12] or the variation of energy and the momentum [14]. In particular, for ac and dc forces we have recovered the results obtained in [9,10,23,24,25]. We would like to remark that this approach, presented here for the perturbed sG and ϕ^4 models with the Rice *Ansatz* can be extended to others perturbed nonlinear Klein-Gordon systems, for example, DsG and ADSG equations [6]. Furthermore, the Lagrangian formalism is not restricted to one dimensional system. Indeed, it recently have been applied to solve approximately a problem concerning to the vortex theory [26].

Finally for N CCs, we have shown the equivalence between this method and the $GTWA$, regardless of the *Ansatz* that we use for the approximated solution of Eq. (1).

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