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EXISTENCE AND UNIQUENESS OF SOLUTIONS, AND PULLBACK ATTRACTOR FOR A SYSTEM OF GLOBALLY MODIFIED 3D–NAVIER-STOKES EQUATIONS WITH FINITE DELAY

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Abstract

We first study the existence and uniqueness of strong solutions of a three dimensional system of globally modified Navier-Stokes equations with finite delay in the locally Lipschitz case. The asymptotic behaviour of solutions, and the existence of pullback attractor are also analyzed.

Key words: *3-dimensional Navier-Stokes equations, Galerkin approximations, weak solutions, existence and uniqueness of strong solutions, pullback attractors.*
AMS subject classifications: *35Q30, 35K90, 37L30.*

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be an open bounded set with regular boundary Γ, and consider the following system of *globally modified Navier-Stokes equations (GMNSE)* on Ω with a homogeneous Dirichlet boundary condition

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + F_N\left(\|u\|\right)[(u\cdot\nabla)u] + \nabla p = f(t), \qquad \text{in } \ (0,+\infty)\times\Omega, \\ \nabla\cdot u \ = 0 \quad \text{in } \ (0,+\infty)\times\Omega, \\ u = 0 \ \text{ on } (0,+\infty)\times\Gamma, \\ u(0,x) = u^0(x), \quad x\in\Omega, \end{cases} \tag{1}$$

where $\nu > 0$ is the kinematic viscosity, u is the velocity field of the fluid, p the pressure, u^0 the initial velocity field, $f(t)$ a given external force field, and F_N : $\mathbb{R}^+ \to \mathbb{R}^+$ is defined by

$$F_N(r) := \min\left\{1, \frac{N}{r}\right\}, \quad r\in\mathbb{R}^+,$$

for some $N \in \mathbb{R}^+$.

The GMNSE (1) has been introduced and studied in [1] (see also [2], [3], [8] and [9]). In this paper we are interested in the case in which terms containing finite delays appear. We consider the following version of GMNSE (we will refer to it as GMNSED):

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + F_N\left(\|u\|\right)[(u\cdot\nabla)u] + \nabla p \\ = G(t, u(t-\rho(t))) \quad \text{in} \quad (\tau, +\infty)\times\Omega, \\ \nabla\cdot u \;= 0 \quad \text{in} \quad (\tau, +\infty)\times\Omega, \\ u = 0 \;\; \text{on } (\tau, +\infty)\times\Gamma, \\ u(\tau, x) = u^0(x), \quad x\in\Omega, \\ u(t,x) = \phi(t-\tau, x), \;\; \text{in } (\tau - h, \tau)\times\Omega, \end{cases} \tag{2}$$

where $\tau \in \mathbb{R}$ is an initial time, the term $G(t, u(t-\rho(t)))$ is an external force depending eventually on the value $u(t-\rho(t))$, $\rho(t) \geq 0$ is a delay function and ϕ is a given velocity field defined in $(-h, 0)$, with $h > 0$ a fixed time such that $\rho(t) \leq h$.

The aim of this paper is to report on some recent results concerning the existence, uniqueness and asymptotic behaviour of solutions of (2). The detailed proofs of these results can be found in [4]. In the next section we state some preliminaries, establish the framework for our problem, and the existence and uniqueness of weak and strong solutions. In Section 3 we analyze the asymptotic behaviour of solutions, obtaining finally the existence of pullback attractor for our model.

2 Preliminaries

To set our problem in the abstract framework, we consider the following usual abstract spaces (see [12] and [14, 15]):

$$\mathcal{V} = \left\{ u \in (C_0^\infty(\Omega))^3 : \operatorname{div} u \;= 0 \right\},$$

$H =$ the closure of $\mathcal{V}$ in $(L^2(\Omega))^3$ with inner product $(\cdot,\cdot)$ and associate norm $|\cdot|$, where for $u, v \in (L^2(\Omega))^3$,

$$(u, v) = \sum_{j=1}^{3} \int_\Omega u_j(x) v_j(x) \mathrm{d}x,$$

$V =$ the closure of $\mathcal{V}$ in $(H_0^1(\Omega))^3$ with scalar product $((\cdot,\cdot))$ and associate norm $\|\cdot\|$, where for $u, v \in (H_0^1(\Omega))^3$,

$$((u, v)) = \sum_{i,j=1}^{3} \int_\Omega \frac{\partial u_j}{\partial x_i}\frac{\partial v_j}{\partial x_i} \mathrm{d}x.$$

It follows that $V \subset H \equiv H' \subset V'$, where the injections are dense and compact. Finally, we will use $\|\cdot\|_*$ for the norm in V' and $\langle \cdot, \cdot \rangle$ for the duality pairing between V and V'.

Now we define the trilinear form b on $V \times V \times V$ by

$$b(u,v,w) = \sum_{i,j=1}^{3} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \, dx, \qquad \forall\, u,v,w \in V,$$

and we denote

$$b_N(u,v,w) = F_N(\|v\|) b(u,v,w), \qquad \forall\, u,v,w \in V,$$

and

$$\langle B_N(u,v), w \rangle = b_N(u,v,w), \qquad \forall\, u,v,w \in V.$$

The form b_N is linear in u and w, but it is nonlinear in v.

We also consider $A : V \to V'$ defined by $\langle Au, v \rangle = ((u,v))$. Denoting $D(A) = (H^2(\Omega))^3 \cap V$, then $Au = -P\Delta u, \forall\, u \in D(A)$, is the Stokes operator (P is the ortho-projector from $(L^2(\Omega))^3$ onto H). Moreover, we assume $G : \mathbb{R} \times H \longrightarrow H$, is such that

c1) $G(\cdot, u) : \mathbb{R} \longrightarrow H$ is measurable, $\forall u \in H$,

c2) there exists a nonnegative function $g \in L^p_{loc}(\mathbb{R})$ for some $1 \le p \le +\infty$, and a nondecreasing function $L : (0,\infty) \to (0,\infty)$, such that for all $R > 0$ if $|u|, |v| \le R$, then

$$|G(t,u) - G(t,v)| \le L(R) g^{1/2}(t) |u - v|,$$

for all $t \in \mathbb{R}$, and

c3) there exists a nonnegative function $f \in L^1_{loc}(\mathbb{R})$, such that for any $u \in H$,

$$|G(t,u)|^2 \le g(t) |u|^2 + f(t), \quad \forall\, t \in \mathbb{R}.$$

Finally, we suppose $\phi \in L^{2p'}(-h,0;H)$ and $u^0 \in H$, where $\frac{1}{p} + \frac{1}{p'} = 1$. In this situation, we consider a delay function $\rho \in C^1(\mathbb{R})$ such that $0 \le \rho(t) \le h$ for all $t \in \mathbb{R}$, and there exists a constant ρ_* satisfying

$$\rho'(t) \le \rho_* < 1 \quad \forall\, t \in \mathbb{R}.$$

Definition 1 *Let $\tau \in \mathbb{R}$, $u^0 \in H$ and $\phi \in L^{2p'}(-h,0;H)$ be given. A weak solution of (2) is a function*

$$u \in L^{2p'}(\tau - h, T; H) \cap L^2(\tau, T; V) \cap L^\infty(\tau, T; H) \quad \textit{for all } T > \tau,$$

such that

$$\begin{cases} \dfrac{d}{dt} u(t) + \nu A u(t) + B_N(u(t), u(t)) = G(t, u(t - \rho(t))) \ \textit{in } \mathcal{D}'(\tau, +\infty; V'), \\ u(\tau) = u^0, \\ u(t) = \phi(t - \tau) \quad t \in (\tau - h, \tau). \end{cases}$$

Remark 1 *We suppose u is a weak solution of (2) and we define $\tilde{g}(t) = g \circ \theta^{-1}(t)$, where $\theta : [\tau, +\infty) \longrightarrow [\tau - \rho(\tau), +\infty)$ is the differentiable and strictly increasing function given by $\theta(s) = s - \rho(s)$. Then, taking into account that $\tilde{g} \in L^p(\tau - \rho(\tau), T)$ for all $T > \tau$, we have that $G(t, u(t - \rho(t)))$ belongs to $L^2(\tau, T; H)$ for all $T > \tau$.*

Then, $\dfrac{d}{dt}u(t) \in L^2(\tau, T; V')$, and consequently (see [15]) $u \in C([\tau, +\infty); H)$ and satisfies the energy equality, for all $\tau \leq s \leq t$,

$$|u(t)|^2 - |u(s)|^2 + 2\nu \int_s^t \|u(r)\|^2 \, dr = 2 \int_s^t (G(r, u(r - \rho(r))), u(r)) \, dr. \qquad (3)$$

The following theorem, which improves Theorem 3 in [5], states the existence and uniqueness of weak and/or strong solutions.

Theorem 1 *Under the conditions c1)-c3) in the previous section, assume that $\tau \in \mathbb{R}$, $u^0 \in H$ and $\phi \in L^{2p'}(-h, 0; H)$ are given. Then, there exists a unique weak solution u of (2) which is, in fact, a strong solution in the sense that*

$$u \in C([\tau + \varepsilon, T]; V) \cap L^2(\tau + \varepsilon, T; D(A)), \quad \text{for all } T - \tau > \varepsilon > 0. \qquad (4)$$

Moreover, if $u^0 \in V$, then

$$u \in C([\tau, T]; V) \cap L^2(\tau, T; D(A)), \quad \text{for all } T > \tau. \qquad (5)$$

3 Asymptotic behaviour of solutions

In this section we first establish a result about the asymptotic behavior of the solutions of problem (2) when t goes to $+\infty$.

Theorem 2 *Let us suppose that c1)-c3) hold with $g \in L^\infty(\mathbb{R})$, and assume also that $\nu^2 \lambda_1^2 (1 - \rho_*) > |g|_\infty$, where $|g|_\infty := \|g\|_{L^\infty(\mathbb{R})}$.*

Let us denote $\varepsilon > 0$ the unique root of $\varepsilon - \nu\lambda_1 + \frac{|g|_\infty e^{\varepsilon h}}{\nu\lambda_1(1-\rho_*)} = 0$. Then, for any $(u^0, \phi) \in V \times L^2(-h, 0; H)$, and any $\tau \in \mathbb{R}$, the corresponding solution $u(t; \tau, u^0, \phi)$ of problem (2) satisfies

$$\begin{aligned} \left|u(t; \tau, u^0, \phi)\right|^2 \leq & \left(\left|u^0\right|^2 + \frac{|g|_\infty e^{\varepsilon h}}{\nu\lambda_1(1 - \rho_*)} \int_{-h}^0 e^{\varepsilon s} \left|\phi(s)\right|^2 ds \right) e^{\varepsilon(\tau - t)} \\ & + \frac{e^{-\varepsilon t}}{\nu\lambda_1} \int_\tau^t e^{\varepsilon s} f(s) \, ds, \quad \text{for all } t \geq \tau. \end{aligned}$$

In particular, if $\int_\tau^\infty e^{\varepsilon s} f(s) \, ds < \infty$, then every solution $u(t; \tau, u^0, \phi)$ of (2) converges exponentially to 0 as $t \to +\infty$.

Now, we study the existence of global attractor for the dynamical system generated by our problem. As this model is non-autonomous, our analysis requires of the theory of pullback attractor which we will introduced below (see [7], [10] and [11]).

Let X be a metric space.

Definition 2 *A family of mappings $\{U(t,\tau) : X \to X : t,\tau \in \mathbb{R}, t \geq \tau\}$ is said to be a process (or a two–parameter semigroup, or an evolution semigroup) in X if*

$$\begin{aligned} U(t,r)U(r,\tau) &= U(t,\tau) \qquad \text{for all } t \geq r \geq \tau, \\ U(\tau,\tau) &= Id \qquad\qquad \text{for all } \tau \in \mathbb{R}. \end{aligned}$$

The process $U(\cdot,\cdot)$ is said to be continuous if the mapping $x \to U(t,\tau)x$ is continuous on X for all $t,\tau \in \mathbb{R}, t \geq \tau$.

Recall that $dist(A,B)$ denotes the Hausdorff semidistance between the sets A and B, which is given by

$$dist(A,B) = \sup_{a\in A} \inf_{b\in B} d(a,b), \qquad \text{for } A,B \subset X.$$

Definition 3 *Let $U(\cdot,\cdot)$ be a process in the metric space X. A family of compact sets $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ is said to be a (global) pullback attractor for $U(\cdot,\cdot)$ if, for every $t \in \mathbb{R}$, if follows*

(i) $U(t,\tau)\mathcal{A}(\tau) = \mathcal{A}(t)$ *for all $\tau \leq t$ (invariance), and*

(ii) $\lim_{\tau\to-\infty} dist(U(t,\tau)D, \mathcal{A}(t)) = 0$ *(pullback attraction) for all bounded subset $D \subset X$.*

The concept of pullback attractor is related to that of pullback absorbing set.

Definition 4 *The family of subsets $\{B(t)\}_{t\in\mathbb{R}}$ of X is said to be pullback absorbing with respect to the process $U(\cdot,\cdot)$ if, for every $t \in \mathbb{R}$ and all bounded subset $D \subset X$, there exists $\tau_D(t) \leq t$ such that*

$$U(t,\tau)D \subset B(t), \quad \text{for all } \tau \leq \tau_D(t).$$

In fact, as happens in the autonomous case, the existence of compact pullback attracting sets is enough to ensure the existence of pullback attractors. The following result can be found in [7] and [13] (see also [6]).

Theorem 3 *Let $U(\cdot,\cdot)$ be a continuous process on the metric space X. If there exists a family of compact pullback attracting sets $\{B(t)\}_{t\in\mathbb{R}}$, then there exists a pullback attractor $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$, with $\mathcal{A}(t) \subset B(t)$ for all $t \in \mathbb{R}$, given by*

$$\mathcal{A}(t) = \overline{\bigcup_{\substack{D\subset X \\ bounded}} \Lambda_D(t)}, \quad \text{where } \Lambda_D(t) = \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{\tau\leq t-n} U(t,\tau)D}.$$

Now we will establish the existence of the pullback attractor for our GMNSED model (2).

First we construct the associated process. To this end, assume that $G:\mathbb{R}\times H\to H$ satisfies c1), c2) and c3) with $g\in L^\infty(\mathbb{R})$. Thus, without loss of generality we can assume that G satisfies c2) with $g\equiv 1$, and there exists a nonnegative constant a such that

$$|G(t,u)|^2 \le a|u|^2 + f(t) \quad \forall\,(t,u)\in\mathbb{R}\times H. \tag{6}$$

We will assume moreover that

$$f\in L^\infty_{loc}(\mathbb{R}). \tag{7}$$

Under these assumptions, for each initial time $\tau\in\mathbb{R}$, and any $\phi\in C(-h,0;H)$, Theorem 1 ensures that if we take $u^0=\phi(0)$, problem (2) possesses a unique solution $u(\cdot;\tau,\phi)=u(\cdot;\tau,\phi(0),\phi)$, which belongs to the space $C([\tau-h,T];H)\cap L^2(\tau,T;V)\cap C([\tau+\epsilon,T];V)\cap L^2(\tau+\epsilon,T;D(A))$ for all $T>\tau+\epsilon>\tau$.

Then, we define a process in the phase space $C_H=C([-h,0];H)$ with sup norm, $\|\phi\|_{C_H}=\sup_{s\in[-h,0]}|\phi(s)|$, as the family of mappings $U(t,\tau):C_H\to C_H$ given by

$$U(t,\tau)\phi = u_t(\cdot;\tau,\phi), \tag{8}$$

for any $\phi\in C_H,$ and any $\tau\le t$, where $u_t(\cdot;\tau,\phi)\in C_H$ is defined by

$$u_t(s;\tau,\phi)=u(t+s;\tau,\phi) \quad \forall\, s\in[-h,0]. \tag{9}$$

Proposition 4 *It is easy to check that if G satisfies c1), c2) with $g=1$, (6) and (7), then the family of mappings $U(\tau,t)$, $\tau\le t$, defined by (8) and (9) is a continuous process on C_H.*

Now, we will obtain that, under suitable assumptions, there exists a family of bounded pullback absorbing sets in C_H and then, another one in C_V, for the process $U(t,\tau)$.

Theorem 5 *Assume that G satisfies c1), c2) with $g=1$, (6), (7), and $\nu^2\lambda_1^2(1-\rho_*)>a$.*
Let $\varepsilon>0$ denote the unique solution of $\ \varepsilon-\nu\lambda_1+\frac{ae^{\varepsilon h}}{\nu\lambda_1(1-\rho_*)}=0.$
Let us suppose that $\ \int_{-\infty}^0 e^{\varepsilon r}f(r)\,dr<\infty$, and define

$$\rho_H(t)=1+\frac{e^{\varepsilon(1+h-t)}}{\nu\lambda_1}\int_{-\infty}^t e^{\varepsilon r}f(r)\,dr \quad t\in\mathbb{R}.$$

Then, for every bounded subset $D\subset C_H$ there exists a $T_D>1+h$ such that for any $t\in\mathbb{R}$ and all $\phi\in D$ one has

$$|u(s;\tau,\phi)|^2\le\rho_H(t) \quad \forall\, s\in[t-h-1,t], \quad \textit{for all}\ \ \tau\le t-T_D.$$

As a direct consequence of the preceding result, we get the existence of the family of bounded absorbing sets in C_H.

In fact, one can prove the following result of existence of an absorbing family of bounded sets in $C_V = C([-h,0];V)$ and a necessary bound on the term $\int_{t+\theta_1}^{t+\theta_2} |Au(r)|^2\,dr$.

Theorem 6 *Under the assumptions in Theorem 5, there exist two positive functions $\rho_V, F \in C(\mathbb{R})$ such that for any bounded set $D \subset C_H$ and for any $t \in \mathbb{R}$,*

$$\|u(t;\tau,\phi)\|^2 \leq \rho_V(t) \ \ \forall\, \tau \leq t - T_D, \ \forall\, \phi \in D,$$

and

$$\int_{t+\theta_1}^{t+\theta_2} |Au(r;\tau,\phi)|^2\,dr \leq F(t), \quad \forall\, \tau \leq t - T_D - h, \ \forall\, \theta_1 \leq \theta_2 \in [-h,0], \quad \forall\, \phi \in D,$$

where T_D is given in Theorem 5.

Finally, under an additional assumption, we can ensure the existence of the pullback attractor.

Theorem 7 *Under the assumptions in Theorem 5, suppose moreover that*

$$\sup_{s\leq 0} e^{-\varepsilon s} \int_{-\infty}^{s} e^{\varepsilon r} f(r)\,dr < \infty.$$

Then, there exists a pullback attractor $\{\mathcal{A}_{C_H}(t)\}_{t\in\mathbb{R}}$ for the process $U(\cdot,\cdot)$ in C_H defined by (8) *and* (9). *Moreover, $\mathcal{A}_{C_H}(t)$ is a bounded subset of C_V for any $t \in \mathbb{R}$.*

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