

LAGRANGIAN SUBMANIFOLDS IN COMPLEX SPACE FORMS SATISFYING AN IMPROVED EQUALITY INVOLVING $\delta(2, 2)$

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ABSTRACT. It was proved in [8, 9] that every Lagrangian submanifold M of a complex space form $\tilde{M}^5(4c)$ of constant holomorphic sectional curvature $4c$ satisfies the following optimal inequality:

$$\delta(2, 2) \leq \frac{25}{4}H^2 + 8c, \quad (\text{A})$$

where H^2 is the squared mean curvature and $\delta(2, 2)$ is a δ -invariant on M introduced by the first author. This optimal inequality improves a special case of an earlier inequality obtained in [B.-Y. Chen, Japan. J. Math. **26** (2000), 105–127].

The main purpose of this paper is to classify Lagrangian submanifolds of $\tilde{M}^5(4c)$ satisfying the equality case of the improved inequality (A).

1. INTRODUCTION

Let \tilde{M}^n be a Kähler n -manifold with the complex structure J , a Kähler metric g and the Kähler 2-form ω . An isometric immersion $\psi : M \rightarrow \tilde{M}^n$ of a Riemannian n -manifold M into \tilde{M}^n is called *Lagrangian* if $\psi^*\omega = 0$.

Let $\tilde{M}^n(4c)$ denote a Kähler n -manifold with constant holomorphic sectional curvature $4c$, called a *complex space form*. A complete simply-connected complex space form $\tilde{M}^n(4c)$ is holomorphically isometric to the complex Euclidean n -plane \mathbf{C}^n , the complex projective n -space $CP^n(4c)$, or a complex hyperbolic n -space $CH^n(4c)$ according to $c = 0$, $c > 0$ or $c < 0$, respectively.

B.-Y. Chen introduced in 1990s new Riemannian invariants $\delta(n_1, \dots, n_k)$. For any n -dimensional submanifold M in a real space form $R^m(c)$ of constant

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curvature c , he proved the following sharp general inequality (see [5, 7] for details):

$$\begin{aligned} \delta(n_1, \dots, n_k) &\leq \frac{n^2(n+k-1-\sum n_j)}{2(n+k-\sum n_j)} H^2 \\ &\quad + \frac{1}{2} \left(n(n-1) - \sum_{j=1}^k n_j(n_j-1) \right) c. \end{aligned} \quad (1.1)$$

For Lagrangian submanifolds in a complex space form $\tilde{M}^n(4c)$, we have

Theorem A. *Let M be an n -dimensional Lagrangian submanifold in a complex space form $\tilde{M}^n(4c)$ of constant holomorphic sectional curvature $4c$. Then inequality (1.1) holds for each k -tuple $(n_1, \dots, n_k) \in \mathcal{S}(n)$.*

The following result from [6] extends a result in [10] on $\delta(2)$.

Theorem B. *Every Lagrangian submanifold of a complex space form $\tilde{M}^n(4c)$ is minimal if it satisfies the equality case of (1.1) identically.*

Theorem B was improved recently in [8, 9] to the following inequality.

Theorem C. *Let M be an n -dimensional Lagrangian submanifold of $\tilde{M}^n(4c)$. Then, for an $(n_1, \dots, n_k) \in \mathcal{S}(n)$ with $\sum_{i=1}^k n_i < n$, we have*

$$\begin{aligned} \delta(n_1, \dots, n_k) &\leq \frac{n^2 \left\{ \left(n - \sum_{i=1}^k n_i + 3k - 1 \right) - 6 \sum_{i=1}^k (2 + n_i)^{-1} \right\}}{2 \left\{ \left(n - \sum_{i=1}^k n_i + 3k + 2 \right) - 6 \sum_{i=1}^k (2 + n_i)^{-1} \right\}} H^2 \\ &\quad + \frac{1}{2} \left\{ n(n-1) - \sum_{i=1}^k n_i(n_i-1) \right\} c. \end{aligned} \quad (1.2)$$

The equality sign holds at a point $p \in M$ if and only if there is an orthonormal basis $\{e_1, \dots, e_n\}$ at p such that the second fundamental form h satisfies

$$\begin{aligned} h(e_{\alpha_i}, e_{\beta_i}) &= \sum_{\gamma_i} h_{\alpha_i \beta_i}^{\gamma_i} J e_{\gamma_i} + \frac{3\delta_{\alpha_i \beta_i}}{2+n_i} \lambda J e_{N+1}, \quad \sum_{\alpha_i=1}^{n_i} h_{\alpha_i \alpha_i}^{\gamma_i} = 0, \\ h(e_{\alpha_i}, e_{\alpha_j}) &= 0, \quad i \neq j; \quad h(e_{\alpha_i}, e_{N+1}) = \frac{3\lambda}{2+n_i} J e_{\alpha_i}, \quad h(e_{\alpha_i}, e_u) = 0, \\ h(e_{N+1}, e_{N+1}) &= 3\lambda J e_{N+1}, \quad h(e_{N+1}, e_u) = \lambda J e_u, \quad N = n_1 + \dots + n_k, \\ h(e_u, e_v) &= \lambda \delta_{uv} J e_{N+1}, \quad i, j = 1, \dots, k; \quad u, v = N+2, \dots, n. \end{aligned} \quad (1.3)$$

For simplicity, we call a Lagrangian submanifold of a complex space form $\delta(n_1, \dots, n_k)$ -ideal (resp., *improved* $\delta(n_1, \dots, n_k)$ -ideal) if it satisfies the equality case of (1.1) (resp., the equality case of (1.2)) identically.

For $k = 2$ and $n_1 = n_2 = 2$, Theorem C reduces to the following.

Theorem D. *Let M be a Lagrangian submanifold in a complex space form $\tilde{M}^5(4c)$ of constant holomorphic sectional curvature $4c$. Then we have*

$$\delta(2, 2) \leq \frac{25}{4}H^2 + 8c. \quad (1.4)$$

If the equality sign of (1.4) holds identically, then with respect some suitable orthonormal frame $\{e_1, \dots, e_5\}$ the second fundamental form h satisfies

$$\begin{aligned} h(e_1, e_1) &= \alpha J e_1 + \beta J e_2 + \mu J e_5, & h(e_1, e_2) &= \beta J e_1 - \alpha J e_2, \\ h(e_2, e_2) &= -\alpha J e_1 - \beta J e_2 + \mu J e_5, \\ h(e_3, e_3) &= \gamma J e_3 + \delta J e_4 + \mu J e_5, & h(e_3, e_4) &= \delta J e_3 - \gamma J e_4, \\ h(e_4, e_4) &= -\gamma J e_3 - \delta J e_4 + \mu J e_5, & h(e_5, e_5) &= 4\mu J e_5, \\ h(e_i, e_5) &= \mu J e_i, \quad i \in \Delta; & h(e_i, e_j) &= 0, \quad \text{otherwise,} \end{aligned} \quad (1.5)$$

for some functions $\alpha, \beta, \gamma, \delta, \mu$, where $\Delta = \{1, 2, 3, 4\}$.

The classification of $\delta(2, 2)$ -ideal Lagrangian submanifolds in complex space forms $\tilde{M}^5(4c)$ is done in [13]. In this paper we classify improved $\delta(2, 2)$ -ideal Lagrangian submanifolds in $\tilde{M}^5(4c)$. The main results of this paper are stated as Theorem 6.1, Theorem 7.1 and Theorem 8.1.

2. PRELIMINARIES

2.1. Basic formulas. Let $\tilde{M}^n(4c)$ denote a complete simply-connected Kähler n -manifold with constant holomorphic sectional curvature $4c$. Then $\tilde{M}^n(4c)$ is holomorphically isometric to the complex Euclidean n -plane \mathbf{C}^n , the complex projective n -space $CP^n(4c)$, or a complex hyperbolic n -space $CH^n(-4c)$ according to $c = 0, c > 0$ or $c < 0$.

Let M be a Lagrangian submanifold of $\tilde{M}^n(4c)$. We denote the Levi-Civita connections of M and $\tilde{M}^n(4c)$ by ∇ and $\tilde{\nabla}$, respectively. The formulas of Gauss and Weingarten are given respectively by (cf. [7])

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \quad (2.1)$$

for tangent vector fields X and Y and normal vector fields ξ , where h is the second fundamental form, A is the shape operator and D is the normal connection.

The second fundamental form and the shape operator are related by

$$\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle.$$

The mean curvature vector \vec{H} of M is defined by $\vec{H} = \frac{1}{n} \text{trace } h$ and the *squared mean curvature* is given by $H^2 = \langle \vec{H}, \vec{H} \rangle$.

For Lagrangian submanifolds, we have (cf. [7, 12])

$$D_X JY = J\nabla_X Y, \quad (2.2)$$

$$A_{JX} Y = -Jh(X, Y) = A_{JY} X. \quad (2.3)$$

Formula (2.3) implies that $\langle h(X, Y), JZ \rangle$ is totally symmetric.

The equations of Gauss and Codazzi are given respectively by

$$\begin{aligned} \langle R(X, Y)Z, W \rangle &= \langle A_{h(Y, Z)} X, W \rangle - \langle A_{h(X, Z)} Y, W \rangle \\ &\quad + c(\langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle), \end{aligned} \quad (2.4)$$

$$(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z), \quad (2.5)$$

where R is the curvature tensor of M and ∇h is defined by

$$(\nabla_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \quad (2.6)$$

For an orthonormal basis $\{e_1, \dots, e_n\}$ of $T_p M$, we put

$$h_{jk}^i = \langle h(e_j, e_k), J e_i \rangle, \quad i, j, k = 1, \dots, n.$$

It follows from (2.3) that $h_{jk}^i = h_{ik}^j = h_{ij}^k$.

2.2. δ -invariants. Let M be a Riemannian n -manifold. Denote by $K(\pi)$ the sectional curvature of a plane section $\pi \subset T_p M$, $p \in M$. For any orthonormal basis e_1, \dots, e_n of $T_p M$, the scalar curvature τ at p is $\tau(p) = \sum_{i < j} K(e_i \wedge e_j)$.

Let L be a r -subspace of $T_p M$ with $r \geq 2$ and $\{e_1, \dots, e_r\}$ an orthonormal basis of L . The scalar curvature $\tau(L)$ of L is defined by

$$\tau(L) = \sum_{\alpha < \beta} K(e_\alpha \wedge e_\beta), \quad 1 \leq \alpha, \beta \leq r. \quad (2.7)$$

For given integers $n \geq 3$, $k \geq 1$, we denote by $\mathcal{S}(n, k)$ the finite set consisting of k -tuples (n_1, \dots, n_k) of integers satisfying $2 \leq n_1, \dots, n_k < n$ and $\sum_{j=1}^k n_j \leq n$.

Put $\mathcal{S}(n) = \cup_{k \geq 1} \mathcal{S}(n, k)$. For each k -tuple $(n_1, \dots, n_k) \in \mathcal{S}(n)$, the first author introduced in 1990s the Riemannian invariant $\delta(n_1, \dots, n_k)$ by

$$\delta(n_1, \dots, n_k)(p) = \tau(p) - \inf\{\tau(L_1) + \dots + \tau(L_k)\}, \quad p \in M, \quad (2.8)$$

where L_1, \dots, L_k run over all k mutually orthogonal subspaces of $T_p M$ such that $\dim L_j = n_j$, $j = 1, \dots, k$ (cf. [7] for details).

2.3. Horizontal lift of Lagrangian submanifolds. The following link between Legendrian submanifolds and Lagrangian submanifolds is due to [16] (see also [7, pp. 247–248]).

Case (i): $CP^n(4)$. Consider Hopf's fibration $\pi : S^{2n+1} \rightarrow CP^n(4)$. For a given point $u \in S^{2n+1}(1)$, the horizontal space at u is the orthogonal complement of ιu , $\iota = \sqrt{-1}$, with respect to the metric on S^{2n+1} induced from the metric on \mathbf{C}^{n+1} . Let $\iota : N \rightarrow CP^n(4)$ be a Lagrangian isometric immersion. Then there is a covering map $\tau : \hat{N} \rightarrow N$ and a horizontal immersion $\hat{\iota} : \hat{N} \rightarrow S^{2n+1}$ such that $\iota \circ \tau = \pi \circ \hat{\iota}$. Thus each Lagrangian immersion can be lifted locally (or globally if N is simply-connected) to a Legendrian immersion of the same Riemannian manifold. In particular, a minimal Lagrangian submanifold of $CP^n(4)$ is lifted to a minimal Legendrian submanifold of the Sasakian $S^{2n+1}(1)$.

Conversely, suppose that $f : \hat{N} \rightarrow S^{2n+1}$ is a Legendrian isometric immersion. Then $\iota = \pi \circ f : N \rightarrow CP^n(4)$ is again a Lagrangian isometric immersion. Under this correspondence the second fundamental forms h^f and h^ι of f and ι satisfy $\pi_* h^f = h^\iota$. Moreover, h^f is horizontal with respect to π .

Case (ii): $CH^n(-4)$. We consider the complex number space \mathbf{C}_1^{n+1} equipped with the pseudo-Euclidean metric: $g_0 = -dz_1 d\bar{z}_1 + \sum_{j=2}^{n+1} dz_j d\bar{z}_j$.

Consider $H_1^{2n+1}(-1) = \{z \in \mathbf{C}_1^{2n+1} : \langle z, z \rangle = -1\}$ with the canonical Sasakian structure, where $\langle \cdot, \cdot \rangle$ is the induced inner product.

Put $T'_z = \{u \in \mathbf{C}^{n+1} : \langle u, z \rangle = 0\}$, $H_1^1 = \{\lambda \in \mathbf{C} : \lambda \bar{\lambda} = 1\}$. Then there is an H_1^1 -action on $H_1^{2n+1}(-1)$, $z \mapsto \lambda z$ and at each point $z \in H_1^{2n+1}(-1)$, the vector $\xi = -iz$ is tangent to the flow of the action. Since the metric g_0 is Hermitian, we have $\langle \xi, \xi \rangle = -1$. The quotient space $H_1^{2n+1}(-1)/\sim$, under the identification induced from the action, is the complex hyperbolic space $CH^n(-4)$ with constant holomorphic sectional curvature -4 whose complex structure J is induced from the complex structure J on \mathbf{C}_1^{n+1} via Hopf's fibration $\pi : H_1^{2n+1}(-1) \rightarrow CH^n(4c)$.

Just like case (i), suppose that $\iota : N \rightarrow CH^n(-4)$ is a Lagrangian immersion, then there is an isometric covering map $\tau : \hat{N} \rightarrow N$ and a Legendrian immersion $f : \hat{N} \rightarrow H_1^{2n+1}(-1)$ such that $\iota \circ \tau = \pi \circ f$. Thus every Lagrangian immersion into $CH^n(-4)$ can be lifted locally (or globally if N is

simply-connected) to a Legendrian immersion into $H_1^{2n+1}(-1)$. In particular, Lagrangian minimal submanifolds of $CH^n(-4)$ are lifted to Legendrian minimal submanifolds of $H_1^{2n+1}(-1)$. Conversely, if $f : \hat{N} \rightarrow H_1^{2n+1}(-1)$ is a Legendrian immersion, then $\iota = \pi \circ f : N \rightarrow CH^n(-4)$ is a Lagrangian immersion. Under this correspondence the second fundamental forms h^f and h^ι are related by $\pi_* h^f = h^\iota$. Also, h^f is horizontal with respect to π .

Let h be the second fundamental form of M in $S^{2n+1}(1)$ (or in $H_1^{2n+1}(-1)$). Since $S^{2n+1}(1)$ and $H_1^{2n+1}(-1)$ are totally umbilical with one as its mean curvature in \mathbf{C}^{n+1} and in \mathbf{C}_1^{n+1} , respectively, we have

$$\hat{\nabla}_X Y = \nabla_X Y + h(X, Y) - \varepsilon L, \quad (2.9)$$

where $\varepsilon = 1$ if the ambient space is \mathbf{C}^{n+1} ; and $\varepsilon = -1$ if it is \mathbf{C}_1^{n+1} .

3. H -UMBILICAL LAGRANGIAN SUBMANIFOLDS AND COMPLEX EXTENSORS

3.1. H -umbilical Lagrangian submanifolds.

Definition 3.1. A non-totally geodesic Lagrangian submanifold of a Kähler n -manifold is called *H -umbilical* if its second fundamental form satisfies

$$\begin{aligned} h(e_j, e_j) &= \mu J e_n, \quad h(e_j, e_n) = \mu J e_j, \quad j = 1, \dots, n-1, \\ h(e_n, e_n) &= \varphi J e_n, \quad h(e_j, e_k) = 0, \quad 1 \leq j \neq k \leq n-1, \end{aligned} \quad (3.1)$$

for some functions μ, φ with respect to an orthonormal frame $\{e_1, \dots, e_n\}$. If the ratio of $\varphi : \mu$ is a constant r , the H -umbilical submanifold is said to be of ratio r .

If $G : N^{n-1} \rightarrow \mathbb{E}^n$ is a hypersurface of a Euclidean n -space \mathbb{E}^n and $\gamma : I \rightarrow \mathbf{C}^*$ is a unit speed curve in $\mathbf{C}^* = \mathbf{C} - \{0\}$, then we may extend $G : N^{n-1} \rightarrow \mathbb{E}^n$ to an immersion $I \times N^{n-1} \rightarrow \mathbf{C}^n$ by $\gamma \otimes G : I \times N^{n-1} \rightarrow \mathbf{C} \otimes \mathbb{E}^n = \mathbf{C}^n$, where $(\gamma \otimes G)(s, p) = F(s) \otimes G(p)$ for $s \in I$, $p \in N^{n-1}$. This extension of G via tensor product \otimes is called the *complex extensor* of G via the *generating curve* γ .

H -umbilical Lagrangian submanifolds in complex space forms were classified in a series of papers by the first author (cf. [2, 3, 4]). In particular, the following two results were proved in [2].

Theorem E. *Let $\iota : S^{n-1} \subset \mathbb{E}^n$ be the unit hypersphere in \mathbb{E}^n centered at the origin. Then every complex extensor of ι via a unit speed curve*

$\gamma : I \rightarrow \mathbf{C}^*$ is an H -umbilical Lagrangian submanifold of \mathbf{C}^n unless γ is contained in a line through the origin (which gives a totally geodesic Lagrangian submanifold).

Theorem F. *Let M be an H -umbilical Lagrangian submanifold of \mathbf{C}^n with $n \geq 3$. Then M is either a flat space or congruent to an open part of a complex extensor of $\iota : S^{n-1} \subset \mathbb{E}^n$ via a curve $\gamma : I \rightarrow \mathbf{C}^*$.*

3.2. Legendre curves. A unit speed curve $z : I \rightarrow S^3(1) \subset \mathbf{C}^2$ (resp., $z : I \rightarrow H_1^3(-1) \subset \mathbf{C}_1^2$) is called *Legendre* if $\langle z', iz \rangle = 0$. It was proved in [3] that a unit speed curve z in $S^3(1)$ (resp., in $H_1^3(-1)$) is Legendre if and only if it satisfies

$$z'' = i\lambda z' - z \quad (\text{resp., } z'' = i\lambda z' + z) \quad (3.2)$$

for a real-valued function λ . It is known in [3] that λ is the curvature function of z in $S^3(1)$ (resp., in $H_1^3(-1)$) (see also [1, Lemmas 3.1 and 3.2]).

3.3. H -umbilical submanifolds with arbitrary ratio. We provide a general method to construct H -umbilical Lagrangian submanifolds with any given ratio in $CP^n(4)$ via curves in $S^2(\frac{1}{2})$ (resp., in $CH^n(-4)$ via curves in $H^2(-\frac{1}{2})$).

Proposition 3.2. *For any real number r there exist H -umbilical Lagrangian submanifolds of ratio r in $CP^n(4)$ and in $CH^n(-4)$.*

Proof. If $r = 2$ this was done in [3, Theorems 5.1 and 6.1]. If $r \neq 2$, H -umbilical Lagrangian submanifolds of ratio r can be constructed as follows:

Case (a): $CP^n(4)$. Let $S^2(\frac{1}{2}) = \{\mathbf{x} \in \mathbb{E}^3; \langle \mathbf{x}, \mathbf{x} \rangle = \frac{1}{4}\}$. The Hopf fibration π from $S^3(1)$ onto $S^2(\frac{1}{2}) \cong CP^1(4)$ is given by (cf. [1])

$$\pi(z_1, z_2) = \left(z_1 \bar{z}_2, \frac{1}{2}(|z_1|^2 - |z_2|^2) \right), \quad (z_1, z_2) \in S^3(1) \subset \mathbf{C}^2. \quad (3.3)$$

For a Legendre curve z in $S^3(1)$, the projection $\gamma_z = \pi \circ z$ is a curve in $S^2(\frac{1}{2})$. Conversely, each curve γ in $S^2(\frac{1}{2})$ gives rise to a horizontal lift $\tilde{\gamma}$ in $S^3(1)$ via π which is unique up to a factor $e^{i\theta}$, $\theta \in \mathbf{R}$. Notice that each horizontal lift of γ is a Legendre curve in $S^3(1)$. Moreover, since the Hopf fibration is a Riemannian submersion, each unit speed Legendre curve z in $S^3(1)$ is projected to a unit speed curve γ_z in $S^2(\frac{1}{2})$ with the same curvature.

It was known in [3, Lemma 7.2] that, for a given H -umbilical Lagrangian submanifold of ratio $r \neq 2$ in $\tilde{M}^n(4c)$, the function μ in (3.1) satisfies

$$\mu\mu'' - \left(\frac{r-3}{r-2}\right)\mu'^2 + (r-2)\mu^2((r-1)\mu^2 + c) = 0. \quad (3.4)$$

If μ is a non-trivial solution of (3.4) with $c = 1$, then there is a unit speed curve γ in $S^2(\frac{1}{2})$ whose curvature equals to $r\mu$. Let z be a horizontal lift of γ in $S^3(1)$. Then z is a unit speed Legendre curve satisfying $z''(x) = ir\mu z'(x) - z(x)$ (cf. [3, Theorem 4.1] or [1, Lemma 3.1]).

Consider the map $\psi : M^5 \rightarrow S^{11}(1) \subset \mathbf{C}^6$ defined by

$$\psi(x, y_1, \dots, y_5) = (z_1(x), z_2(x)y_1, \dots, z_2(x)y_5), \sum_{j=1}^5 y_j^2 = 1). \quad (3.5)$$

It follows from [3, Theorem 4.1 and Lemma 7.2] that $\pi \circ \psi$ is a H -umbilical Lagrangian submanifold of ratio r in $CP^n(4)$ such that

$$\begin{aligned} h(e_j, e_j) &= \mu J e_5, & h(e_j, e_n) &= \mu J e_j, \\ h(e_n, e_n) &= r\mu J e_n, & h(e_j, e_k) &= 0, \quad 1 \leq j \neq k \leq n-1, \end{aligned} \quad (3.6)$$

with respect to suitable orthonormal frame $\{e_1, \dots, e_5\}$.

Case (b): $CH^n(-4)$. For a non-trivial solution of (3.4) with $c = -1$, we can construct an H -umbilical Lagrangian submanifold of $CH^n(-4)$ via the Hopf fibration $\pi : H_1^3(-1) \rightarrow CH^1(-4) \equiv H^2(-\frac{1}{2})$ in a similar way as case (a), where

$$\pi(z_1, z_2) = (z_1 \bar{z}_2, \frac{1}{2}(|z_1|^2 + |z_2|^2)), \quad (z_1, z_2) \in H_1^3(-1) \subset \mathbf{C}_1^2, \quad (3.7)$$

and $H^2(-\frac{1}{2}) = \{(x_1, x_2, x_3) \in \mathbb{E}_1^3 : x_1^2 - x_2^2 - x_3^2 = \frac{1}{4}, x_1 \geq \frac{1}{2}\}$ is the model of the real projective plane of curvature -4 . \square

3.4. Classification of H -umbilical submanifolds of ratio 4. The equation of Gauss and (3.1) imply that H -umbilical Lagrangian submanifolds of ratio $r \neq 4$ in complex space forms contain no open subsets of constant sectional curvature. Hence we conclude from [3, Theorems 4.1 and 7.1] and §3.3 the following results.

Lemma 3.3. *An H -umbilical Lagrangian submanifold M of ratio 4 in $CP^5(4)$ is congruent to an open portion of $\pi \circ \psi$, where $\pi : S^{11}(1) \rightarrow CP^5(4)$ is Hopf's fibration, $\psi : M \rightarrow S^{11}(1) \subset \mathbf{C}^6$ is given by*

$$\psi(t, y_1, \dots, y_5) = (z_1(t), z_2(t)\mathbf{y}), \quad \{\mathbf{y} \in \mathbb{E}^5 : \langle \mathbf{y}, \mathbf{y} \rangle = 1\}, \quad (3.8)$$

and $z = (z_1, z_2) : I \rightarrow S^3(1) \subset \mathbf{C}^2$ is a unit speed Legendre curve satisfying $z'' = 4i\mu z' - z$, and μ is a nonzero solution of $2\mu\mu'' - \mu'^2 + 4\mu^2(3\mu^2 + 1) = 0$.

Let M be an H -umbilical Lagrangian submanifold in $CH^5(-4)$ satisfying (3.1). We may assume that μ is defined on an open interval $I \ni 0$. Since H -umbilical submanifolds of ratio 4 in $CH^5(-4)$ contain no open subsets of constant curvature, Theorems 4.2 and 9.1 of [3] and results in §3.3 imply the following classification of H -umbilical submanifolds of ratio 4 in $CH^5(-4)$.

Lemma 3.4. *An H -umbilical Lagrangian submanifold M of ratio 4 in $CH^5(-4)$ is congruent to an open part of $\pi \circ \psi$, where $\pi : H_1^{11}(-1) \rightarrow CH^5(-4)$ is Hopf's fibration and $\psi : M \rightarrow H_1^{11}(-1) \subset \mathbf{C}_1^6$ is either one of*

$$\psi(t, y_1, \dots, y_4) = (z_1(t), z_2(t)\mathbf{y}), \quad \{\mathbf{y} \in \mathbb{E}^5 : \langle \mathbf{y}, \mathbf{y} \rangle = 1\}, \quad (3.9)$$

$$\psi(t, y_1, \dots, y_4) = (z_1(t)\mathbf{y}, z_2(t)), \quad \{\mathbf{y} \in \mathbb{E}_1^5 : \langle \mathbf{y}, \mathbf{y} \rangle = -1\}, \quad (3.10)$$

where z is a unit speed Legendre curve in $H_1^3(-1)$ satisfying $z'' = 4i\mu z' + z$ and μ is a non-trivial solution of $2\mu\mu'' - \mu'^2 + 4\mu^2(3\mu^2 - 1) = 0$; or ψ is

$$\begin{aligned} \psi(t, u_1, \dots, u_4) = \sqrt{\mu} e^{i \int_0^t \mu(t) dt} & \left(1 + \frac{1}{2} \sum_{j=1}^4 u_j^2 - it + \frac{1}{2\mu} - \frac{1}{2\mu(0)}, \right. \\ & \left. \left(i\mu(0) - \frac{\mu'(0)}{2\mu(0)} \right) \left(\frac{1}{2} \sum_{j=1}^4 u_j^2 - it + \frac{1}{2\mu} - \frac{1}{2\mu(0)} \right), u_1, \dots, u_4 \right), \end{aligned} \quad (3.11)$$

where $z = (z_1, z_2) : I \rightarrow H_1^3(-1) \subset \mathbf{C}_1^2$ is a unit speed Legendre curve and μ is a non-trivial solution of $\mu'^2 = 4\mu^2(1 - \mu^2)$.

Example. It is easy to verify that $\mu = \operatorname{sech} 2t$ is a non-trivial solution of $\mu'^2 = 4\mu^2(1 - \mu^2)$. Using $\mu = \operatorname{sech} 2t$, (3.11) reduces to

$$\begin{aligned} \psi(t, u_1, \dots, u_4) = \frac{e^{i \tan^{-1}(\tanh t)}}{\sqrt{\cosh 2t}} & \left(\frac{1}{2} - it + \frac{1}{2} \sum_{j=1}^4 u_j^2 + \frac{\cosh 2t}{2}, \right. \\ & \left. t - \frac{i}{2} + \frac{i}{2} \sum_{j=1}^4 u_j^2 + \frac{i \cosh 2t}{2}, u_1, \dots, u_4 \right). \end{aligned} \quad (3.12)$$

It is direct to verify that (3.12) satisfies $\langle \psi, \psi \rangle = -1$ and the composition $\pi \circ \psi$ gives rise to an H -umbilical Lagrangian submanifold of ratio 4 in $CH^5(-4)$.

4. SOME LEMMAS

We need the following lemmas for the proof of the main theorems.

Lemma 4.1. *Let M be an improved $\delta(2, 2)$ -ideal Lagrangian submanifold of $\tilde{M}^5(4c)$. Then with respect to some orthonormal frame $\{e_1, \dots, e_5\}$ we have*

$$\begin{aligned}
h(e_1, e_1) &= aJe_1 + \mu Je_5, \quad h(e_1, e_2) = -aJe_2, \\
h(e_2, e_2) &= -aJe_1 + \mu Je_5, \quad h(e_3, e_3) = bJe_3 + \mu Je_5, \\
h(e_3, e_4) &= -bJe_4, \quad h(e_4, e_4) = -bJe_3 + \mu Je_5, \\
h(e_i, e_5) &= \mu Je_i, \quad i \in \Delta, \quad h(e_5, e_5) = 4\mu Je_5, \\
h(e_i, e_j) &= 0, \quad \text{otherwise.}
\end{aligned} \tag{4.1}$$

Proof. Under the hypothesis, we have (1.5) with respect to an orthonormal frame $\{e_1, \dots, e_5\}$. Thus, after applying [6, Lemma 1] to $V = \text{Span}\{e_1, e_2\}$ and $V = \text{Span}\{e_3, e_4\}$, we obtain (4.1). \square

Let us put

$$\nabla_X e_i = \sum_{j=1}^5 \omega_i^j(X) e_j, \quad i = 1, \dots, 5, \quad X \in TM^5. \tag{4.2}$$

Then $\omega_i^j = -\omega_j^i$, $i, j = 1, \dots, 5$.

If $\mu = 0$, then M is a minimal Lagrangian submanifold according (4.1). Such submanifolds in complex space forms $\tilde{M}^5(4c)$ have been classified in [13].

If $a = b = 0$ and $\mu \neq 0$, then M is an H -umbilical Lagrangian submanifold with ratio 4. Therefore, from now on we assume that $a, \mu \neq 0$.

Lemma 4.2. *Let M be a Lagrangian submanifold of $\tilde{M}^5(4c)$ whose second fundamental form satisfies (4.1) with $a, b, \mu \neq 0$. Then we have*

$$\begin{aligned}
\nabla_{e_1} e_1 &= \frac{e_2 a}{3a} e_2 - \nu e_5, \quad \nabla_{e_1} e_2 = -\frac{e_2 a}{3a} e_1, \quad \nabla_{e_2} e_1 = -\frac{e_1 a}{3a} e_2, \\
\nabla_{e_2} e_2 &= \frac{e_1 a}{3a} e_1 - \nu e_5, \quad \nabla_{e_3} e_3 = \frac{e_4 b}{3b} e_4 - \nu e_5, \quad \nabla_{e_3} e_4 = -\frac{e_4 b}{3b} e_3, \\
\nabla_{e_4} e_3 &= -\frac{e_3 b}{3b} e_4, \quad \nabla_{e_4} e_4 = \frac{e_3 b}{3b} e_3 - \nu e_5, \quad \nabla_{e_i} e_5 = \nu e_i, \quad i \in \Delta, \\
\nabla_{e_k} e_j &= 0, \quad \text{otherwise,}
\end{aligned} \tag{4.3}$$

with $\nu = \frac{1}{2}e_5(\ln \mu) = -e_5(\ln a) = -e_5(\ln b)$, where $\Delta = \{1, 2, 3, 4\}$. Moreover, we have

$$e_j \mu = 0, j \in \Delta, \quad e_1 b = e_2 b = e_3 a = e_4 a = 0. \quad (4.4)$$

Proof. This lemma is obtained from Codazzi's equations via Lemma 4.1 and (4.2) and long computations. \square

Lemma 4.3. *Under the hypothesis of Lemma 4.2, we have*

- (a) T_0 is a totally geodesic distribution, i.e. T_0 is integrable whose leaves are totally geodesic submanifolds;
- (b) $T_0 \oplus T_1$ and $T_0 \oplus T_2$ are totally geodesic distributions;
- (c) T_1 and T_2 are spherical distributions, i.e. T_1, T_2 are integrable distributions whose leaves are totally umbilical submanifolds with parallel mean curvature vector,

where $T_0 = \text{Span}\{e_5\}$, $T_1 = \text{Span}\{e_1, e_2\}$ and $T_2 = \text{Span}\{e_3, e_4\}$.

Proof. Since the distribution T_0 is of rank one, it is integrable. Moreover, since $\nabla_{e_5} e_5 = 0$ by Lemma 4.2, the integral curves of e_5 are geodesics in M . Thus we have statement (a). Statement (b) follows easily from (4.3).

To prove statement (c), first we observe that $[e_1, e_2] \in T_1$ and $[e_3, e_4] \in T_2$ follow from (4.3). Thus T_1, T_2 are integrable. Also, it follows from (4.3) that the second fundamental form h_1 of a leaf \mathcal{L}_1 of T_1 in M is given by

$$h_1(X, Y) = -\nu g_1(X_1, Y_1)e_5, \quad X_1, Y_1 \in T\mathcal{L}_1, \quad (4.5)$$

where g_1 is the metric of \mathcal{L}_1 . From (4.3) we obtain $\nabla_{e_i} e_5 = \nu e_i$, $i = 1, 2$. Thus $D_{e_1}^1 e_5 = D_{e_2}^1 e_5 = 0$, where D^1 is the normal connection of \mathcal{L}_1 in M . It follows from Gauss' equation and Lemma 4.1 that the curvature tensor R satisfies

$$\langle R(e_1, e_2)e_1, e_j \rangle = 0, \quad j = 3, 4, 5. \quad (4.6)$$

Thus (4.6) and Lemma 4.2 imply that $0 \equiv R(e_1, e_2)e_1 \equiv (e_2 \nu)e_5 \pmod{T_1}$. Hence $e_2 \nu = 0$. Similarly, by considering $R(e_2, e_1)e_2$, we also have $e_1 \nu = 0$. After combining these with $D^1 e_5 = 0$, we conclude that \mathcal{L}_1 has parallel mean curvature vector in M . Hence T_1 is a spherical distribution. Similarly, T_2 is also a spherical distribution. Consequently, we obtain statement (c). \square

Lemma 4.4. *Under the hypothesis of Lemma 4.2, M is locally a warped product $I \times_{\rho_1(t)} M_1^2 \times_{\rho_2(t)} M_2^2$, where t is function such that $e_5 = \partial_t$ (i.e., $e_5 =$*

$\frac{\partial}{\partial t}$), ρ_1 and ρ_2 are two positive functions in t and M_1^2, M_2^2 are Riemannian 2-manifolds.

Proof. This lemma follows from Lemma 4.3 and a result of Hiepko [15] (see also [7, Theorem 4.4, p. 90]). \square

Lemma 3.3 and (4.4) imply that μ depends only on t . Thus $\mu = \mu(t)$.

Lemma 4.5. *Let M be a Lagrangian submanifold of $\tilde{M}^5(4c)$ whose second fundamental form satisfies (4.1) with $a, b, \mu \neq 0$. Then we have $c = -\nu^2 - \mu^2 < 0$. Thus μ satisfies $\mu'(t)^2 = -4\mu^2(t)(c + \mu^2(t))$.*

Proof. Under the hypothesis, it follows from Gauss' equation and Lemma 4.1 that $\langle R(e_1, e_3)e_3, e_1 \rangle = c + \mu^2$. On the other hand, the definition of curvature tensor and Lemma 4.2 imply that $\langle R(e_1, e_3)e_3, e_1 \rangle = -\nu^2$. Thus $c = -\nu^2 - \mu^2 < 0$. By combining this with the definition of ν , we obtain the lemma. \square

5. MORE LEMMAS

Next, we consider the case $a, \mu \neq 0$ and $b = 0$.

Lemma 5.1. *Let M be a Lagrangian submanifold of $\tilde{M}^5(4c)$ whose second fundamental form satisfies (4.1) with $a, \mu \neq 0$ and $b = 0$. Then we have*

$$\begin{aligned}
\nabla_{e_1}e_1 &= \frac{e_2a}{3a}e_2 + \frac{e_3a}{a}e_3 + \frac{e_4a}{3a}e_4 - \nu e_5, \\
\nabla_{e_1}e_2 &= -\frac{e_2a}{3a}e_1 - 3\omega_1^2(e_3)e_3 - 3\omega_1^2(e_4)e_4, \\
\nabla_{e_1}e_3 &= -\frac{e_3a}{a}e_1 + 3\omega_1^2(e_3)e_2 + \omega_3^4(e_1)e_4, \\
\nabla_{e_1}e_4 &= -\frac{e_4a}{a}e_1 + 3\omega_1^2(e_4)e_2 - \omega_3^4(e_1)e_3, \\
\nabla_{e_2}e_1 &= -\frac{e_1a}{3a}e_2 + 3\omega_1^2(e_3)e_3 + \omega_1^4(e_2)e_4, \\
\nabla_{e_2}e_2 &= \frac{e_1a}{3a}e_1 + \frac{e_3a}{a}e_3 + \frac{e_4a}{a}e_4 - \nu e_5, \\
\nabla_{e_2}e_3 &= -3\omega_1^2(e_3)e_1 - \frac{e_3a}{a}e_2 + \omega_3^4(e_2)e_4, \\
\nabla_{e_2}e_4 &= -\omega_1^4(e_2)e_1 - \frac{e_4a}{a}e_2 - \omega_3^4(e_2)e_3, \\
\nabla_{e_3}e_1 &= \omega_1^2(e_3)e_2, \quad \nabla_{e_3}e_2 = -\omega_1^2(e_3)e_1, \\
\nabla_{e_3}e_3 &= \omega_3^4(e_3)e_4 - \nu e_5, \quad \nabla_{e_3}e_4 = -\omega_3^4(e_3)e_3, \\
\nabla_{e_4}e_1 &= \omega_1^2(e_4)e_2, \quad \nabla_{e_4}e_2 = -\omega_1^2(e_4)e_1, \\
\nabla_{e_4}e_3 &= \omega_3^4(e_4)e_4, \quad \nabla_{e_4}e_4 = -\omega_3^4(e_4)e_3 - \nu e_5, \\
\nabla_{e_5}e_3 &= \omega_3^4(e_5)e_4, \quad \nabla_{e_5}e_4 = -\omega_3^4(e_5)e_5, \\
\nabla_{e_i}e_5 &= \nu e_i, \quad i \in \Delta, \quad \nabla_{e_k}e_j = 0, \quad \text{otherwise.}
\end{aligned} \tag{5.1}$$

with $\nu = \frac{1}{2}e_5(\ln \mu) = -e_5(\ln a)$. Moreover, we have

$$e_j\mu = 0, \quad j \in \Delta = \{1, 2, 3, 4\}. \tag{5.2}$$

Proof. Follows from Codazzi's equations via Lemma 4.1 and (4.2). \square

Lemma 5.2. *Under the hypothesis of Lemma 5.1, we have*

- (i) T_0 is a totally geodesic distribution;
- (ii) T_3 is a spherical distribution,

where $T_0 = \text{Span}\{e_5\}$ and $T_3 = \text{Span}\{e_1, e_2, e_3, e_4\}$.

Proof. Clearly, T_0 is integrable. Moreover, since $\nabla_{e_5}e_5 = 0$ by Lemma 5.1, integral curves of e_5 are geodesics in M^5 . Thus statement (i) follows. To prove statement (ii), we observe that the integrability of T_3 follows from (5.1). Also, (5.1) implies that the second fundamental form \hat{h} of a leaf \mathcal{L} of T_3 in M^5 is given by $\hat{h}(X, Y) = -\nu\hat{g}(X, Y)e_5$ for $X, Y \in T\mathcal{L}$, where \hat{g} is the

metric of \mathcal{L} . Since $[e_j, e_5]\mu = 0$ by (5.1) and $e_j\mu = 0$, for $j \in \Delta$, we find $e_i e_5 \mu - e_5 e_i \mu = 2e_1 \nu = 0$. Therefore T_3 is a spherical distribution. \square

Lemma 5.3. *Under the hypothesis of Lemma 5.1, M is locally a warped product $I \times_{\rho(t)} N^4$, where t is function such that $e_5 = \frac{\partial}{\partial t}$ and ρ is a positive function in t and N^4 is a Riemannian 4-manifold.*

Proof. Follows from Lemma 5.2 and Hiepko's theorem. \square

It follows from (5.2) and the definition of ν that $\mu = \mu(t)$ and $\nu = \nu(t)$.

Lemma 5.4. *Under the hypothesis of Lemma 5.1, we have*

$$\frac{d\nu}{dt} = -3\mu^2 - \nu^2 - c, \quad \frac{d\mu}{dt} = 2\mu\nu. \quad (5.3)$$

Proof. From Gauss' equation and (5.1) we find $\langle R(e_1, e_5)e_5, e_1 \rangle = 3\mu^2 + c$. On the other hand, (5.1) of Lemma 5.1 yields $\langle R(e_1, e_5)e_5, e_1 \rangle = -e_5 \nu - \nu^2$. Thus we find the first equation of (5.3). The second one follows immediately from the definition of ν given in Lemma 5.1. \square

6. IMPROVED $\delta(2, 2)$ -IDEAL LAGRANGIAN SUBMANIFOLDS OF \mathbf{C}^5

Theorem 6.1. *Let M be an improved $\delta(2, 2)$ -ideal Lagrangian submanifold in \mathbf{C}^5 . Then it is one of the following Lagrangian submanifolds:*

- (a) a $\delta(2, 2)$ -ideal Lagrangian minimal submanifold;
- (b) an H -umbilical Lagrangian submanifold of ratio 4;
- (c) a Lagrangian submanifold defined by

$$L(\mu, u_2, \dots, u_n) = \frac{e^{\frac{4}{3}i \tan^{-1} \sqrt{\mu^3/(c^2 - \mu^3)}}}{\sqrt{c^2 \mu^{-1} - \mu^2 + i\mu}} \phi(u_2, \dots, u_n), \quad (6.1)$$

where c is a positive real number and $\phi(u_2, \dots, u_n)$ is a horizontal lift of a non-totally geodesic $\delta(2)$ -ideal Lagrangian minimal immersion in $CP^4(4)$.

Proof. Assume that M is an improved $\delta(2, 2)$ -ideal Lagrangian submanifold in \mathbf{C}^5 . Then there exists an orthonormal frame $\{e_1, \dots, e_5\}$ such that (4.1) holds. If $\mu = 0$, then M is a minimal $\delta(2, 2)$ -ideal Lagrangian submanifold. Thus, we obtain case (a). If $\mu \neq 0$ and $a = b = 0$, we obtain case (b).

Now, let us assume $a, \mu \neq 0$. Then Lemma 4.5 implies $b = 0$. So, by Lemmas 5.1 we have (5.1) and $e_j \mu = 0$, $j \in \Delta$. Further, by Lemma 5.3, M

is locally a warped product $I \times_{\rho(t)} N^4$ with $e_5 = \partial_t$. Moreover, 4.1 shows that the second fundamental form satisfies

$$\begin{aligned} h(e_1, e_1) &= aJe_1 + \mu Je_5, \quad h(e_1, e_2) = -aJe_2, \\ h(e_2, e_2) &= -aJe_1 + \mu Je_5, \\ h(e_3, e_3) &= h(e_4, e_4) = \mu Je_5, \\ h(e_i, e_5) &= \mu Je_i, \quad i \in \Delta, \\ h(e_5, e_5) &= 4\mu Je_5, \quad h(e_i, e_j) = 0, \quad \textit{otherwise}. \end{aligned} \tag{6.2}$$

From Lemma 5.4 we have the following differential system:

$$\frac{d\nu}{dt} = -3\mu^2 - \nu^2, \quad \frac{d\mu}{dt} = 2\mu\nu. \tag{6.3}$$

Let $\varphi(t)$ be a function satisfying $\frac{d\varphi}{dt} = -4\mu$. Consider the map

$$\phi = e^{i\varphi} e_5. \tag{6.4}$$

Then $\langle \phi, \phi \rangle = 1$. It follows from $\nabla_{e_5} e_5 = 0$, $\frac{d\varphi}{dt} = -4\mu$ and (6.2) that $\tilde{\nabla}_{e_5} \phi = 0$, where $\tilde{\nabla}$ is the Levi-Civita connection of \mathbf{C}^5 . Thus ϕ is independent of t .

Let L denote the Lagrangian immersion of M in \mathbf{C}^5 . Then (6.4) yields

$$e_5 = L_t = e^{-i\varphi} \phi(u_1, \dots, u_4), \tag{6.5}$$

where u_1, \dots, u_4 are local coordinates of N^4 . For each $j \in \Delta$, we obtain from $\nabla_{e_j} e_5 = \nu e_j$ of Lemma 5.1 and the first equation of (6.3) that

$$\phi_*(e_j) = \tilde{\nabla}_{e_j} \phi = e^{i\varphi} \tilde{\nabla}_{e_j} e_5 = e^{i\varphi} (\nu + i\mu) e_j. \tag{6.6}$$

Thus

$$\tilde{\nabla}_{e_j} (\phi_*(e_i)) = e^{i\varphi} (\nu + i\mu) \tilde{\nabla}_{e_j} e_i. \tag{6.7}$$

In view of $\nabla_{e_j} e_5 = \nu e_j$ and (6.2), we may put

$$\tilde{\nabla}_{e_i} e_j = \left(\sum_{k=1}^4 \Gamma_{ij}^k + ih_{ij}^k \right) e_k - (\nu - i\mu) \delta_{ij} e_5, \quad i, j \in \Delta, \tag{6.8}$$

for some functions Γ_{ij}^k . Now, it follows from (6.4), (6.6), (6.7), and (6.8) that

$$\begin{aligned} \tilde{\nabla}_{e_j} (\phi_*(e_i)) &= \sum_{\gamma=2}^n \left(\Gamma_{ij}^k + ih_{ij}^k \right) \phi_*(e_k) - (\mu^2 + \nu^2) \delta_{ij} \phi \\ &= \sum_{\gamma=2}^n \left(\Gamma_{ij}^k + ih_{ij}^k \right) \phi_*(e_k) - \langle \phi_*(e_i), \phi_*(e_j) \rangle \phi. \end{aligned} \tag{6.9}$$

Since M is a Lagrangian submanifold in \mathbf{C}^5 , (6.4) and (6.6) show that $i\phi$ is perpendicular to each tangent space of M . Hence ϕ is a horizontal immersion in the unit hypersphere $S^9(1) \subset \mathbf{C}^5$. Moreover, it follows from (6.9) that the second fundamental form of ϕ is the original second fundamental form of M respect to to the second factor N^4 of the warped product $I \times_{\rho(t)} N^4$. Hence, ϕ is a minimal horizontal immersion in $S^9(1)$. Therefore, ϕ is a horizontal lift of a minimal Lagrangian immersion in $CP^4(4)$. Now, it follows from (6.2) that ϕ is a horizontal lift of a $\delta(2)$ -ideal minimal Lagrangian submanifold of $CP^4(4)$.

By direct computation we find

$$\tilde{\nabla}_{e_\alpha} \left(L - \frac{e_5}{\nu + i\mu} \right) = 0, \quad \alpha = 1, \dots, 5. \quad (6.10)$$

Thus, by (6.4), up to translations the Lagrangian immersion L is

$$L = \frac{e^{-i\varphi}}{\nu + i\mu} \phi(u_1, \dots, u_4), \quad (6.11)$$

where ϕ is a horizontal minimal immersion in $S^9(1)$ and ν, φ, μ satisfy

$$\frac{d\nu}{dt} = -3\mu^2 - \nu^2, \quad \frac{d\varphi}{dt} = -4\mu, \quad \frac{d\mu}{dt} = 2\mu\nu. \quad (6.12)$$

From (6.12) we find

$$\frac{d\nu}{d\mu} + \frac{\nu}{2\mu} = -\frac{3\mu}{2\nu}. \quad (6.13)$$

After solving (6.13) we get $\nu = \pm \sqrt{c^2\mu^{-1} - \mu^2}$ for some real number $c > 0$. Replacing e_5 by $-e_5$ if necessary, we have

$$\nu = \sqrt{c^2\mu^{-1} - \mu^2}. \quad (6.14)$$

It follows from (6.12) and (6.14) that $\varphi'(\mu) = -2/\sqrt{c^2\mu^{-1} - \mu^2}$. By solving the last equation we find $\varphi = -\frac{4}{3}i \tan^{-1} \sqrt{\mu^3/(c^2 - \mu^3)} + c_0$ for some constant c_0 . Therefore, we have the theorem after applying a suitable translation in μ . \square

Remark 6.2. Minimal $\delta(2, 2)$ -ideal Lagrangian submanifolds in complex space forms \mathbf{C}^5 , CP^5 and CH^5 are classified in [13]. Also $\delta(2)$ -ideal minimal Lagrangian submanifolds in CP^4 and CH^4 have been classified recently in [14].

Let $\gamma(t)$ be a unit speed curve in \mathbf{C}^* . We put

$$\gamma(t) = r(t)e^{i\theta(t)}, \quad \gamma'(t) = e^{i\zeta(t)}. \quad (6.15)$$

The following result gives H -umbilical submanifolds of \mathbf{C}^5 with ratio 4.

Proposition 6.3. *If M is an H -umbilical Lagrangian submanifold of \mathbf{C}^5 of ratio 4, then M is an open part of a complex extensor $\gamma \otimes \iota$ of the unit hypersphere $\iota : S^4(1) \subset \mathbb{E}^5$ via a generating curve $\gamma : I \rightarrow \mathbf{C}^*$ whose curvature satisfies $\kappa = 4\theta'$.*

Proof. If M is an H -umbilical Lagrangian submanifold of \mathbf{C}^5 with ratio 4, then the second fundamental form satisfies

$$\begin{aligned} h(e_j, e_j) &= \mu J e_5, \quad h(e_j, e_5) = \mu J e_j, \quad j \in \Delta, \\ h(e_5, e_5) &= 4\mu J e_5, \quad h(e_j, e_k) = 0, \quad 1 \leq j \neq k \leq 4, \end{aligned}$$

for a nonzero function μ . Thus Gauss' equation yields $K(e_1 \wedge e_5) = 3\mu^2$. Hence M is non-flat. Therefore, according to Theorem F, M is an open part of a complex extensor of $\iota : S^{n-1}(1) \subset \mathbb{E}^n$ via a generating curve $\gamma : I \rightarrow \mathbf{C}^*$. It follows from [2] that the functions φ and μ in (4.1) are related with the two angle functions ζ and θ by $\varphi = \zeta'(t) = \kappa$ and $\mu = \theta'(t)$. Thus whenever γ is a unit speed curve satisfying $\kappa = 4\theta'$, the complex extensor $\gamma \otimes \iota$ is an H -umbilical Lagrangian submanifold of ratio 4. Conversely, every H -umbilical Lagrangian submanifold of ratio 4 in \mathbf{C}^n can be obtained in such way. \square

7. IMPROVED $\delta(2, 2)$ -IDEAL LAGRANGIAN SUBMANIFOLDS OF CP^5

Theorem 7.1. *Let M be an improved $\delta(2, 2)$ -ideal Lagrangian submanifold in $CP^5(4)$. Then it is one of the following Lagrangian submanifolds:*

- (1) a $\delta(2, 2)$ -ideal Lagrangian minimal submanifold;
- (2) an H -umbilical Lagrangian submanifold of ratio 4;
- (3) a Lagrangian submanifold defined by

$$L(\mu, u_2, \dots, u_4) = \frac{1}{c} \left(\sqrt{\mu} e^{i\theta} \phi, e^{3i\theta} (\sqrt{c^2 - \mu^3 - \mu} - i\mu^{\frac{3}{2}}) \right), \quad (7.1)$$

where c is a positive real number, $\phi : N^4 \rightarrow S^9(1) \subset \mathbf{C}^5$ is a horizontal lift of a non-totally geodesic $\delta(2)$ -ideal Lagrangian minimal immersion in $CP^4(4)$, and $\theta(\mu)$ satisfies

$$\frac{d\theta}{d\mu} = \frac{1}{2\sqrt{c^2\mu^{-1} - \mu^2 - 1}}. \quad (7.2)$$

Proof. Under the hypothesis there is an orthonormal frame $\{e_1, \dots, e_5\}$ such that (4.1) holds. If $\mu = 0$, then M is a $\delta(2, 2)$ -ideal Lagrangian minimal submanifold. Thus we obtain case (1). If $\mu \neq 0$ and $a, b = 0$, then M is an H -umbilical Lagrangian submanifold of ratio 4, which gives case (2).

Next, assume that $a, \mu \neq 0$. Then Lemma 4.5 implies $b = 0$. So, by Lemmas 5.1 we obtain (5.1) and (5.2). Also, in this case M is locally a warped product $I \times_{\rho(t)} N^4$ with $e_5 = \partial_t$ according to Lemma 5.3. From Lemma 4.1, we find

$$\begin{aligned} h(e_1, e_1) &= aJe_1 + \mu Je_5, & h(e_1, e_2) &= -aJe_2, \\ h(e_2, e_2) &= -aJe_1 + \mu Je_5, & & \\ h(e_3, e_3) &= h(e_4, e_4) = \mu Je_5, & h(e_5, e_5) &= 4\mu Je_5, \\ h(e_i, e_5) &= \mu Je_i, \quad i \in \Delta, & h(e_i, e_j) &= 0, \text{ otherwise.} \end{aligned} \tag{7.3}$$

By Lemma 5.4 we have the following ODE system:

$$\frac{d\nu}{dt} = -1 - \nu^2 - 3\mu^2, \quad \frac{d\mu}{dt} = 2\mu\nu. \tag{7.4}$$

Let $\theta(t)$ be a function on M satisfying

$$\theta'(t) = \mu. \tag{7.5}$$

Let L denote the horizontal lift in $S^{11}(1) \subset \mathbf{C}^6$ of the Lagrangian immersion of M in $CP^5(4)$ via Hopf's fibration. Consider the maps:

$$\xi = \frac{e^{-3i\theta}(e_5 - (\nu + i\mu)L)}{\sqrt{1 + \mu^2 + \nu^2}}, \quad \phi = \frac{e^{-i\theta}(L + (\nu - i\mu)e_5)}{\sqrt{1 + \mu^2 + \nu^2}}. \tag{7.6}$$

Then $\langle \xi, \xi \rangle = \langle \phi, \phi \rangle = 1$. From $\nabla_{e_j} e_5 = \nu e_j$, $j \in \Delta$, and (7.4), we find $\tilde{\nabla}_{e_j} \xi = 0$. Moreover, it follows from Lemma 5.1 and (7.3) that $\tilde{\nabla}_{e_5} e_5 = 4i\mu e_5 - L$. Thus we also have $\tilde{\nabla}_{e_5} \xi = 0$. Hence ξ is a constant unit vector in \mathbf{C}^6 . Similarly, we also have $\tilde{\nabla}_{e_5} \phi = 0$. So ϕ is independent of t . Therefore, by combining (7.6) we find

$$L = \frac{e^{i\theta}\phi - e^{3i\theta}(\nu - i\mu)\xi}{\sqrt{1 + \mu^2 + \nu^2}}. \tag{7.7}$$

Since ϕ is orthogonal to ξ , $i\xi$, after choosing $\xi = (0, \dots, 0, 1) \in \mathbf{C}^6$ we obtain

$$L = \frac{1}{\sqrt{1 + \mu^2 + \nu^2}} \left(e^{i\theta}\phi, e^{3i\theta}(\nu - i\mu) \right) \tag{7.8}$$

It follows from (7.4) and (7.5) that

$$\frac{d\nu}{d\mu} = -\frac{1 + \nu^2 + 3\mu^2}{2\mu\nu}, \quad \frac{d\theta}{d\mu} = \frac{1}{2\nu}. \quad (7.9)$$

Solving the first differential equation in (7.9) gives

$$\nu = \pm \sqrt{c^2\mu^{-1} - \mu^2 - 1}, \quad c \in \mathbf{R}^+. \quad (7.10)$$

By replacing e_5 by $-e_5$ if necessary, we have $\nu = \sqrt{c^2\mu^{-1} - \mu^2 - 1}$. Consequently,

$$L = \frac{1}{c} \left(\sqrt{\mu} e^{i\theta} \phi, e^{3i\theta} (\sqrt{c^2 - \mu^3 - \mu} - i\mu^{\frac{3}{2}}) \right), \quad (7.11)$$

It follows from (5.1), (7.3) and the second formula in (7.6) that

$$\hat{\nabla}_{e_j} \phi = \frac{ce^{-i\theta}}{\sqrt{\mu}} e_j, \quad j \in \Delta. \quad (7.12)$$

Thus after applying (6.11) and (7.12) we derive that

$$\hat{\nabla}_{e_\beta} \hat{\nabla}_{e_\alpha} \phi = \sum_{\gamma=2}^n \left(\Gamma_{ij}^k + ih_{ij}^k \right) \phi_*(e_k) - \langle \phi_*(e_i), \phi_*(e_j) \rangle \phi, \quad i, j \in \Delta. \quad (7.13)$$

Hence ϕ is a horizontal immersion in $S^9(1)$. Moreover, it follows from (7.13) that the second fundamental form of ϕ is a scalar multiple of the original second fundamental form of M restricted to the second factor of the warped product $I \times_\rho N$. Consequently, ϕ is a minimal horizontal immersion in $S^9(1)$ of a non-totally geodesic $\delta(2)$ -ideal Lagrangian minimal submanifold of $CP^4(4)$.

The converse is easy to verify. \square

8. IMPROVED $\delta(2, 2)$ -IDEAL LAGRANGIAN SUBMANIFOLDS OF CH^5

Theorem 8.1. *Let M be an improved $\delta(2, 2)$ -ideal Lagrangian submanifold in $CH^5(-4)$. Then M is one of the following Lagrangian submanifolds:*

- (i) a $\delta(2, 2)$ -ideal Lagrangian minimal submanifold;
- (ii) an H -umbilical Lagrangian submanifold of ratio 4;
- (iii) a Lagrangian submanifold defined by

$$L(\mu, u_1, \dots, u_4) = \frac{1}{c} \left(\sqrt{\mu} e^{i\theta} \phi(u_2, \dots, u_4), e^{-i\theta} (\sqrt{\mu - \mu^3 - c^2} - i\mu^{\frac{3}{2}}) \right), \quad (8.1)$$

where c is a positive number, $\phi : N^4 \rightarrow H_1^9(-1) \subset \mathbf{C}_1^5$ is a horizontal lift of a non-totally geodesic $\delta(2)$ -ideal minimal Lagrangian immersion in $CH^4(-4)$, and $\theta(t)$ satisfies $\frac{d\theta}{d\mu} = \frac{1}{2}\sqrt{1 - \mu^2 - c^2\mu^{-1}}$;

(iv) a Lagrangian submanifold defined by

$$L(\mu, u_1, \dots, u_4) = \frac{1}{c} \left(e^{-i\theta}(\sqrt{\mu - \mu^3 + c^2} - i\mu^{\frac{3}{2}}), \sqrt{\mu}e^{i\theta}\phi(u_2, \dots, u_4) \right), \quad (8.2)$$

where c is a positive number, $\phi : N^4 \rightarrow S^9(1) \subset \mathbf{C}^5$ is a horizontal lift of a non-totally geodesic $\delta(2)$ -ideal minimal Lagrangian immersion in $CP^4(4)$, and $\theta(t)$ satisfies $\frac{d\theta}{d\mu} = \frac{1}{2}\sqrt{1 - \mu^2 + c^2\mu^{-1}}$;

(v) a Lagrangian submanifold defined by

$$L(t, u_1, \dots, u_4) = \frac{1}{\cosh t - i \sinh t} \left(2t + w + i \left(\cosh 2t - \langle \psi, \psi \rangle - \frac{1}{4} \right), \right. \\ \left. \psi, 2t + w + i \left(\cosh 2t - \langle \psi, \psi \rangle + \frac{1}{4} \right) \right), \quad (8.3)$$

where $\psi(u_1, \dots, u_4)$ is a non-totally geodesic $\delta(2)$ -ideal Lagrangian minimal immersion in \mathbf{C}^4 and up to a constant $w(u_1, \dots, u_4)$ is the unique solution of the PDE system: $w_{u_j} = 2 \langle \psi_{u_j}, i\psi \rangle$, $j = 1, 2, 3, 4$;

(vi) a Lagrangian submanifold defined by

$$L(t, u_1, \dots, u_4) = \frac{1}{\cosh t - i \sinh t} \left(2t + w + i \left(\cosh 2t - \langle \psi, \psi \rangle - \frac{1}{4} \right), \right. \\ \left. \psi_1, \psi_2, 2t + w + i \left(\cosh 2t - \langle \psi, \psi \rangle + \frac{1}{4} \right) \right), \quad (8.4)$$

where $\psi = (\psi_1, \psi_2)$ is the direct product immersion of two non-totally geodesic Lagrangian minimal immersions $\psi_\alpha : N_\alpha^2 \rightarrow \mathbf{C}^2$, $\alpha = 1, 2$, and up to a constant $w(u_1, \dots, u_4)$ is the unique solution of the PDE system: $w_{u_j} = 2 \langle \psi_{u_j}, i\psi \rangle$, $j = 1, 2, 3, 4$.

Proof. Under the hypothesis there exists an orthonormal frame $\{e_1, \dots, e_5\}$ such that (4.1) holds.

Case (1) $\mu = 0$. In this case, we obtain case (i) of the theorem.

Case (2): $\mu \neq 0$ and $a, b = 0$. In this case M is an H -umbilical Lagrangian submanifold with ratio 4, which gives case (ii).

Case (3): $\mu \neq 0$ and at least one of a, b is nonzero. Without loss of generality, we may assume $a \neq 0$ and $\mu > 0$. We divide this into two cases.

Case (3.a): $a, \mu \neq 0$ and $b = 0$. By Lemmas 5.1 we obtain (5.1) and (5.2). Also, M is locally a warped product $I \times_{\rho(t)} N^4$ with $e_5 = \partial_t$ according to Lemma 5.3. From Lemma 4.1 we find

$$\begin{aligned} h(e_1, e_1) &= aJe_1 + \mu Je_5, & h(e_1, e_2) &= -aJe_2, \\ h(e_2, e_2) &= -aJe_1 + \mu Je_5, \\ h(e_3, e_3) &= h(e_4, e_4) = \mu Je_5, & h(e_5, e_5) &= 4\mu Je_5, \\ h(e_i, e_5) &= \mu Je_i, \quad i \in \Delta, & h(e_i, e_j) &= 0, \text{ otherwise.} \end{aligned} \tag{8.5}$$

Let L be a horizontal immersion of M in $H_1^{11}(-1) \subset \mathbf{C}_1^6$ of the Lagrangian immersion of M in $CH^5(-4)$ via Hopf's fibration and $\theta(t)$ a function satisfying

$$\frac{d\theta}{dt} = \mu. \tag{8.6}$$

From Lemma 5.4 we obtain the following ODE system:

$$\frac{d\nu}{dt} = 1 - 3\mu^2 - \nu^2, \quad \frac{d\mu}{dt} = 2\mu\nu. \tag{8.7}$$

It follows from (8.6) and (8.7) that

$$\frac{d\nu}{d\mu} = \frac{1 - 3\mu^2 - \nu^2}{2\mu\nu}, \quad \frac{d\theta}{d\mu} = \frac{1}{2\nu}. \tag{8.8}$$

Solving the first differential equation in (8.8) gives $\nu = \pm\sqrt{1 - \mu^2 - k\mu^{-1}}$ for some real number k . By replacing e_5 by $-e_5$ if necessary, we find

$$\nu = \sqrt{1 - \mu^2 - k\mu^{-1}}, \quad \frac{d\theta}{d\mu} = \frac{1}{2\sqrt{1 - \mu^2 - k\mu^{-1}}}. \tag{8.9}$$

It follows from (8.7) that $\frac{d}{dt}(1 - \mu^2 - \nu^2) = -2\nu(1 - \mu^2 - \nu^2)$. Since this equation for $y(t) = 1 - \mu^2 - \nu^2 = k\mu^{-1}$ has a unique solution for each given initial condition, each solution either vanishes identically or is nowhere zero.

Case (3.a.1): $\mu^2 + \nu^2 < 1$. In this case, (8.9) implies $k > 0$. Thus we may put $k = c^2$, $c > 0$. Consider the maps:

$$\eta = \frac{e^{-3i\theta}(e_5 - (\nu + i\mu)L)}{\sqrt{1 - \mu^2 - \nu^2}}, \quad \phi = \frac{e^{-i\theta}((\nu - i\mu)e_5 - L)}{\sqrt{1 - \mu^2 - \nu^2}}. \tag{8.10}$$

Then $\langle \eta, \eta \rangle = 1$ and $\langle \phi, \phi \rangle = -1$. From $\nabla_{e_j} e_5 = \nu e_j$, $j \in \Delta$, and (8.5), we obtain $\tilde{\nabla}_{e_j} \xi = 0$, where $\tilde{\nabla}$ is the Levi-Civita connection of \mathbf{C}_1^6 . Lemma 5.1 and (8.5) give $\tilde{\nabla}_{e_5} e_5 = 4i\mu e_5 + L$. Thus we find $\tilde{\nabla}_{e_5} \xi = 0$. So η is a constant

unit vector. Also, we find $\tilde{\nabla}_{e_5}\phi = 0$. Hence ϕ is independent of t . From (8.10) we get

$$L = -\frac{e^{i\theta}\phi + e^{-i\theta}(\nu - i\mu)\eta}{\sqrt{1 - \mu^2 - \nu^2}}. \quad (8.11)$$

Since ϕ is orthogonal to $\eta, i\eta$ and η is a constant unit space-like vector, we conclude from (8.9) and (8.11) that L is congruent to (8.1). Next, by applying the same method of the proof of Theorem 7.1, we conclude that ϕ is a horizontal immersion in $H_1^9(-1)$ whose second fundamental form is a scalar multiple of the original second fundamental form restricted to the second factor of $I \times_\rho N$. Consequently, ϕ is a minimal horizontal immersion in $H_1^9(-1)$ of a non-totally geodesic $\delta(2)$ -ideal Lagrangian minimal submanifold of $CH^4(-4)$. This gives case (iii).

Case (3.a.2): $\mu^2 + \nu^2 > 1$. In this case (8.8) implies $k < 0$. Thus we may put $k = -c^2$, $c > 0$. Now, we consider the maps:

$$\eta = \frac{e^{-3i\theta}(e_5 - (\nu + i\mu)L)}{\sqrt{\mu^2 + \nu^2 - 1}}, \quad \phi = \frac{e^{-i\theta}((\nu - i\mu)e_5 - L)}{\sqrt{\mu^2 + \nu^2 - 1}} \quad (8.12)$$

instead. Then $\langle \phi, \phi \rangle = -\langle \eta, \eta \rangle = 1$. By applying similar arguments as case (3.a.1), we know that η is a constant time-like vector and ϕ is independent of t and orthogonal to $\eta, i\eta$. Moreover, we may prove that ϕ is a minimal Legendre immersion in $S^9(1)$. Therefore we have case (iv) after choosing $\eta = (1, 0, \dots, 0)$.

Case (3.a.3): $\mu^2 + \nu^2 = 1$. In this case system (8.7) gives $\frac{d\nu}{dt} = 2(\nu^2 - 1)$ and $\mu = \pm\sqrt{1 - \nu^2}$. Solving these and applying a suitable translations in t , we find

$$\mu = \operatorname{sech} 2t, \quad \nu = -\tanh 2t. \quad (8.13)$$

It follows from $\nabla_{e_5}e_5 = 0$, (8.5) and (8.13) that the horizontal lift L of the Lagrangian immersion of M in $CH^5(-4) \subset \mathbf{C}_1^6$ satisfies

$$L_{tt} - 4i(\operatorname{sech} 2t)L_t - L = 0. \quad (8.14)$$

Solving this second order differential equation gives

$$L = \frac{\phi(u_1, \dots, u_4) + B(u_1, \dots, u_4)(2t + i \cosh 2t)}{\cosh t - i \sinh t}, \quad (8.15)$$

where $\phi(u_1, \dots, u_4)$ and $B(u_1, \dots, u_4)$ are \mathbf{C}_1^6 -valued functions.

On the other hand, it follows from Lemma 5.1, (8.5) and (8.13) that

$$L_{tu_j} = (i \operatorname{sech} 2t - \tanh 2t)L_{u_j}, \quad j \in \Delta. \quad (8.16)$$

Substituting (8.15) into (8.16) shows that B is a constant vector ζ . Thus

$$L(t, u_1, \dots, u_4) = \frac{\phi(u_1, \dots, u_4)}{\cosh t - i \sinh t} + \frac{(2t + i \cosh 2t)}{\cosh t - i \sinh t} \zeta, \quad (8.17)$$

Since $\langle L, L \rangle = -1$, (8.17) implies

$$-\cosh 2t = \langle \phi, \phi \rangle + \langle \phi, (4t + 2i \cosh 2t)\zeta \rangle + (4t^2 + \cosh^2(2t)) \langle \zeta, \zeta \rangle. \quad (8.18)$$

Since $\phi_t = 0$, by differentiating (8.18) with respect t we find

$$-\sinh 2t = 2t \langle \phi, \zeta \rangle + 2 \sinh 2t \langle \phi, i\zeta \rangle + (4t + \sinh 4t) \langle \zeta, \zeta \rangle. \quad (8.19)$$

We find from (8.19) at $t = 0$ that $\langle \phi, \zeta \rangle = 0$. Thus (8.19) gives

$$0 = \sinh 2t(1 + \langle \phi, i\zeta \rangle) + (4t + \sinh 4t) \langle \zeta, \zeta \rangle. \quad (8.20)$$

Differentiating (8.20) gives $\langle \phi, i\zeta \rangle = -\frac{1}{2} - 2 \langle \zeta, \zeta \rangle$. Thus (8.17) yields $\langle \phi, i\zeta \rangle = -\frac{1}{2}$ and $\langle \zeta, \zeta \rangle = 0$. Now, we find from (8.18) that $\langle \phi, \phi \rangle = 0$. Consequently we have

$$\langle \phi, \phi \rangle = \langle \zeta, \zeta \rangle = \langle \phi, \zeta \rangle = 0, \quad \langle \phi, i\zeta \rangle = -\frac{1}{2}. \quad (8.21)$$

Since ζ is a constant light-like vector, we may put

$$\zeta = (1, 0, \dots, 0, 1), \quad \phi = (a_1 + ib_1, \dots, a_6 + ib_6). \quad (8.22)$$

It follows from (8.21) and (8.22) that $a_6 = a_1$ and $b_6 = b_1 + \frac{1}{2}$. Therefore

$$\phi = (a_1 + ib_1, a_2 + ib_2, \dots, a_1 + i(b_1 + \frac{1}{2})). \quad (8.23)$$

Now, by using $\langle \phi, \phi \rangle = 0$ and (8.23), we find $\psi = (a_2 + ib_2, \dots, a_5 + ib_5)$ and $b_1 = -\frac{1}{4} - \langle \psi, \psi \rangle$. Combining these with (8.23) yields

$$\phi = \left(w - i \langle \psi, \psi \rangle - \frac{i}{4}, \psi, w - i \langle \psi, \psi \rangle + \frac{i}{4} \right) \quad (8.24)$$

with $w = a_1$. It follows from (8.22) and (8.24) that $\langle \phi_{u_j}, \zeta \rangle = \langle \phi_{u_j}, i\zeta \rangle = 0$. Thus, by applying $\langle L_{u_j}, iL \rangle = 0$, $j \in \Delta$, we find from (8.17) that $\langle \phi_{u_j}, i\phi \rangle = 0$.

On the other hand, (8.24) implies that

$$\langle \phi_{u_j}, i\phi \rangle = -\frac{1}{2}w_{u_j} + \langle \psi_{u_j}, i\psi \rangle \quad (8.25)$$

with $w_{u_j} = \frac{\partial w}{\partial u_j}$. Therefore w satisfies the PDE system: $w_{u_j} = 2 \langle \psi_{u_j}, i\psi \rangle$.

Now, we derive from (8.17), (8.22) and (8.23) that

$$L = \frac{1}{\cosh t - i \sinh t} \left(2t + w + i \left(\cosh 2t - \langle \psi, \psi \rangle - \frac{1}{4} \right), \right. \\ \left. \psi, 2t + w + i \left(\cosh 2t - \langle \psi, \psi \rangle + \frac{1}{4} \right) \right). \quad (8.26)$$

It follows from (8.26) that

$$L_{u_j} = \frac{1}{\cosh t - i \sinh t} \left(w_{u_j} - i \langle \psi, \psi \rangle_{u_j}, \psi_{u_j}, w_{u_j} - i \langle \psi, \psi \rangle_{u_j} \right). \quad (8.27)$$

Thus we find $\langle \psi_{u_j}, \psi_{u_k} \rangle = \cosh 2t \langle L_{u_j}, L_{u_k} \rangle$ which implies that ψ is an immersion in \mathbf{C}^4 . Also, we find from (8.27) and $\langle L_{u_j}, iL_{u_k} \rangle = 0$ that $\langle \psi_{u_j}, i\psi_{u_k} \rangle = 0$. Thus ψ is a Lagrangian immersion. Now, by applying an argument similar to the last part of the proof of [11, Theorem 6.1], we conclude that

$$\psi_{u_j u_k} = \sum_{i=1}^4 (\Gamma_{jk}^i + ih_{jk}^i) \phi_{u_i}, \quad j, k \in \Delta.$$

Therefore, according to (8.5), ψ is a $\delta(2)$ -ideal minimal Lagrangian immersion in \mathbf{C}^4 . Consequently, we obtain case (v) of the theorem.

Case (3.b): $a, b, \mu \neq 0$. We obtain case (vi) of the theorem by applying the same argument as case (3.a.3). \square

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