A FIXED POINT THEOREM FOR WEAKLY ZAMFIRESCU MAPPINGS

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ABSTRACT: In [13] Zamfirescu gave a fixed point theorem that generalizes the classical fixed point theorems by Banach, Kannan and Chatterjea. In this paper, we follow the ideas of Dugundji and Granas to extend Zamfirescu's fixed point theorem to the class of weakly Zamfirescu maps. A continuation method for this class of maps is also given.

1 Introduction

Throughout all this paper, (X, d) will be a metric space, D a subset of X and $f : D \to X$ will be a map. We say that f is contractive if there exists $\alpha \in [0, 1)$ such that, for all $x, y \in D$,

$$d(f(x), f(y)) \leq \alpha\, d(x, y). \tag{C}$$

The well known Banach's fixed point theorem asserts that if $D = X$, f is contractive and (X, d) is complete, then f has a unique fixed point $x \in X$, and

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for any $x_0 \in X$ the sequence $\{T^n(x_0)\}$ converges to x. This result has been extended by several authors to some classes of maps which do not satisfy the contractive condition (C). For instance, two conditions that can replace (C) in Banach's theorem are the following:

(Kannan, [9]) There exists $\alpha \in [0, 1)$ such that, for all $x, y \in X$,

$$d(f(x), f(y)) \le \frac{\alpha}{2} \left[d(x, f(x)) + d(y, f(y))\right]. \tag{K}$$

(Chatterjea [3]) There exists $\alpha \in [0, 1)$ such that, for all $x, y \in X$,

$$d(f(x), f(y)) \le \frac{\alpha}{2} \left[d(x, f(y)) + d(y, f(x))\right]. \tag{Ch}$$

The conditions (C), (K) and (Ch) are independent as it will be showed in Section 2 (see also [11] and [5]).

After these three results, many papers have been written generalizing some of the conditions (C), (K) or (Ch), or even the three conditions simultaneously. In 1972, Zamfirescu [13], combining the conditions (C), (K) and (Ch), obtained a fixed point theorem for the class of maps $f : X \to X$ for which there exists $\zeta \in [0, 1)$ such that

$$\begin{aligned} d(f(x), f(y)) \le \zeta \max\Big\{ & d(x, y), \frac{1}{2}\big[d(x, f(x)) + d(y, f(y))\big], \\ & \frac{1}{2}\big[d(x, f(y)) + d(y, f(x))\big]\Big\}. \end{aligned} \tag{Z}$$

A mapping satisfying (Z) is commonly called a Zamfirescu map. Note that the class of Zamfirescu maps is a subclass of the class of mappings f satisfying the following condition: there exists $0 \le q < 1$ such that

$$\begin{aligned} d(f(x), f(y)) \le q \max\Big\{ & d(x, y), d(x, f(x)), d(y, f(y)), \\ & \frac{1}{2}\big[d(x, f(y)) + d(y, f(x))\big]\Big\}. \end{aligned} \tag{Q}$$

The class of mappings satisfying (Q) was introduced and investigated by Ćirić [4] in 1971, who obtained a fixed point theorem under this contractive condition. A somewhat different way of generalizing Banach's theorem was followed by Dugundji and Granas [6], who extended Banach's theorem to the class of weakly contractive maps. The concept of weakly contractive map was introduced in [6] by replacing the constant α, in (C), by a function $\alpha = \alpha(x, y)$: we say that $f : X \to X$ is weakly contractive if there exists $\alpha : X \times X \to [0, 1]$, satisfying that $\theta(a, b) := \sup\{\alpha(x, y) : a \le d(x, y) \le b\} < 1$ for every $0 < a \le b$, such that, for all $x, y \in X$,

$$d(f(x), f(y)) \le \alpha(x, y)\, d(x, y).$$

Following this fashion, it was shown in [2] that Kannan's fixed point theorem for Kannan maps (those satisfying (K)) could be extended to the new class of weakly Kannan maps. An immediate question is whether these arguments also work for maps of type Chatterjea (those satisfying (Ch)). More generally, we may pose the question of whether condition (Q) in Ćirić's theorem could be replaced by the corresponding weak concept, as done by Dugundji and Granas with the concept of weakly contractive maps. But this question has a negative answer, as it was shown by Sastry [12] by giving a simple example. Nevertheless, we are able to show that the above question has an affirmative answer for the class of Zamfirescu maps. This will be done in Section 3. In Section 2 we include several examples that separate each of the conditions (C), (K) and (Ch) from the others, and the same is done for the corresponding weak concepts. Finally, in Section 4 we give a continuation method for the class of maps introduced in this paper.

2 Weakly Zamfirescu maps

Although Rhoades [11] showed that the conditions (C) and (K) are independent, as far as we know nothing more is known on this question. The aim of this section is to separate definitively the concepts (C), (K) and (Ch), as well as the corresponding weak concepts.

The concept of contractive map was generalized by Dugundji and Granas [6] as follows.

Definition 1. We say that $f : D \to X$ is a weakly contractive map if there exists $\alpha : D \times D \to [0, 1]$, satisfying that

$$\theta(a, b) := \sup\{\alpha(x, y) : a \leq d(x, y) \leq b\} < 1$$

for every $0 < a \leq b$, such that, for all $x, y \in D$,

$$d(f(x), f(y)) \leq \alpha(x, y)\, d(x, y). \tag{1}$$

Remark 2. As in the case of a contractive map, any weakly contractive map has at most one fixed point.

Remark 3. Notice that if $f : D \subset X \to X$ is a weakly contractive map and we define $\alpha_f(x, y)$ on $D \times D$ as

$$\alpha_f(x, y) = \begin{cases} \frac{d(f(x), f(y))}{d(x,y)}, & \text{if } x \neq y; \\ \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

then α_f is well defined, takes values in $[0, 1]$, satisfies $\sup\{\alpha_f(x, y) : a \leq d(x, y) \leq b\} < 1$ for all $0 < a \leq b$, and α_f is smaller than any α associated to f, and also satisfies (1), with α replaced by α_f, for all $x, y \in D$. Conversely, if α_f is defined as in (2) and satisfies the above set of conditions, then f is a weakly contractive map, establishing in this way an equivalent definition for weakly contractive maps.

The following two examples show that the class of weakly contractive maps is larger than the class of contractive maps.

Example 4. Consider the subset $D = [0, \pi/2]$ of the metric space $X = \mathbb{R}$ with the usual metric $d(x, y) = |x - y|$, and let $f : D \to X$ be the function defined as $f(x) = \sin(x)$. Then f is a weakly contractive map, but not a contractive map.

First of all, we will see that the map f does not satisfy the contractive condition (C). To do this, simply note that

$$\lim_{x\to 0} \frac{d(f(x), f(0))}{d(x,0)} = \lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$

To check that f is a weakly contractive map, consider the function $\alpha_f : [0,\pi/2]\times[0,\pi/2] \to \mathbb{R}$ given by (2). This function is well defined and also takes values in $[0,1]$, since for every $x, y \in [0,\pi/2]$, we have $|\sin(x) - \sin(y)| \le |x-y|$.

Next, assume that $0 < a \le b$ and let us see that $\theta(a,b) = \sup\{\alpha(x,y) : a \le d(x,y) \le b,\ x,y \in [0,\pi/2]\} < 1$. To this end, just notice that if $x, y \in [0,\pi/2]$ with $0 < a \le |x-y| \le b$, then

$$\frac{a}{2} \le \frac{x+y}{2},$$

and use the monotonicity of the cosine function to obtain

$$\begin{aligned} \alpha_f(x,y) &= \frac{|\sin(x) - \sin(y)|}{|x-y|} \\ &= \frac{\left|2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)\right|}{|x-y|} \\ &\le \cos\left(\frac{x+y}{2}\right) \le \cos\left(\frac{a}{2}\right) < 1. \end{aligned}$$

The following class of mappings was introduced by Kannan [9] in 1968.

Definition 5. We say that a mapping $f : D \to X$ is a Kannan map if there exists $\kappa \in [0,1)$ such that

$$d(f(x), f(y)) \le \frac{\kappa}{2}\big[d(x, f(x)) + d(y, f(y))\big], \tag{3}$$

for all $x, y \in D$.

Remark 6. Rhoades proved that the concepts of contractive map and Kannan map are independent (see [11], Theorem 1, (iii)).

Recently this concept has been generalized in [2], obtaining the so-called weakly Kannan maps.

Definition 7. We say that $f : D \to X$ is a weakly Kannan map if there exists $\kappa : D \times D \to [0,1]$, satisfying that

$$\theta(a,b) := \sup\{\kappa(x,y) : a \le d(x,y) \le b\} < 1$$

for every $0 < a \le b$, such that, for all $x, y \in D$,

$$d(f(x), f(y)) \le \frac{\kappa(x,y)}{2}\big[d(x, f(x)) + d(y, f(y))\big]. \tag{4}$$

Remark 8. In [9], Kannan noted that if a Kannan map has a fixed point then it is unique. Using the same reasoning we have that any weakly Kannan map has at most one fixed point.

Remark 9. Notice that if $f : D \subset X \to X$ is a weakly Kannan map and we define $\kappa_f(x, y)$ on $D \times D$ as

$$\kappa_f(x,y) = \begin{cases} \frac{2\,d(f(x),f(y))}{d(x,f(x))+d(y,f(y))}, & \text{if } d(x, f(x)) + d(y, f(y)) \neq 0; \\ 0, & \text{otherwise,} \end{cases} \tag{5}$$

then κ_f is well defined, takes values in $[0, 1]$, satisfies $\sup\{\kappa_f(x, y) : a \leq d(x, y) \leq b\} < 1$ for all $0 < a \leq b$, and κ_f is smaller than any α associated to f, and also satisfies (4), with κ replaced by κ_f, for all $x, y \in D$. Conversely, if κ_f is defined as in (5) and satisfies the above set of conditions, then f is a weakly Kannan map, establishing in this way an equivalent definition for weakly Kannan maps.

The following example shows that the class of weakly Kannan maps is larger than the class of Kannan maps.

Example 10 (see [2], example 2.5)**.** Let $D = [0, \infty)$ the subset of the metric space $X = \mathbb{R}$ with the usual metric $d(x, y) = |x - y|$. The map $f : D \to X$ defined as $f(x) = \frac{1}{3}\log(1 + e^x)$ is a weakly Kannan map, but not a Kannan map.

The following two examples show that the concepts of weakly contractive map and weakly Kannan map are independent.

Example 11. Let f be the mapping of Example 4, i.e., $f : [0, \pi/2] \subset \mathbb{R} \to \mathbb{R}$ with $f(x) = \sin(x)$. We know that f is weakly contractive. Moreover, f is not weakly Kannan, since

$$\kappa_f(0, \tfrac{\pi}{2}) = \frac{4}{\pi - 2} > \frac{7}{2}\,.$$

Example 12 (see [2], Example 3.4)**.** Consider the subset $D = [-1, 1]$ of the metric space (X, d), where $X = \mathbb{R}$ and $d(x, y) = |x - y|$, and let $f : [-1, 1] \to \mathbb{R}$ be the map given as

$$f(x) = \begin{cases} -\sin(x) & \text{if } -1 \leq x < 1, \\ 0 & \text{if } x = 1. \end{cases}$$

This mapping is weakly Kannan, but not weakly contractive.

In 1972, Chatterjea [3] considered a type or contractive condition similar to that of Kannan but independent of it, and which does not imply the continuity of the operator.

Definition 13. We say that a mapping $f : D \to X$ is a Chatterjea map if there exists $\xi \in [0, 1)$ such that, for all $x, y \in D$,

$$d(f(x), f(y)) \leq \frac{\xi}{2}\big[d(x, f(y)) + d(y, f(x))\big]. \tag{6}$$

The following example shows that a Chatterjea map need not be of contractive type neither of Kannan type.

Example 14. Consider the metric space (X, d), where $X = [0, 1]$ and d is the usual metric in $\mathbb{R}$. The mapping $f : [0, 1] \to [0, 1]$ given as

$$f(x) = \begin{cases} 1 & \text{if } 0 < x \leq 1, \\ \\ 2/3 & \text{if } x = 0. \end{cases}$$

is a Chatterjea map, but not a contractive map neither a Kannan map.

That f is not contractive is immediate, since f is not continuous. To see that f is not a Kannan map, observe that

$$\lim_{x \to 1^-} \frac{2d(f(x), f(0))}{d(x, f(x)) + d(0, f(0))} = \lim_{x \to 1^-} \frac{2\left|1 - \frac{2}{3}\right|}{|x - 1| + \left|0 - \frac{2}{3}\right|} = \lim_{x \to 1^-} \frac{\frac{2}{3}}{\frac{5}{3} - x} = 1.$$

To see that f is a Chatterjea map, observe that for $x \in (0, 1]$ and $y = 0$,

$$\frac{2d(f(x), f(y))}{d(x, f(y)) + d(y, f(x))} = \frac{\frac{2}{3}}{\left|x - \frac{2}{3}\right| + 1} \leq \frac{2}{3}.$$

Hence, taking $\xi \in [2/3, 1)$, we have

$$d(f(x), f(y)) \leq \frac{\xi}{2}\big[d(x, f(y)) + d(y, f(x))\big],$$

for every $x, y \in [0, 1]$.

The following two examples show that the concept of Chatterjea map is independent of the concepts of contractive map and Kannan map.

Example 15. Let X be $\mathbb{R}^2$ with its usual euclidean norm, $\|\cdot\|_2$, and let D be the closed unit ball, $D = \{(x_1, x_2) \in \mathbb{R}^2 : \|(x_1, x_2)\|_2 \leq 1\}$. Define $f : D \to X$ as $f(x_1, x_2) = \alpha\,(-x_2, x_1)$, where $\alpha \in [1/\sqrt{3}, 1)$. Then f is a contractive map, but not a Kannan map neither a Chatterjea map.

Obviously, f is a contractive map, since $0 \leq \alpha < 1$, and any rotation is an isometry. To see that f is not a Kannan map neither a Chatterjea map, just consider $x = (1, 0)$ and $y = (-1, 0)$, and use that $\alpha \geq \frac{1}{\sqrt{3}}$ to obtain

$$d(f(x), f(y)) \geq \frac{1}{2}\left[d(x, f(x)) + d(y, f(y))\right]$$

and

$$d(f(x), f(y)) \geq \frac{1}{2}\big[d(x, f(y)) + d(y, f(x))\big].$$

Example 16. Consider the metric space (X, d), where $X = [0, 1]$ and d is the usual metric in $\mathbb{R}$. The mapping $f : [0, 1] \to [0, 1]$ given as

$$f(x) = \begin{cases} 1/3 & \text{if } 0 \leq x < 1, \\ \\ 0 & \text{if } x = 1, \end{cases}$$

is a Kannan map, but not a Chatterjea map.

To see that f is not a Chatterjea map, it suffices to note that

$$\frac{2d(f(0),f(1))}{d(0,f(1))+d(1,f(0))} = \frac{2\left|\frac{1}{3}-0\right|}{|0-0|+\left|1-\frac{1}{3}\right|} = 1.$$

To show that f is a Kannan map, observe that if $x \in [0,1)$ and $y = 1$,

$$\frac{2d(f(x),f(1))}{d(x,f(x))+d(1,f(1))} = \frac{\frac{2}{3}}{\left|x-\frac{1}{3}\right|+1} \leq \frac{2}{3},$$

and hence, taking $\kappa \in [2/3, 1)$, we have

$$d(f(x),f(y)) \leq \frac{\kappa}{2}\big[d(x,f(x))+d(y,f(y))\big],$$

for all $x, y \in [0,1]$.

Next, as done for contractive maps and Kannan maps, we introduce the corresponding weak concept for Chatterjea maps, and study its relations with the previous concepts.

Definition 17. We say that $f : D \to X$ is a weakly Chatterjea map if there exists $\xi : D \times D \to [0,1]$, satisfying that

$$\theta(a,b) := \sup\{\xi(x,y) : \; a \leq d(x,y) \leq b\} < 1$$

for every $0 < a \leq b$, such that, for all $x, y \in D$,

$$d(f(x),f(y)) \leq \frac{\xi(x,y)}{2}\big[d(x,f(y))+d(y,f(x))\big]. \tag{7}$$

Remark 18. As in the case of Chatterjea maps, any weakly Chatterjea map has at most one fixed point. Indeed, suppose that u and v are fixed points of a weakly Chatterjea map f, with $u \neq v$. Then $\xi(u,v) \leq \theta(\frac{r}{2}, r) < 1$, where $r = d(u,v) > 0$. So, by (7), we have

$$\begin{aligned} d(u,v) &= d(f(u),f(v)) \\ &\leq \frac{\xi(u,v)}{2}\big[d(u,f(v))+d(v,f(u))\big] \\ &\leq \xi(u,v)\,d(u,v) \\ &\leq \theta(\tfrac{r}{2}, r)\,d(u,v). \end{aligned}$$

which is a contradiction, since $\theta(\frac{r}{2}, r) < 1$.

Proposition 19. Let D be a nonempty subset of a metric space (X,d). A map $f : D \to X$ is weakly Chatterjea if only if f satisfies the following conditions:

(P_0) if $x, y \in D$, with $y = f(x)$ and $x = f(y)$, then $x = y$;

(P_1) the mapping $\xi_f : D \times D \to \mathbb{R}$ given as

$$\xi_f(x,y) = \begin{cases} \frac{2d(f(x),f(y))}{d(x,f(y))+d(y,f(x))}, & \text{if } d(x,f(y))+d(y,f(x)) \neq 0; \\ 0, & \text{otherwise,} \end{cases} \tag{8}$$

takes values in $[0,1]$ and satisfies $\sup\{\xi_f(x,y) : \; a \leq d(x,y) \leq b\} < 1$ for all $0 < a \leq b$.

Remark 20. Condition (P_0) is is essential. For example, if we take the metric space $X = \{-r, r\}$, where r is any fixed positive real number, with the Euclidean metric and the mapping $f : X \to X$ is given as $f(t) = -t$, then f satisfies (P_1) (since ξ_f is null mapping), but not (P_0), and moreover f is not a weakly Chatterjea map.

Example 21. Consider the metric space (X, d), where $X = [0, 1]$ and d is the usual metric. The mapping $f : [0, 1] \to [0, 1]$ given as

$$f(x) = \begin{cases} \frac{2}{3}x & \text{if } 0 \le x < 1, \\ 0 & \text{if } x = 1, \end{cases}$$

is a weakly Chatterjea map, but not a Chatterjea map. Moreover, f is not weakly contractive neither weakly Kannan.

Since

$$\lim_{x \to 1^-} \frac{2d(f(x), f(1))}{d(x, f(1)) + d(f(x), 1)} = \lim_{x \to 1^-} \frac{\frac{4}{3}x}{x + 1 - \frac{2}{3}x} = 1,$$

there exists no $\xi \in [0, 1)$ such that $d(f(x), f(y)) \le \frac{\xi}{2}\left[d(x, f(y)) + d(y, f(x))\right]$ for every $x, y \in [0, 1]$. Hence, f is not a Chatterjea map.

That f is not weakly contractive is immediate, since f is not continuous. To see that f is not a weakly Kannan map, just observe that

$$\kappa_f(0, \tfrac{1}{2}) = \frac{2\left|f(0) - f(\frac{1}{2})\right|}{|0 - f(0)| + \left|\frac{1}{2} - f(\frac{1}{2})\right|} = \frac{2\left|0 - \frac{1}{3}\right|}{|0 - 0| + \left|\frac{1}{2} - \frac{1}{3}\right|} = 4.$$

Next, we shall check that f is weakly Chatterjea. To this end, we shall use Proposition 19. First we show that f satisfies (P_0). Suppose $x, y \in [0, 1]$ with $x = f(y)$ and $y = f(x)$. Without loss of generality we can assume that $x \le y$. Note that if $y = 1$, then $x = f(1) = 0$ and so $y = f(0) = 0$, which is a contradiction. Thus, $0 \le x \le y < 1$. From the definition of f, we have $x = f(y) = \frac{2}{3}y = \frac{2}{3}f(x) = \frac{4}{9}x$. Thus, $x = 0$ and $y = 0$. Therefore, f satisfies (P_0). Now we shall show that f verifies (P_1). We need to check that ξ_f only takes values in $[0, 1]$ and that $\theta(a, b) := \sup\left\{\xi_f(x, y) : a \le |x - y| \le b,\ x, y \in [0, 1]\right\} < 1$ for all $0 < a \le b$. In fact, all this follow if we just show that, for $0 < a \le 1$,

$$\theta(a, 1) \le \min\left\{\frac{4}{5}, \frac{4(1 - a)}{4 - a}\right\}.$$

Thus, assume that $0 < a \le 1$ and that $x, y \in [0, 1]$, with $a \le |x - y| \le 1$. Without loss of generality we may also assume that $x < y$, since ξ_f is symmetric. Note that, in this case,

$$\xi_f(x, y) = \begin{cases} \frac{4x}{3+x}, & \text{if } y = 1; \\ \frac{4(y-x)}{x+y}, & \text{if } 0 < \frac{2}{3}y \le x < y < 1; \\ \frac{4}{5}, & \text{if } 0 \le x < \frac{2}{3}y. \end{cases}$$

Hence, if $y = 1$ we have $x \le 1 - a$, and then

$$\xi_f(x,1) = \frac{4x}{3+x} = 4 - \frac{12}{x+3} \le 4 - \frac{12}{3+(1-a)} = \frac{4(1-a)}{4-a}\,.$$

On the other hand, if $0 < \frac{2}{3}y \le x < y < 1$ we have

$$\xi_f(x,y) = \frac{4(y-x)}{x+y} \le \frac{4(y-\frac{2}{3}y)}{\frac{2}{3}y+y} = \frac{4}{5}\,.$$

The following two examples, together with the previous one show that the concepts of weakly Chatterjea map, weakly Kannan map and weakly contractive map are independent.

Example 22. Fix $\omega > 0$. Let $D = [0,\infty)$ be the subset of the metric space $X = \mathbb{R}$ with the usual metric $d(x,y) = |x-y|$. Consider the mapping $f : D \to X$ defined as $f(x) = \omega^2/(\omega + x)$. Then f is a weakly contractive map, but not a weakly Chatterjea map. Moreover f is not a contractive map, since

$$\lim_{x\to 0^+} \frac{d(f(x)), d(f(0))}{d(x,0)} = \left| \lim_{x\to 0^+} \frac{\omega - \frac{\omega^2}{\omega+x}}{x} \right| = \lim_{x\to 0^+} \frac{\omega}{\omega + x} = 1\,.$$

To check that f is a weakly contractive map, consider the function $\alpha_f : [0,\infty)\times[0,\infty) \to \mathbb{R}$ given by (2). This function is well defined and also takes values in $[0,1]$, since for every $x, y \in [0,\infty)$, with $x \ne y$, we have

$$\alpha_f(x,y) = \frac{\omega^2}{(\omega+x)(\omega+y)} \le 1.$$

Moreover, for $0 < a \le b$ we have that

$$\theta(a,b) = \sup\{\alpha_f(x,y) : a \le d(x,y) \le b,\ x,y \in [0,\infty)\} \le \frac{\omega}{\omega + \frac{a}{2}} < 1.$$

Finally, since

$$\xi_f(0,\omega) = \frac{2\,|f(0) - f(\omega)|}{|0 - f(\omega)| + |\omega - f(0)|} = \frac{2\left|\omega - \frac{\omega}{2}\right|}{\left|0 - \frac{\omega}{2}\right| + |\omega - \omega|} = 2,$$

then f is not weakly Chatterjea.

Example 23. Let be $f : [-1,1] \to \mathbb{R}$ the map of Example 12, i.e.,

$$f(x) = \begin{cases} -\sin(x) & \text{if } -1 \le x < 1, \\ 0 & \text{if } x = 1. \end{cases}$$

As mentioned in Example 12, this mapping is weakly Kannan. However, f is not weakly Chatterjea, since

$$\xi_f(-1,1) = \frac{2\,|f(-1) - f(1)|}{|-1 - f(1)| + |1 - f(-1)|} = \frac{2\sin(1)}{2 - \sin(1)} \ge 1.$$

3 A fixed point theorem

Our aim in this section is to give a fixed point theorem for the class of weakly Chatterjea maps. Indeed, this will be done for a wider class of maps: those which satisfy the weak condition associated to the condition (Z) introduced by Zamfirescu [13].

Definition 24. Let (X,d) be a metric space, $D \subset X$ and $f : D \to X$. We say that f is a weakly Zamfirescu map if there exists $\alpha : D \times D \to [0,1]$, with $\theta(a,b) := \sup\{\alpha(x,y) : a \le d(x,y) \le b\} < 1$ for every $0 < a \le b$, such that, for all $x, y \in D$,

$$d(f(x), f(y)) \le \alpha(x,y)\, M_f(x,y), \tag{Z_w}$$

where

$$M_f(x,y) := \max\Big\{ d(x,y), \frac{1}{2}\big[d(x,f(x)) + d(y,f(y))\big], \frac{1}{2}\big[d(x,f(y)) + d(y,f(x))\big]\Big\}.$$

We first give three propositions in which some properties of those functions f satisfying (Z_w) are established.

Proposition 25. Let (X,d) be a metric space and $D \subset X$. If $f : D \to X$ is a weakly Zamfirescu map, then f has at most one fixed point in D.

Proof. Suppose that u and v are fixed points of f, with $u \neq v$. Then $\alpha(u,v) \le \theta(\frac{r}{2}, r) < 1$, where $r = d(u,v) > 0$. So, by (Z_w), we have

$$d(u,v) = d(f(u), f(v)) \le \alpha(u,v)\, M_f(u,v) = \alpha(u,v)\, d(v,u) \le \theta\left(\tfrac{r}{2}, r\right)\, d(u,v),$$

which is a contradiction. □

Recall that a self-mapping f on a metric space (X,d) is said to be asymptotically regular at $x_0 \in X$ if

$$\lim_{n\to\infty} d(f^n(x_0), f^{n+1}(x_0)) = 0.$$

Proposition 26. Let (X,d) be a metric space. If $f : X \to X$ is a weakly Zamfirescu map, then f is asymptotically regular at each point in X.

Proof. Let $x_0 \in X$ and define the Picard iterates $x_n = f(x_{n-1}) = f^n(x_0)$ for $n = 1, 2, \ldots$. We first prove that, for all $n \ge 1$,

$$d(x_n, x_{n+1}) \le \alpha(x_{n-1}, x_n)\, d(x_{n-1}, x_n), \tag{9}$$

where α is the function associated to f by the condition (Z_w).

Observe that, for all $n \ge 1$,

$$\begin{aligned} M_f(x_{n-1}, x_n) &= \max\Big\{ d(x_{n-1}, x_n), \tfrac{1}{2}\big[d(x_{n-1}, f(x_{n-1})) + d(x_n, f(x_n))\big], \\ &\qquad \tfrac{1}{2}\big[d(x_{n-1}, f(x_n)) + d(x_n, f(x_{n-1}))\big]\Big\} \\ &= \max\Big\{ d(x_{n-1}, x_n), \tfrac{1}{2}\big[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big], \\ &\qquad \tfrac{1}{2}\big[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)\big]\Big\} \\ &= \max\Big\{ d(x_{n-1}, x_n), \tfrac{1}{2}\big[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big], \\ &\qquad \tfrac{1}{2}\, d(x_{n-1}, x_{n+1})\Big\}. \end{aligned}$$

To end the proof of (9), we shall appeal to condition (Z_w), and hence, we consider the following three cases:

Case 1. If $M_f(x_{n-1}, x_n) = d(x_{n-1}, x_n)$,

$$d(x_n, x_{n+1}) \leq \alpha(x_{n-1}, x_n)\, d(x_{n-1}, x_n).$$

Case 2. If $M_f(x_{n-1}, x_n) = \frac{1}{2}\big[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big]$,

$$d(x_n, x_{n+1}) \leq \frac{\alpha(x_{n-1}, x_n)}{2}\big[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big],$$

i.e.,

$$d(x_n, x_{n+1}) \leq \frac{\alpha(x_{n-1}, x_n)}{2 - \alpha(x_{n-1}, x_n)}\, d(x_{n-1}, x_n) \leq \alpha(x_{n-1}, x_n)\, d(x_{n-1}, x_n).$$

Case 3. If $M_f(x_{n-1}, x_n) = \frac{1}{2}\, d(x_{n-1}, x_{n+1})$,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \frac{\alpha(x_{n-1}, x_n)}{2} d(x_{n-1}, x_{n+1}) \\ &\leq \frac{\alpha(x_{n-1}, x_n)}{2}\big[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big], \end{aligned}$$

i.e.,

$$d(x_n, x_{n+1}) \leq \frac{\alpha(x_{n-1}, x_n)}{2 - \alpha(x_{n-1}, x_n)}\, d(x_{n-1}, x_n) \leq \alpha(x_{n-1}, x_n)\, d(x_{n-1}, x_n).$$

Thus, relation (9) is proved

As a consequence, we obtain that the sequence $\{d(x_n, x_{n+1})\}$ is nonincreasing, since $0 \leq \alpha(x_{n-1}, x_n) \leq 1$. Then, it is convergent to the real number

$$d = \inf\big\{d(x_{n-1}, x_n) : n = 1, 2, \dots\big\}.$$

It suffices to prove that $d = 0$. Suppose that $d > 0$ and arrive to a contradiction as follows: use that for all $n \in \mathbb{N}$

$$0 < d \leq d(x_n, x_{n+1}) \leq d(x_0, x_1)$$

and the definition of $\theta = \theta(d, d(x_0, x_1))$ to obtain that $\alpha(x_{n-1}, x_n) \leq \theta$. This, together with (9) gives that

$$d \leq d(x_n, x_{n+1}) \leq \theta^n d(x_0, x_1)$$

for all $n \in \mathbb{N}$, which is impossible since $d > 0$ and $0 \leq \theta < 1$. Therefore, f is asymptotically regular at x_0. □

Although a weakly Zamfirescu map f may be discontinuous at some point, the following proposition shows that the discontinuity cannot occur at a fixed point for f.

Proposition 27. Let (X, d) be a metric space and $f : X \to X$ a weakly Zamfirescu map. If f has a fixed point, say u, then f is continuous at u.

Proof. Let $\{x_n\}$ be a convergent sequence to $u = f(u)$. For any $n \in \mathbb{N}$, we have that

$$\begin{aligned} d(f(x_n), f(u)) &\le \alpha(x_n, u)\, M_f(x_n, u) \\ &\le \max\Big\{ d(x_n, u), \tfrac{1}{2}\big[d(x_n, f(x_n)) + d(u, f(u))\big], \\ &\qquad\qquad \tfrac{1}{2}\big[d(x_n, f(u)) + d(u, f(x_n))\big] \Big\} \\ &= \max\Big\{ d(x_n, u), \tfrac{1}{2}\, d(x_n, f(x_n)), \\ &\qquad\qquad \tfrac{1}{2}\big[d(x_n, u) + d(f(u), f(x_n))\big] \Big\} \\ &\le \max\Big\{ d(x_n, u), \tfrac{1}{2}\big[d(x_n, u) + d(f(u), f(x_n))\big], \\ &\qquad\qquad \tfrac{1}{2}\big[d(x_n, u) + d(f(u), f(x_n))\big] \Big\}. \end{aligned}$$

Hence, for all $n \in \mathbb{N}$,

$$0 \le d(f(u), f(x_n)) \le d(x_n, u),$$

so that $\{f(x_n)\}$ converges to $f(u)$. Therefore, f is continuous at u. □

Now proceed to prove the main result of this section.

Theorem 28. Let (X, d) be a complete metric space and $f : X \to X$ a weakly Zamfirescu map. Then, f has a unique fixed point $u \in X$ and at this point u the mapping f is continuous. Moreover, for each $x_0 \in X$, the sequence $\{f^n(x_0)\}$ converges to u.

Proof. Let $x_0 \in X$ and define $x_{n+1} = f(x_n)$ for $n \in \mathbb{N}$. We may assume that $d(x_0, x_1) > 0$ because otherwise we have finished. We shall prove that $\{x_n\}$ is a Cauchy sequence and that its limit is a fixed point for f. To do it, let us prove that

$$d(x_{n+k+1}, x_{n+1}) \le \alpha(x_{n+k}, x_n)\, d(x_{n+k}, x_n) + 2d(x_{n+1}, x_n) \tag{10}$$

for all $n, k \in \mathbb{N}$.

Let $n, k \in \mathbb{N}$. By (Z_w),

$$\begin{aligned} d(x_{n+k+1}, x_{n+1}) &= d(f(x_{n+k}), f(x_n)) \\ &\le \alpha(x_{n+k}, x_n)\, M_f(x_{n+k}, x_n), \end{aligned}$$

where

$$\begin{aligned} M_f(x_{n+k}, x_n) &= \max\Big\{ d(x_{n+k}, x_n), \frac{1}{2}\big[d(x_{n+k}, f(x_{n+k})) + d(x_n, f(x_n))\big], \\ &\qquad\qquad \frac{1}{2}\big[d(x_{n+k}, f(x_n)) + d(x_n, f(x_{n+k}))\big] \Big\} \\ &= \max\Big\{ d(x_{n+k}, x_n), \frac{1}{2}\big[d(x_{n+k}, x_{n+k+1}) + d(x_n, x_{n+1})\big], \\ &\qquad\qquad \frac{1}{2}\big[d(x_{n+k}, x_{n+1}) + d(x_n, x_{n+k+1})\big] \Big\}. \end{aligned}$$

We consider the following three cases:

Case 1. If $M_f(x_{n+k}, x_n) = d(x_{n+k}, x_n)$, then (10) is obvious.

Case 2. If $M_f(x_{n+k}, x_n) = \frac{1}{2}\big[d(x_{n+k}, x_{n+k+1}) + d(x_n, x_{n+1})\big]$, then

$$d(x_{n+k+1}, x_{n+1}) \leq \frac{\alpha(x_{n+k}, x_n)}{2}\big[d(x_{n+k}, x_{n+k+1}) + d(x_n, x_{n+1})\big].$$

Applying (9)

$$d(x_{n+k}, x_{n+k+1}) \leq d(x_n, x_{n+1}).$$

So,

$$d(x_{n+k+1}, x_{n+1}) \leq \alpha(x_{n+k}, x_n)\, d(x_{n+k}, x_n) + 2d(x_{n+1}, x_n).$$

Case 3. If $M_f(x_{n+k}, x_n) = \frac{1}{2}\big[d(x_{n+k}, x_{n+1}) + d(x_n, x_{n+k+1})\big]$, then

$$\begin{aligned} d(x_{n+k+1}, x_{n+1}) &\leq \frac{\alpha(x_{n+k}, x_n)}{2}\big[d(x_{n+k}, x_{n+1}) + d(x_n, x_{n+k+1})\big] \\ &\leq \frac{\alpha(x_{n+k}, x_n)}{2}\big[d(x_{n+k}, x_{n+1}) + d(x_n, x_{n+1}) \\ &\qquad\qquad + d(x_{n+1}, x_{n+k+1})\big]. \end{aligned}$$

Then,

$$\left(1 - \frac{\alpha(x_{n+k}, x_n)}{2}\right) d(x_{n+k+1}, x_{n+1}) \leq \frac{\alpha(x_{n+k}, x_n)}{2}\big[d(x_{n+k}, x_{n+1}) \\ + d(x_{n+1}, x_n)\big],$$

i.e.,

$$\begin{aligned} d(x_{n+k+1}, x_{n+1}) &\leq \frac{\alpha(x_{n+k}, x_n)}{2 - \alpha(x_{n+k}, x_n)}\big[d(x_{n+k}, x_{n+1}) + d(x_{n+1}, x_n)\big] \\ &\leq \alpha(x_{n+k}, x_n)\big[d(x_{n+k}, x_{n+1}) + d(x_{n+1}, x_n)\big] \\ &\leq \alpha(x_{n+k}, x_n)\big[d(x_{n+k}, x_n) + 2d(x_n, x_{n+1})\big] \\ &\leq \alpha(x_{n+k}, x_n)\, d(x_{n+k}, x_n) + 2d(x_{n+1}, x_n). \end{aligned}$$

To prove that $\{x_n\}$ is a Cauchy sequence, suppose that $\varepsilon > 0$ and use Proposition 26 to obtain $N \in \mathbb{N}$ such that

$$d(x_{N+1}, x_N) < \frac{1}{6}\left(1 - \theta\left(\tfrac{\varepsilon}{2}, \varepsilon\right)\right) \cdot \varepsilon. \tag{11}$$

We will prove inductively that $d(x_{N+k}, x_N) < \varepsilon$ for all $k \in \mathbb{N}$. It is obvious for $k = 1$, and assuming $d(x_{N+k}, x_N) < \varepsilon$, let us see $d(x_{N+k+1}, x_N) < \varepsilon$.

Note that using (10) we have that

$$\begin{aligned} d(x_{N+k+1}, x_N) &\leq d(x_{N+k+1}, x_{N+1}) + d(x_{N+1}, x_N) \\ &\leq \alpha(x_{N+k}, x_N)\, d(x_{N+k}, x_N) + 3d(x_{N+1}, x_N) \end{aligned} \tag{12}$$

Thus, if $d(x_{N+k}, x_N) < \varepsilon/2$ it follows from (11) and (12) that

$$\begin{aligned} d(x_{N+k+1}, x_{N+1}) &\leq d(x_{N+k}, x_N) + 3d(x_{N+1}, x_N) \\ &< \frac{\varepsilon}{2} + 3 \cdot \frac{1}{6}\left(1 - \theta\left(\tfrac{\varepsilon}{2}, \varepsilon\right)\right) \cdot \varepsilon \\ &\leq \varepsilon. \end{aligned}$$

And if $d(x_{N+k}, x_N) \geq \varepsilon/2$, applying the induction hypothesis, we have that $\alpha(x_{N+k}, x_N) \leq \theta(\frac{\varepsilon}{2}, \varepsilon)$. Then, from (11) and (12), we conclude that

$$\begin{aligned} d(x_{N+k+1}, x_{N+1}) &\leq \alpha(x_{N+k}, x_N)\, d(x_{N+k}, x_N) + 3d(x_{N+1}, x_N) \\ &< \theta\left(\tfrac{\varepsilon}{2}, \varepsilon\right) \cdot \varepsilon + 3 \cdot \frac{1}{6}\left(1 - \theta\left(\tfrac{\varepsilon}{2}, \varepsilon\right)\right) \cdot \varepsilon \\ &\leq \varepsilon. \end{aligned}$$

Since (X, d) is complete, then $\{x_n\}$ is convergent, say to $u \in X$. That u is a fixed point for f follows from standard arguments which we include for the sake of completeness: for any $n \in \mathbb{N}$, we have that

$$\begin{aligned} d(u, f(u)) &= \lim_{n\to\infty} d(x_{n+1}, f(u)) \\ &= \lim_{n\to\infty} d(f(x_n), f(u)) \\ &\leq \limsup_{n\to\infty} \alpha(x_n, u)\, M_f(x_n, u) \\ &\leq \limsup_{n\to\infty} M_f(x_n, u) \\ &= \frac{1}{2}\, d(u, f(u))\ , \end{aligned}$$

since

$$\begin{aligned} \limsup_{n\to\infty} M_f(x_n, u) &= \limsup_{n\to\infty} \max\Big\{ d(x_n, u), \frac{1}{2}\big[d(x_n, f(x_n)) + d(u, f(u))\big], \\ &\qquad\qquad \frac{1}{2}\big[d(x_n, f(u)) + d(u, f(x_n))\big]\Big\} \\ &= \limsup_{n\to\infty} \max\Big\{ d(x_n, u), \frac{1}{2}\big[d(x_n, x_{n+1}) + d(u, f(u))\big], \\ &\qquad\qquad \frac{1}{2}\big[d(x_n, f(u)) + d(u, x_{n+1})\big]\Big\} \\ &= \max\Big\{0, \frac{1}{2}\big[0 + d(u, f(u))\big], \frac{1}{2}\big[d(u, f(u)) + 0\big]\Big\}\ . \end{aligned}$$

By Proposition 25 and Proposition 27, u is the unique fixed point of f and f is continuous at u. □

As a consequence of the previous theorem, we obtain the following local result, which will be used in the next section to prove the continuation method.

Corollary 29. Assume that (X, d) is a complete metric space, $x_0 \in X$, $r > 0$ and that $f : \overline{B(x_0, r)} \to X$ is a weakly Zamfirescu map with associated function α satisfying (Z_w). If θ is defined as usual, and

$$d(x_0, f(x_0)) < \frac{1}{3} \min\left\{\frac{r}{2}, r\left[1 - \theta\left(\tfrac{r}{2}, r\right)\right]\right\}\ ,$$

then f has a fixed point.

Proof. Bearing in mind Theorem 28, it suffices to show that the closed ball $\overline{B(x_0, r)}$ is invariant under f. Consider any $x \in \overline{B(x_0, r)}$ and obtain the relation

$$\begin{aligned} d(x_0, f(x)) &\leq d(x_0, f(x_0)) + d(f(x_0), f(x)) \\ &\leq d(x_0, f(x_0)) + \alpha(x_0, x)\, M_f(x_0, x)\,, \end{aligned}$$

where

$$M_f(x_0,x) = \max\Big\{d(x_0,x), \tfrac{1}{2}\big[d(x_0,f(x_0)) + d(x,f(x))\big],$$
$$\tfrac{1}{2}\big[d(x_0,f(x)) + d(x,f(x_0))\big]\Big\}.$$

We now consider three cases:

Case 1. If $M_f(x_0,x) = d(x_0,x)$,

$$d(x_0,f(x)) \le d(x_0,f(x_0)) + \alpha(x_0,x)\, d(x_0,x).$$

Case 2. If $M_f(x_0,x) = \frac{1}{2}\big[d(x_0,f(x_0)) + d(x,f(x))\big]$,

$$\begin{aligned} d(x_0,f(x)) &\le d(x_0,f(x_0)) + \frac{\alpha(x_0,x)}{2}\big[d(x_0,f(x_0)) + d(x,f(x))\big] \\ &\le d(x_0,f(x_0)) + \frac{\alpha(x_0,x)}{2}\big[d(x_0,f(x_0)) + d(x,x_0) \\ &\qquad\qquad + d(x_0,f(x_0))\big]\,, \end{aligned}$$

from which, having in mind that $\alpha(x_0,x) \le 1$,

$$d(x_0,f(x)) \le 2d(x_0,f(x_0)) + \alpha(x_0,x)\, d(x_0,x)\,.$$

Case 3. If $M_f(x_0,x) = \frac{1}{2}\big[d(x_0,f(x)) + d(x,f(x_0))\big]$,

$$\begin{aligned} d(x_0,f(x)) &\le d(x_0,f(x_0)) + \frac{\alpha(x_0,x)}{2}\big[d(x_0,f(x)) + d(x,f(x_0))\big] \\ &\le d(x_0,f(x_0)) + \frac{\alpha(x_0,x)}{2}\big[d(x_0,f(x)) + d(x,x_0) \\ &\qquad\qquad + d(x_0,f(x_0))\big]\,, \end{aligned}$$

from which, having in mind that $\alpha(x_0,x) \le 1$,

$$d(x_0,f(x)) \le 3d(x_0,f(x_0)) + \alpha(x_0,x)\, d(x_0,x)\,.$$

Therefore, in any case,

$$d(x_0,f(x)) \le 3d(x_0,f(x_0)) + \alpha(x_0,x)\, d(x_0,x)\,.$$

To end the proof, obtain that $d(x_0,f(x)) \le r$ through the above inequality by considering two cases: if $d(x_0,x) \le r/2$, then $d(x_0,f(x)) \le r$ because $d(x_0,f(x_0)) \le r/6$. Otherwise, we would have $r/2 \le d(x_0,x) \le r$, and consequently $\alpha(x_0,x) \le \theta(r/2,r)$, from which

$$d(x_0,f(x)) \le r\left[1 - \theta\left(\tfrac{r}{2},r\right)\right] + r\,\theta\left(\tfrac{r}{2},r\right) = r\,.$$

□

4 Homotopy invariance

The property of having a fixed point is an invariant by homotopy for some classes of nonlinear operators, such as contractive maps and compact maps. Usually, these maps are defined on a subset of a Banach space, the jump from the Banach space setting to the metric space setting was given by Granas [8] in 1994, who gave a homotopy result known as a continuation method for contractive maps. After Granas, Frigon [7] gave a similar result for weakly contractive maps, and the corresponding result for weakly Kannan maps was given in [2]. In this section, we give an analogous result for weakly Zamfirescu maps. For multivalued mappings this property has been studied by Altun [1], Moţ and Petruşel [10], among others.

Theorem 30. Let (X, d) be a complete metric space, U a bounded open subset of X and $H : \overline{U} \times [0,1] \to X$ satisfying the following properties:

(P1) $H(x,\lambda) \neq x$ for all $x \in \partial U$ and all $\lambda \in [0,1]$;

(P2) there exists $\alpha : \overline{U} \times \overline{U} \to [0,1]$ satisfying

$$\theta(a,b) := \sup\{\alpha(x,y) : a \le d(x,y) \le b\} < 1 \text{ for all } 0 < a \le b,$$

such that for all $x, y \in \overline{U}$ and $\lambda \in [0,1]$ we have

$$d(H(x,\lambda), H(y,\lambda)) \le \alpha(x,y)\, M_H^{\lambda}(x,y), \tag{H}$$

where

$$M_H^{\lambda}(x,y) := \max\Big\{d(x,y), \frac{1}{2}\big[d(x,H(x,\lambda)) + d(y,H(y,\lambda))\big], \frac{1}{2}\big[d(x,H(y,\lambda)) + d(y,H(x,\lambda))\big]\Big\};$$

(P3) $H(x,\lambda)$ is continuous in λ, uniformly for $x \in \overline{U}$. That is, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $d(H(x,t), H(x,s)) \le \varepsilon$ for all $x \in \overline{U}$ and $t, s \in [0,1]$ with $|t-s| < \delta$, where δ is independent of x.

If $H(\cdot, 0)$ has a fixed point in U, then $H(\cdot,\lambda)$ also has a fixed point in U for all $\lambda \in [0,1]$.

Proof. Consider the nonempty set

$$A = \{\lambda \in [0,1] : H(x,\lambda) = x \text{ for some } x \in U\}.$$

We just need to prove that $A = [0,1]$, and for this it suffices to show that A is both closed and open in $[0,1]$.

We first prove that A is closed in $[0,1]$: suppose that $\{\lambda_n\}$ is a sequence in A converging to $\lambda \in [0,1]$ and let us show that $\lambda \in A$. By definition of A, there exists a sequence $\{x_n\}$ in U with $x_n = H(x_n, \lambda_n)$. We shall prove that $\{x_n\}$ converges to a point $x_0 \in U$ with $H(x_0, \lambda) = x_0$, therefore $\lambda \in A$.

In the first place, we shall prove that, for all $n, m \in \mathbb{N}$,

$$d(x_n, x_m) \le \alpha(x_n, x_m)\, d(x_n, x_m) + \left(1 + \frac{\alpha(x_n,x_m)}{2}\right)\Big[d(H(x_m,\lambda_m), H(x_m,\lambda)) + d(H(x_n,\lambda_n), H(x_n,\lambda))\Big]. \tag{13}$$

To do it, observe that, if $n, m \in \mathbb{N}$,

$$\begin{aligned}
d(x_n, x_m) &= d(H(x_n, \lambda_n), H(x_m, \lambda_m)) \\
&\le d(H(x_n, \lambda_n), H(x_n, \lambda)) + d(H(x_n, \lambda), H(x_m, \lambda)) \\
&\quad + d(H(x_m, \lambda), H(x_m, \lambda_m)) \\
&\le d(H(x_n, \lambda_n), H(x_n, \lambda)) + \alpha(x_n, x_m)\, M_H^\lambda(x_n, x_m) \\
&\quad + d(H(x_m, \lambda), H(x_m, \lambda_m)),
\end{aligned}$$

where

$$\begin{aligned}
M_H^\lambda(x_n, x_m) = \max\Big\{ d(x_n, x_m), \frac{1}{2}\big[d(x_n, H(x_n, \lambda)) + d(x_m, H(x_m, \lambda))\big], \\
\frac{1}{2}\big[d(x_n, H(x_m, \lambda)) + d(x_m, H(x_n, \lambda))\big] \Big\}.
\end{aligned}$$

To continue with the above chain of inequalities, just consider the following three possibilities for $M_H^\lambda(x_n, x_m)$:

Case 1. If $M_H^\lambda(x_n, x_m) = d(x_n, x_m)$, then (13) is obvious.

Case 2. If $M_H^\lambda(x_n, x_m) = \frac{1}{2}\big[d(x_n, H(x_n, \lambda)) + d(x_m, H(x_m, \lambda))\big]$,

$$\begin{aligned}
d(x_n, x_m) &\le d(H(x_n, \lambda_n), H(x_n, \lambda)) \\
&\quad + \frac{\alpha(x_n, x_m)}{2}\big[d(x_n, H(x_n, \lambda)) + d(x_m, H(x_m, \lambda))\big] \\
&\quad + d(H(x_m, \lambda), H(x_m, \lambda_m)) \\
&= \left(1 + \frac{\alpha(x_n, x_m)}{2}\right)\big[d(H(x_m, \lambda_m), H(x_m, \lambda)) \\
&\qquad + d(H(x_n, \lambda_n), H(x_n, \lambda))\big].
\end{aligned}$$

Case 3. If $M^\lambda_H(x_n, x_m) = \frac{1}{2}\big[d(x_n, H(x_m, \lambda)) + d(x_m, H(x_n, \lambda))\big]$,

$$\begin{aligned}
d(x_n, x_m) &\le d(H(x_n, \lambda_n), H(x_n, \lambda)) \\
&\quad + \frac{\alpha(x_n, x_m)}{2}\Big[d(x_n, H(x_m, \lambda)) + d(x_m, H(x_n, \lambda))\Big] \\
&\quad + d(H(x_m, \lambda), H(x_m, \lambda_m)) \\[1em]
&\le d(H(x_n, \lambda_n), H(x_n, \lambda)) \\
&\quad + \frac{\alpha(x_n, x_m)}{2}\Big[2d(x_n, x_m) + d(H(x_m, \lambda_m), H(x_m, \lambda)) \\
&\qquad\qquad\qquad + d(H(x_n, \lambda_n), H(x_n, \lambda))\Big] \\
&\quad + d(H(x_m, \lambda), H(x_m, \lambda_m)) \\[1em]
&= \alpha(x_n, x_m)\, d(x_n, x_m) \\
&\quad + \left(1 + \frac{\alpha(x_n, x_m)}{2}\right)\Big[d(H(x_m, \lambda_m), H(x_m, \lambda)) \\
&\qquad\qquad\qquad + d(H(x_n, \lambda_n), H(x_n, \lambda))\Big].
\end{aligned}$$

Hence, (13) is proved.

We claim that $\{x_n\}$ is a Cauchy sequence. Otherwise, there exist a positive constant δ and two subsequences of $\{x_n\}$, $\{x_{n_k}\}$ and $\{x_{m_k}\}$, such that $d(x_{n_k}, x_{m_k}) \ge \delta$ for all $k \in \mathbb{N}$. Consequently, if $M = \operatorname{diam} U$, we have that $\alpha(x_{n_k}, x_{m_k}) \le \theta(\delta, M)$, and then (13) leads to

$$\begin{aligned}
d(x_{n_k}, x_{m_k}) &\le \theta(\delta, M)\, d(x_{n_k}, x_{m_k}) \\
&\quad + \left(1 + \frac{\theta(\delta, M)}{2}\right)\Big[d(H(x_{m_k}, \lambda_{m_k}), H(x_{m_k}, \lambda)) \\
&\qquad\qquad + d(H(x_{n_k}, \lambda_{n_k}), H(x_{n_k}, \lambda))\Big],
\end{aligned}$$

and so

$$\begin{aligned}
\delta \le d(x_{n_k}, x_{m_k}) \le \frac{2 + \theta(\delta, M)}{2\big(1 - \theta(\delta, M)\big)}\Big[&d(H(x_{m_k}, \lambda_{m_k}), H(x_{m_k}, \lambda)) \\
&+ d(H(x_{n_k}, \lambda_{n_k}), H(x_{n_k}, \lambda))\Big].
\end{aligned} \tag{14}$$

Since, by (P3), $d(H(x_{m_k}, \lambda_{m_k}), H(x_{m_k}, \lambda)) \to 0$ as $k \to \infty$, we reach a contradiction from (14). Hence, $\{x_n\}$ is a Cauchy sequence. Write $x_0 = \lim x_n$ and let us see that $x_0 \in U$ and also that $x_0 = H(x_0, \lambda)$. That $x_0 = H(x_0, \lambda)$ is a consequence of the following relation:

$$
\begin{aligned}
d(x_n, H(x_0,\lambda)) &\le d(x_n, H(x_n,\lambda)) + d(H(x_n,\lambda), H(x_0,\lambda)) \\
&\le d(H(x_n,\lambda_n), H(x_n,\lambda)) \\
&\quad + \max\Big\{ d(x_n,x_0), \frac{1}{2}\big[d(x_0,H(x_0,\lambda)) + d(x_n,H(x_n,\lambda))\big], \\
&\qquad\qquad \frac{1}{2}\big[d(x_0,H(x_n,\lambda)) + d(x_n,H(x_0,\lambda))\big] \Big\} \\
&\le d(H(x_n,\lambda_n), H(x_n,\lambda)) \\
&\quad + \max\Big\{ d(x_n,x_0), \frac{1}{2}\big[d(x_0,H(x_0,\lambda)) + d(x_n,H(x_n,\lambda))\big], \\
&\qquad\qquad \frac{1}{2}\big[d(x_0,x_n) + d(H(x_n,\lambda_n),H(x_n,\lambda)) \\
&\qquad\qquad\quad + d(x_n,H(x_0,\lambda))\big] \Big\},
\end{aligned}
$$

and that $x_0 \in U$ is straightforward from (P1).

We now turn to prove that A is open in $[0,1]$. Suppose that $\lambda_0 \in A$ and let us show that $(\lambda_0 - \delta, \lambda_0 + \delta) \cap [0,1] \subset A$, for some $\delta > 0$. Since $\lambda_0 \in A$, there exists $x_0 \in U$ with $x_0 = H(x_0, \lambda_0)$. Consider $r > 0$ with $\overline{B(x_0,r)} \subset U$ and use (P3) to obtain $\delta > 0$ such that

$$
d(H(x_0,\lambda_0), H(x_0,\lambda)) < \frac{1}{3}\min\left\{\frac{r}{2}, r\left[1-\theta\left(\tfrac{r}{2}, r\right)\right]\right\}
$$

for all $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta) \cap [0,1]$.

To show know that any $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta) \cap [0,1]$ is also in A, it suffices to prove that the map $H(\cdot,\lambda) : \overline{B(x_0,r)} \to X$ has a fixed point. This follows by Corollary 29, since

$$
\begin{aligned}
d(x_0, H(x_0,\lambda)) &= d(H(x_0,\lambda_0), H(x_0,\lambda)) \\
&< \frac{1}{3}\min\left\{\frac{r}{2}, r\left[1-\theta\left(\tfrac{r}{2}, r\right)\right]\right\}.
\end{aligned}
$$

□

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