SHARP BOUNDS FOR GENERAL COMMUTATORS ON WEIGHTED LEBESGUE SPACES

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ABSTRACT. We show that if a linear operator T is bounded on weighted Lebesgue space $L^2(w)$ and obeys a linear bound with respect to the A_2 constant of the weight, then its commutator $[b, T]$ with a function b in BMO will obey a quadratic bound with respect to the A_2 constant of the weight. We also prove that the kth-order commutator $T_b^k = [b, T_b^{k-1}]$ will obey a bound that is a power $(k + 1)$ of the A_2 constant of the weight. Sharp extrapolation provides corresponding $L^p(w)$ estimates. In particular these estimates hold for T any Calderón-Zygmund singular integral operator. The results are sharp in terms of the growth of the operator norm with respect to the A_p constant of the weight for all $1 < p < \infty$, all k, and all dimensions, as examples involving the Riesz transforms, power functions and power weights show.

1. INTRODUCTION

Singular integral operators are known to be bounded in weighted Lebesgue spaces $L^p(w)$ if the weight belongs to the A_p class of Muckenhoupt.

Recently there has been renewed interest in understanding the dependence of the operator norm in terms of the A_p constant of the weight, more precisely one seeks estimates of the type,

$$\|T\|_{L^p(w)} \leq \varphi_p([w]_{A_p}) \qquad 1 < p < \infty,$$

where the function $\varphi_p : [1, \infty) \to [0, \infty)$ is optimal in terms of growth. The first result of this type was obtained by S. Buckley [4] who showed that the maximal function obeyed such estimates with $\varphi_p(t) = c_p t^{\frac{1}{p-1}}$ for $1 < p < \infty$, and this is optimal (see [21] for another recent proof). This problem has attracted renewed attention because of the work of Astala, Iwaniec and Saksman [2]. They proved sharp regularity results for solutions to the Beltrami equation, assuming that the operator norm of the Beurling-Ahlfors transform grows linearly in terms of the A_p constant for $p \geq 2$. This linear growth was proved by S. Petermichl and A. Volberg [32] and by Petermichl [30, 31] for the Hilbert transform and the Riesz Transforms. In these papers it has been shown that if T is any of these operators, then

$$(1.1) \qquad \|T\|_{L^p(w)} \leq c_{p,n} [w]_{A_p}^{\max\left\{1, \frac{1}{p-1}\right\}} \qquad 1 < p < \infty,$$

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and the exponent $\max\left\{1, \frac{1}{p-1}\right\}$ is best possible. It has been conjectured, and very recently proved [16], that the same estimate holds for any Calderón-Zygmund operator T. By the sharp version of the Rubio de Francia extrapolation theorem [12], it suffices to prove this inequality for $p = 2$, namely

(1.2) $\|T\|_{L^2(w)} \leq c_n\,[w]_{A_2}.$

We remit the reader to [10] for a new proof and for generalizations and applications. The linear growth in $L^2(w)$ has been shown to hold for dyadic operators (martingale transform [34], dyadic square function [15], dyadic paraproduct [3]), or for operators who have lots of symmetries and can be written as averages of dyadic shift operators (such us the Hilbert transform [30], Riesz transforms [31], Beurling transform [32], [13]). All these estimates were obtained using Bellman functions. Recently all the above results have been recovered using different sets of techniques, and obtaining linear bounds for a larger class of Haar shift operators [20], [8]. In particular, in the latter paper [8], no Bellman function techniques nor any two weight results are used, and the methods can be extended to other important operators in Harmonic Analysis such as dyadic square functions and paraproducts, maximal singular integrals and the vector-valued maximal function as can be found in [9]. The sharp bound (1.2) for any Calderón-Zygmund operator T has been proved in [16] by T. Hytönen. Hytönen's proof is based on approximating T by generalized dyadic Haar shift operators with good bounds combined with the key fact that to prove (1.2) it is enough to prove the corresponding weak type $(2, 2)$ estimate with the same linear bound as proved in [29]. A direct proof avoiding this weak $(2, 2)$ reduction can be found in [17]. A bit earlier, in [22], the sharp $L^p(w)$ bound for T was obtained for values of p outside the interval $(3/2, 3)$ and the proof is based on the corresponding estimates for the intrinsic square functions.

It should be mentioned that until the A_2-conjecture was proved, only the following special case

(1.3) $\|T\|_{L^p(w)} \leq C_p\,[w]_{A_1}   1 < p < \infty,$

had been shown to be true for any Calderón-Zygmund operator. Observe that the condition imposed on the weight is the A_1 weight condition which is stronger than A_p but there is a gain in the exponent since it is linear for any $1 < p < \infty$ (compare with (1.1)). This has been shown in [23, 24] and we remit the reader to [28] for a survey on this topic.

The main purpose of this paper is to prove estimates similar to (1.1) for commutators of appropriate linear operators T with BMO functions b. These operators are defined formally by the expression

$$[b, T]f = bT(f) - T(b\,f).$$

When T is a singular integral operator, these operators were considered by Coifman, Rochberg and Weiss in [7]. Although the original interest in the study of such operators

was related to generalizations of the classical factorization theorem for Hardy spaces many other applications have been found.

The main result from [7] states that $[b, T]$ is a bounded operator on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, when b is a BMO function and T is a singular integral operator. In fact, the BMO condition of b is also a necessary condition for the L^p-boundedness of the commutator when T is the Hilbert transform. Later on a different proof was given by J. O. Strömberg (cf. [33] p.417) with the advantage that it allows to show that these commutators are also bounded on weighted $L^p(w)$, when $w \in A_p$. This approach is based on the use of the classical C. Fefferman-Stein maximal function and it is not precise enough for further developments. Indeed, we may think that these operators behave as Calderón-Zygmund operators, however there are some differences. For instance, an interesting fact is that, unlike what it is done with singular integral operators, the proof of the L^p-boundedness of the commutator does not rely on a weak type $(1, 1)$ inequality. In fact, simple examples show that in general $[b, T]$ fails to be of weak type $(1, 1)$ when $b \in BMO$. This was observed by the third author in [26] where it is also shown that there is an appropriate weak-$L(\log L)$ type estimate replacement. This shows that the operator cannot be a Calderón-Zygmund singular integral operator. To stress this point of view it is also shown by the third author [27] that the right operator controlling $[b, T]$ is $M^2 = M \circ M$, instead of the Hardy-Littlewood maximal function M.

In the present paper we pursue this point of view by showing that commutators have an extra "bad" behavior from the point of view of the A_p theory of weights that it is not reflected in the classical situation. Our argument will be based on the second proof for the L^p-boundedness of the commutator presented in [7]. This proof is interesting because there is no need to assume that T is a singular integral operator, to show the boundedness of the commutator it is enough to assume that the operator T is linear and bounded on $L^p(w)$ for any $w \in A_p$. These ideas were exploited in [1].

The first author showed in [5] that the commutator with the Hilbert transform obeys an estimate of the following type,

(1.4) $$\|[b, H]\|_{L^2(w)} \leq C \, [w]^2_{A_2} \|b\|_{BMO},$$

and he also showed that the quadratic growth with respect to the A_2 constant of the weight is sharp. The techniques used in that paper rely very much in dyadic considerations and the use of Bellman function arguments. Using recent results on Haar shifts operators, he also deduced the quadratic growth for commutators of Haar shift operators and operators in their convex hull, including the Riesz transforms and the Beurling-Ahlfors operator, see [5] for the

Also, it should be mentioned that there is a corresponding version of (1.3) for commutators of any Calderón-Zygmund operator with quadratic growth as in (1.4). This is proved in [25] where an endpoint estimate can also be found.

By completely different methods, we show in this paper that if an operator obeys an initial linear bound in $L^2(w)$, then its commutator will obey a quadratic bound in $L^2(w)$. In light of the positive resolution of the A_2-conjecture, we conclude that the

commutator of any Calderón-Zygmund singular integral operator and a BMO function obeys a quadratic bound in $L^2(w)$. In fact we show that if an operator T obeys a bound in $L^2(w)$ of the form $\varphi([w]_{A_2})$, then its k-th order commutator with $b \in BMO$, $T_b^k := [b, T_b^{k-1}]$, will obey a bound of the form $c_n^k k! \varphi(\gamma_n [w]_{A_2})[w]_{A_2}^k \|b\|_{BMO}^k$. Observe that if we consider the special case of Calderón-Zygmund operators with kernel K then

$$T_b^k(f)(x) \int_{\mathbb{R}^n} (b(x) - b(y))^k K(x, y) f(y)\, dy,$$

and the larger k is, the more singular the operator will be, because the exponent in $[w]_{A_2}^k$ becomes larger.

Corresponding estimates in $L^p(w)$ are deduced by the sharp version of the Rubio de Francia extrapolation theorem found in [12], and are shown to be sharp in the case of the Hilbert and Riesz transforms (in any dimension) for all $1 < p < \infty$, and for all $k \geq 1$.

It will be interesting to recover the result for the higher order commutators with the Haar shift operators (and hence for the Hilbert, Riesz and Beurling transforms) using the dyadic methods, but so far we do not know how to do this.

Recently extensions of our result to two weight settings, fractional integrals and more have been obtained by D. Cruz-Uribe and Kabe Moen, see [11].

The remainder of this paper is organized as follows. In Section 2 we gather some basic results. In Section 3 we give the proof of the main result. In Section 4 we show with examples that the main theorem in the paper, and its corollaries are sharp. Finally, the last section is an appendix where we show a result that it is claimed but never proved in the literature, a sharp reverse Hölder's inequality for A_2 weights.

2. Preliminary results

2.1. **A Sharp John-Nirenberg.** For a locally integrable $b : \mathbb{R}^n \to \mathbb{R}$ we define

$$\|b\|_{BMO} = \sup_Q \frac{1}{|Q|} \int_Q |b(y) - b_Q|\, dy < \infty,$$

where the supremum is taken over all cubes $Q \in \mathbb{R}^n$ with sides parallel to the axes, and

$$b_Q = \frac{1}{|Q|} \int_Q b(y)\, dy.$$

The main relevance of BMO is because of its exponential self-improving property, recorded in the celebrated John-Nirenberg Theorem [18]. We need a very precise version of it, as follows:

Theorem 2.1. *[Sharp John-Nirenberg] There are dimensional constants $0 \leq \alpha_n < 1 < \beta_n$ such that*

$$(2.1) \qquad \sup_Q \frac{1}{|Q|} \int_Q \exp\left(\frac{\alpha_n}{\|b\|_{BMO}} |b(y) - b_Q| \right) dy \leq \beta_n.$$

In fact we can take $\alpha_n = \frac{1}{2^{n+2}}$.

For the proof of this we remit to p. 31-32 of [19] where a proof different from the standard one can be found.

We derive from Theorem 2.1 the following Lemma 2.2. that will be used in the proof of the main theorem. First recall that a weight w satisfies the A_2 condition if

$$[w]_{A_2} = \sup_Q \left(\frac{1}{|Q|} \int_Q w \right) \left(\frac{1}{|Q|} \int_Q w^{-1} \right) < \infty,$$

where the supremum is taken over all cubes $Q \in \mathbb{R}^n$ with sides parallel to the axes. Notice that $[w]_{A_2} \geq 1$.

It is well known that if $w \in A_2$ then $b = \log w \in BMO$. A partial converse also holds, if $b \in BMO$ there is an $s_0 > 0$ such that $w = e^{sb} \in A_p$, $|s| \leq s_0$. As a consequence of the Sharp John-Nirenberg Theorem we can get a more precise version of this partial converse.

Lemma 2.2. *Let* $b \in BMO$ *and let* $\alpha_n < 1 < \beta_n$ *be the dimensional constants from* (2.1). *Then*

$$s \in \mathbb{R}, \quad |s| \leq \frac{\alpha_n}{\|b\|_{BMO}} \implies e^{sb} \in A_2 \quad and \quad [e^{sb}]_{A_2} \leq \beta_n^2.$$

Proof. By Theorem 2.1, if $|s| \leq \frac{\alpha_n}{\|b\|_{BMO}}$ and if Q is fixed

$$\frac{1}{|Q|} \int_Q \exp(|s||b(y) - b_Q|) \, dy \leq \frac{1}{|Q|} \int_Q \exp(\frac{\alpha_n}{\|b\|_{BMO}}|b(y) - b_Q|) \, dy \leq \beta_n,$$

thus

$$\frac{1}{|Q|} \int_Q \exp(s(b(y) - b_Q)) \, dy \leq \beta_n,$$

and

$$\frac{1}{|Q|} \int_Q \exp(-s(b(y) - b_Q)) \, dy \leq \beta_n.$$

If we multiply the inequalities, the b_Q parts cancel out:

$$\left(\frac{1}{|Q|} \int_Q \exp(s(b(y) - b_Q)) \, dy \right) \left(\frac{1}{|Q|} \int_Q \exp(s(b_Q - b(y))) \, dy \right)$$

$$= \left(\frac{1}{|Q|} \int_Q \exp(sb(y)) \, dy \right) \left(\frac{1}{|Q|} \int_Q \exp(-sb(y)) \, dy \right) \leq \beta_n^2$$

namely $e^{sb} \in A_2$ with

$$[e^{sb}]_{A_2} \leq \beta_n^2.$$

$\square$

We remark that it follows easily from minor modifications to the proof of Lemma 2.2 that if $1 < p < \infty$

$$s \in \mathbb{R}, \quad |s| \leq \frac{\alpha_n}{\|b\|_{BMO}} \min\left\{ 1, \frac{1}{p-1} \right\} \implies e^{sb} \in A_p \quad and \quad [e^{sb}]_{A_p} \leq \beta_n^p,$$

where as usual

$$[w]_{A_p} = \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)}dx \right)^{p-1} < \infty.$$

2.2. Sharp reverse Hölder inequality for the A_2 class of weights.

Recall that if $w \in A_2$ then w satisfies a reverse Hölder condition, namely, there are constants $r > 1$ and $c \geq 1$ such that for any cube Q

$$(2.2) \qquad \left(\frac{1}{|Q|} \int_Q w^r dx \right)^{\frac{1}{r}} \leq \frac{c}{|Q|} \int_Q w$$

In the usual proofs, both constants, c and r, depend upon the A_2 constant of the weight. There is a more precise version of (2.2).

Lemma 2.3. *Let $w \in A_2$ and let $r_w = 1 + \frac{1}{2^{n+5}[w]_{A_2}}$. Then*

$$\left(\frac{1}{|Q|} \int_Q w^{r_w} dx \right)^{\frac{1}{r_w}} \leq \frac{2}{|Q|} \int_Q w$$

This result was stated and used in [4] but no proof was given. The author mentioned instead the celebrated work [6] where no explicit statement can be found. We supply in Section 5 a proof taken from [28], where a more general version can be found as well as more information.

3. MAIN RESULT

Theorem 3.1. *Let T be a linear operator bounded on $L^2(w)$ for any $w \in A_2$. Suppose further that there is an increasing function $\varphi : [1, \infty) \to [0, \infty)$ such that*

$$(3.1) \qquad \|T\|_{L^2(w)} \leq \varphi([w]_{A_2}).$$

then there are constants γ_n and c_n independent of $[w]_{A_2}$ such that

$$(3.2) \qquad \|[b, T]\|_{L^2(w)} \leq c_n \, \varphi(\gamma_n \, [w]_{A_2}) \, [w]_{A_2} \|b\|_{BMO}.$$

For the particular case $\varphi(t) = c_0 t^r$, where $r > 0$, and $c_0 > 0$, a simple induction argument shows that if

$$\|T\|_{L^2(w)} \leq a_0 \, [w]_{A_2}^r,$$

then for each integer $k \geq 1$ there is a constant a_k depending on k, the initial value a_0, and the parameters γ_n and c_n in the theorem, such that the kth-order commutator T_b^k defined recursively by $T_b^k := [b, T_b^{k-1}]$, obeys the following weighted estimates

$$\|T_b^k\|_{L^2(w)} \leq a_k \, [w]_{A_2}^{r+k} \|b\|_{BMO}^k.$$

More precisely, the sequence $\{a_k\}_{k \geq 0}$ obeys the following recurrence equation that can be solved easily,

$$a_k = c_n \, a_{k-1} \, \gamma_n^{r+k-1} = c_n^k \, a_0 \, \gamma_n^{kr + \frac{(k-2)(k-1)}{2}}.$$

Using the method of proof of Theorem 3.1 we can obtain a weighted estimate for the kth-order commutator that works for general increasing function $\varphi : [1, \infty) \to [0, \infty)$. Notice the difference in the constants with what we just argued by induction for the particular case $\varphi(t) = a_0 t^r$: in the corollary the constant is $c_n^k \, a_0 \, \gamma_n^r \, k!$, whereas in the induction argument the constant is $c_n^k \, a_0 \, \gamma_n^{kr + \frac{(k-2)(k-1)}{2}}$.

Corollary 3.2. *Let T be a bounded linear operator on $L^2(w)$ with $w \in A_2$ and*

(3.3) $$\|T\|_{L^2(w)} \leq \varphi([w]_{A_2}).$$

then there are constants γ_n and c_n independent of $[w]_{A_2}$ such that

(3.4) $$\|T_b^k\|_{L^2(w)} \leq c_n^k \, k! \, \varphi(\gamma_n [w]_{A_2}) \, [w]_{A_2}^k \|b\|_{BMO}^k.$$

The constants γ_n and c_n that appear in Corollary 3.2 are the same that appeared in Theorem 3.1. We first present the proof of Theorem 3.1, and afterwards we discuss the necessary modifications to obtain Corollary 3.2. As an easy consequence of Corollary 3.2 and the Rubio de Francia extrapolation theorem with sharp constants [12], we have the following.

Corollary 3.3. *Let T be a linear operator bounded on $L^2(w)$ with $w \in A_2$ and*

(3.5) $$\|T\|_{L^2(w)} \leq \varphi([w]_{A_2}).$$

Then, for $1 < p < \infty$, there are constants $\gamma_{n,p}$, and $c_{n,p}$, which only depend on p, and the dimension n, such that for all weights $w \in A_p$

(3.6) $$\|T_b^k\|_{L^p(w)} \leq \sqrt{2} \, c_{n,p}^k \, k! \, \varphi\left(\gamma_{n,p} \, [w]_{A_p}^{\max\{1, \frac{1}{p-1}\}}\right) \, [w]_{A_p}^{k \max\{1, \frac{1}{p-1}\}} \|b\|_{BMO}^k.$$

In the particular case $\varphi(t) = a_0 t^r$ the extrapolated estimate looks like

(3.7) $$\|T_b^k\|_{L^p(w)} \leq \sqrt{2} \, a_0 \, c_n^k \, k! \, \gamma_n^r \, c_p^{r+k} [w]_{A_p}^{(r+k) \max\{1, \frac{1}{p-1}\}} \|b\|_{BMO}^k.$$

where c_p depends only on p, γ_n and c_n are the constants that appeared in Theorem 3.1.

We will show in Section 4 that for $r = 1$, $\varphi(t) = a_0 t$ the power $(1 + k) \max\{1, \frac{1}{p-1}\}$ cannot be decreased for $T = H$ and $T = R_j$ the Hilbert and Riesz transforms, for all $k \geq 1$ and $p > 1$, therefore the theorem is optimal in terms of the rate of the dependence on $[w]_{A_p}$. In [5], examples for $k = 1$, for all $p > 1$, and for T the Hilbert, Beurling and Riesz transforms, were presented.

Proof of Theorem 3.1. We "conjugate" the operator as follows: if z is any complex number we define

$$T_z(f) = e^{zb} T(e^{-zb} f).$$

Then, a computation gives (for instance for "nice" functions),

$$[b, T](f) = \frac{d}{dz} T_z(f)|_{z=0} = \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{T_z(f)}{z^2} \, dz, \qquad \epsilon > 0$$

by the Cauchy integral theorem, see [7], [1].

Now, by Minkowski's inequality

$$(3.8) \qquad \|[b,T](f)\|_{L^2(w)} \leq \frac{1}{2\pi\,\epsilon^2} \int_{|z|=\epsilon} \|T_z(f)\|_{L^2(w)}|dz| \qquad \epsilon > 0.$$

The key point is to find the appropriate radius ϵ. To do this we look at the inner norm $\|T_z(f)\|_{L^2(w)}$

$$\|T_z(f)\|_{L^2(w)} = \|T(e^{-zb}f)\|_{L^2(we^{2Rez\,b})},$$

and try to find appropriate bounds on z. To do this we use the main hypothesis, namely that T is bounded on $L^2(w)$ if $w \in A_2$ with

$$\|T\|_{L^2(w)} \leq \varphi([w]_{A_2}).$$

Hence we should compute

$$[we^{2Rez\,b}]_{A_2} = \sup_Q \left(\frac{1}{|Q|} \int_Q we^{2Rez\,b(x)}\,dx \right) \left(\frac{1}{|Q|} \int_Q w^{-1}e^{-2Rez\,b(x)}\,dx \right).$$

Now, since $w \in A_2$ we use Lemma 2.3: if $r = r_w = 1 + \frac{1}{2^{n+5}[w]_{A_2}} < 2$ then

$$\left(\frac{1}{|Q|} \int_Q w^r dx \right)^{\frac{1}{r}} \leq \frac{2}{|Q|} \int_Q w,$$

and similarly for w^{-1} since $r_w = r_{w^{-1}}$,

$$\left(\frac{1}{|Q|} \int_Q w^{-r} dx \right)^{\frac{1}{r}} \leq \frac{2}{|Q|} \int_Q w^{-1}.$$

Using this and Holder's inequality we have for an arbitrary Q

$$\left(\frac{1}{|Q|} \int_Q w(x)e^{2Rez\,b(x)}\,dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1}e^{-2Rez\,b(x)}\,dx \right) \leq$$

$$\left(\frac{1}{|Q|} \int_Q w^r dx \right)^{\frac{1}{r}} \left(\frac{1}{|Q|} \int_Q e^{2Rez\,r'\,b(x)}\,dx \right)^{\frac{1}{r'}} \left(\frac{1}{|Q|} \int_Q w^{-r} dx \right)^{\frac{1}{r}} \left(\frac{1}{|Q|} \int_Q e^{-2Rez\,r'\,b(x)}\,dx \right)^{\frac{1}{r'}}$$

$$\leq 4 \left(\frac{1}{|Q|} \int_Q w\,dx \right) \left(\frac{1}{|Q|} \int_Q w^{-1}\,dx \right) \left(\frac{1}{|Q|} \int_Q e^{2Rez\,r'\,b(x)}\,dx \right)^{\frac{1}{r'}} \left(\frac{1}{|Q|} \int_Q e^{-2Rez\,r'\,b(x)}\,dx \right)^{\frac{1}{r'}}$$

$$\leq 4\,[w]_{A_2}\,[e^{2Rez\,r'\,b}]_{A_2}^{\frac{1}{r'}}$$

Now, since $b \in BMO$ we are in a position to apply Lemma 2.2,

$$\text{if} \quad |2Rez\,r'| \leq \frac{\alpha_n}{\|b\|_{BMO}} \qquad \text{then} \qquad [e^{2Rez\,r'\,b}]_{A_2} \leq \beta_n^2.$$

Hence for these z, and since $1 < r < 2$,

$$[we^{2Rez\,b}]_{A_2} \leq 4\,[w]_{A_2}\,\beta_n^{\frac{2}{r'}} \leq 4\,[w]_{A_2}\,\beta_n.$$

Using this estimate for these z, and observing that $\|e^{-zb}f\|_{L^2(we^{2Rezb})} = \|f\|_{L^2(w)}$,

$$\|T_z(f)\|_{L^2(w)} = \|T(e^{-zb}f)\|_{L^2(we^{2Rez\,b})} \leq \varphi([we^{2Rez\,b}]_{A_2})\|f\|_{L^2(w)} \leq \varphi(4[w]_{A_2}\,\beta_n)\,\|f\|_{L^2(w)}.$$

Choosing now the radius

$$\epsilon = \frac{\alpha_n}{2r'\|b\|_{BMO}},$$

we can continue estimating the norm in (3.8)

$$\|[b,T](f)\|_{L^2(w)} \le \frac{1}{2\pi\,\epsilon^2} \int_{|z|=\epsilon} \|T_z(f)\|_{L^2(w)} |dz|$$

$$\le \frac{1}{2\pi\,\epsilon^2} \int_{|z|=\epsilon} \varphi(4[w]_{A_2}\,\beta_n)\,\|f\|_{L^2(w)} |dz| = \frac{1}{\epsilon}\,\varphi(4[w]_{A_2}\,\beta_n)\,\|f\|_{L^2(w)},$$

since

$$|2Re z\,r'| \le 2|z|\,r' = 2\epsilon\,r' = \frac{\alpha_n}{\|b\|_{BMO}}.$$

Finally, for this ϵ,

$$\|[b,T](f)\|_{L^2(w)} \le C2^{2n}\,\varphi(4[w]_{A_2}\,\beta_n)\,[w]_{A_2}\,\|b\|_{BMO},$$

because $r' = 1 + 2^{n+5}[w]_2 \approx 2^n[w]_2$, and $\alpha_n = \frac{1}{2^{n+2}}$.

Observe that the optimal radius is essentially the inverse of $[w]_2\|b\|_{BMO}$. This proves the theorem with $c_n \sim 2^{2n}$ and $\gamma_n = 4\,\beta_n$.

□

Proof of Corollary 3.2. In this case a computation gives (for instance for "nice" functions), see [1] for example or the original paper [7],

$$T_b^k(f) = \frac{d^k}{dz^k}T_z(f)|_{z=0} = \frac{k!}{2\pi i}\int_{|z|=\epsilon}\frac{T_z(f)}{z^{k+1}}\,dz \qquad \epsilon > 0$$

by the Cauchy integral theorem. The same calculation as in the case $k = 1$ gives the required estimate, with $c_n^k \approx 2^{2nk}$, and the same $\gamma_n = 4\beta_n$.

□

4. EXAMPLES

In this section, we show that one can not have estimates better than Theorem 3.1, Corollary 3.2, and Corollary 3.3. We present examples which return the same growth with respect to the A_p constant of the weight that appears in our results. First, we discuss the simpler case in dimension one. The following example shows that the quadratic estimate for the first commutator of the Hilbert transform is sharp for $p = 2$.

4.1. Sharp example for the commutator of the Hilbert transform. Consider the Hilbert transform

$$Hf(x) = p.v. \int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy,$$

and consider the BMO function $b(x) = \log|x|$. We know that there is a constant c such that

(4.1) $$\|[b,H]\|_{L^2(w)} \le c\,[w]_{A_2}^2$$

and we show that the result is sharp. More precisely, for any increasing function $\phi : [1, \infty) \to [0, \infty)$ such that $\lim\limits_{t \to \infty} \dfrac{t^2}{\phi(t)} = \infty$ then

$$(4.2) \qquad \sup_{w \in A_2} \frac{1}{\phi([w]_2)} \|[b, T]\|_{L^2(w)} = \infty.$$

In particular if $\phi(t) = t^{2-\epsilon}$ for any $\epsilon > 0$, then (4.2) must hold.

For $0 < \delta < 1$, we let $w(x) = |x|^{1-\delta}$ and it is easy to see that $[w]_{A_2} \sim 1/\delta$. We now consider the function

$$f(x) = x^{-1+\delta} \chi_{(0,1)}(x)$$

and observe that f is in $L^2(w)$ with $\|f\|_{L^2(w)} = 1/\sqrt{\delta}$. To estimate the $L^2(w)$-norm of $[b, H]f$, we claim

$$|[b, H]f(x)| \geq \frac{1}{\delta^2} f(x)$$

and hence

$$\|[b, H]f\|_{L^2(w)} \geq \frac{1}{\delta^2} \|f\|_{L^2(w)}$$

from which the sharpness (4.2) will follow.

We now prove the claim: if $0 < x < 1$,

$$[b, H]f(x) = \int_0^1 \frac{\log(x) - \log(y)}{x - y} y^{-1+\delta} \, dy = \int_0^1 \frac{\log(\frac{x}{y})}{x - y} y^{-1+\delta} \, dy$$

$$= x^{-1+\delta} \int_0^{1/x} \frac{\log(\frac{1}{t})}{1 - t} t^{-1+\delta} \, dt$$

Now,

$$\int_0^{1/x} \frac{\log(\frac{1}{t})}{1 - t} t^{-1+\delta} \, dt = \int_0^1 \frac{\log(\frac{1}{t})}{1 - t} t^{-1+\delta} \, dt + \int_1^{1/x} \frac{\log(\frac{1}{t})}{1 - t} t^{-1+\delta} \, dt$$

and since $\frac{\log(\frac{1}{t})}{1-t}$ is positive for $(0, 1) \cup (1, \infty)$ we have for $0 < x < 1$

$$|[b, H]f(x)| > x^{-1+\delta} \int_0^1 \frac{\log(\frac{1}{t})}{1 - t} t^{-1+\delta} \, dt.$$

But since

$$\int_0^1 \frac{\log(\frac{1}{t})}{1 - t} t^{-1+\delta} \, dt > \int_0^1 \log(\frac{1}{t}) t^{-1+\delta} \, dt = \int_0^\infty s \, e^{-s\delta} \, ds = \frac{1}{\delta^2}$$

and the claim

$$|[b, H]f(x)| \geq \frac{1}{\delta^2} f(x)$$

follows. One can find this example and similar examples which show the commutators with the Beurling-Ahlfors operator and the Riesz transforms obey the quadratic growth in [5].

4.2. Sharp example for the kth-order commutator with the Riesz transforms.

We now consider the higher order commutators with degree $k \geq 1$ in the case of the j-th directional Riesz transform on $\mathbb{R}^n$:

$$R_{j,b}^k f(x) := p.v. \int_{\mathbb{R}^n} \frac{(x_j - y_j)(b(x) - b(y))^k f(y)}{|x - y|^{n+1}} \, dy.$$

To demonstrate the sharpness of the estimate for the higher order commutator with the Riesz transforms, we show sharpness for $1 < p \leq 2$. Then we can extend the sharpness for all $1 < p < \infty$ by using a duality argument, because $R_j^* = -R_j$, so the higher order commutators of the Riesz transforms are almost self-adjoint operators. For $1 < p \leq 2$, we consider weights $w(x) = |x|^{(n-\delta)(p-1)}$, $f(x) = |x|^{\delta-n}\chi_E(x)$ where $E = \{y | y \in (0,1)^n \cap B(0,1)\}$, $b(x) = \log|x|$, and evaluate $L^p(w)$-norm over $\Omega = \{x \in B(0,1)^c \, | \, x_i < 0 \text{ for all } i = 1, 2, ..., n\}$. Note that, for all $y \in E$ and $x \in \Omega$,

$$|x_j - y_j| \geq |x_j| \text{ and } |x - y| \leq |y| + |x|.$$

Then for $x \in \Omega$,

$$\left| R_{j,b}^k f(x) \right| = \left| \int_E \frac{(x_j - y_j)(\log|x| - \log|y|)^k |y|^{\delta-n}}{|x - y|^{n+1}} \, dy \right|$$

$$= \int_E \frac{|x_j - y_j|(\log(|x|/|y|))^k |y|^{\delta-n}}{|x - y|^{n+1}} \, dy \geq |x_j| \int_E \frac{(\log(|x|/|y|))^k |y|^{\delta-n}}{(|y| + |x|)^{n+1}} \, dy$$

$$= |x_j| \int_{E \cap S^{n-1}} \int_0^1 \frac{(\log(|x|/r))^k r^{\delta-n} r^{n-1}}{(r + |x|)^{n+1}} \, dr d\sigma$$

$$= c \, |x_j| \int_0^{1/|x|} \frac{(\log(1/t))^k (t|x|)^{\delta-1}|x|}{(|x| + t|x|)^{n+1}} \, dt = \frac{c \, |x_j|}{|x|^{n+1-\delta}} \int_0^{1/|x|} \frac{(\log(1/t))^k t^{\delta-1}}{(1 + t)^{n+1}} \, dt$$

$$\geq \frac{c \, |x_j|}{|x|^{n+1-\delta}} \left(\frac{|x|}{|x| + 1} \right)^{n+1} \int_0^{1/|x|} (\log(1/t))^k t^{\delta-1} \, dt.$$

Note that the constant $c = c(n)$ is the surface measure of $E \cap S^{n-1}$, depends on the dimension only. On the other hand,

$$\int_0^{1/|x|} (\log(1/t))^k t^{\delta-1} \, dt = \int_{\log|x|}^\infty s^k e^{-\delta s} \, ds = -\frac{1}{\delta} e^{-\delta s} s^k \Big|_{\log|x|}^\infty + \frac{k}{\delta} \int_{\log|x|}^\infty e^{-\delta s} s^{k-1} \, ds$$

$$= \frac{1}{\delta} e^{-\delta \log|x|} (\log|x|)^k + \frac{k}{\delta} \int_{\log|x|}^\infty e^{-\delta s} s^{k-1} \, ds$$

$$= \frac{(\log|x|)^k}{\delta|x|^\delta} + \frac{k}{\delta} \int_{\log|x|}^\infty e^{-\delta s} s^{k-1} \, ds.$$

For $x \in \Omega$, $(\log|x|)^k/\delta|x|^\delta$ is positive, therefore after neglecting some positive terms and applying the integration by parts $k - 1$ times, we get

$$\int_0^{1/|x|} (\log(1/t))^k t^{\delta-1} \, dt \geq \frac{k}{\delta} \int_{\log|x|}^\infty e^{-\delta s} s^{k-1} \, ds$$

$$\geq \cdots \geq \frac{k!}{\delta^{k-1}} \int_{\log|x|}^{\infty} e^{-\delta s} s \, ds \geq \frac{k!}{\delta^{k+1}|x|^{\delta}} \, .$$

Combining the previous computations, we have that for $x \in \Omega$, and recalling that $f(x) = |x|^{\delta-n} \chi_E(x)$,

$$\left| R_{j,b}^k f(x) \right| \geq \frac{k! \, c \, |x_j|}{\delta^{k+1}(|x|+1)^{n+1}} \, .$$

Thus, we estimate

$$
\begin{aligned}
\left\| R_{j,b}^k f \right\|_{L^p(w)}^p &\geq \int_{\Omega} \left(\frac{k! \, c \, |x_j|}{\delta^{k+1}(|x|+1)^{n+1}} \right)^p |x|^{(n-\delta)(p-1)} \, dx \\
&= \left(\frac{k! \, c}{\delta^{k+1}} \right)^p \int_{\Omega} \frac{|x_j|^p |x|^{(n-\delta)(p-1)}}{(|x|+1)^{p(n+1)}} \, dx \\
&\geq \left(\frac{k! \, c}{\delta^{k+1}} \right)^p \int_{\Omega \cap S^{n-1}} \int_1^{\infty} \frac{\gamma_j^p r^p r^{(n-\delta)(p-1)} r^{n-1}}{(r+1)^{p(n+1)}} \, dr d\sigma(\gamma) \\
&= \left(\frac{k! \, c_{n,p}}{\delta^{k+1}} \right)^p \int_1^{\infty} r^{\delta(1-p)-1} \, dr = (k!)^p \frac{c_{n,p}^p}{p-1} \delta^{-p(k+1)-1} \, .
\end{aligned}
$$

Since $\|f\|_{L^p(w)}^p = 1/\delta$, and $[w]_{A_p} \sim 1/\delta^{p-1}$, we conclude that

$$\left\| R_{j,b}^k f \right\|_{L^p(w)} \geq k! \frac{c_{n,p}}{(p-1)^{1/p}} \, \delta^{-(k+1)-1/p} \sim k! \, [w]_{A_p}^{\frac{k+1}{p-1}} \|f\|_{L^p(w)} \, .$$

This shows that Theorem 3.1, Corollary 3.2, and Corollary 3.3 are sharp in the multi-dimensional case. The constant $c_{n,p} \to c_{n,1} > 0$ as $p \to 1$, therefore the estimate blows up as $p \to 1$, as it should since the operators are not bounded in $L^1(w)$.

5. Appendix: the sharp reverse Hölder's inequality for A_2 weights

In this section we give a proof of Lemma 2.3, namely if $w \in A_2$, then

(5.1) $$\left(\frac{1}{|Q|} \int_Q w^{r_w} dx \right)^{\frac{1}{r_w}} \leq \frac{2}{|Q|} \int_Q w,$$

where $r_w = 1 + \frac{1}{2^{5+n}[w]_{A_2}}$.

Proof. Let $w_Q = \frac{1}{|Q|} \int_Q w$ and $\delta > 0$

$$\frac{1}{|Q|} \int_Q w(x)^{1+\delta} \, dx = \frac{1}{|Q|} \int_Q w(x)^{\delta} \, w(x) dx = \frac{\delta}{|Q|} \int_0^{\infty} \lambda^{\delta} w(\{x \in Q : w(x) > \lambda\}) \frac{d\lambda}{\lambda}$$

$$= \frac{\delta}{|Q|} \int_0^{w_Q} \lambda^{\delta} w(\{x \in Q : w(x) > \lambda\}) \frac{d\lambda}{\lambda} + \frac{\delta}{|Q|} \int_{w_Q}^{\infty} \lambda^{\delta} w(\{x \in Q : w(x) > \lambda\}) \frac{d\lambda}{\lambda} = I + II.$$

Observe that $I \leq (w_Q)^{\delta+1}$, where $w_Q = \frac{w(Q)}{|Q|}$.

To estimate II we make two claims. The first is the following observation: if we let

$$E_Q = \{x \in Q : w(x) \leq \frac{1}{2[w]_{A_2}} w_Q\},$$

then

(5.2) $$|E_Q| \leq \frac{1}{2}|Q|.$$

Indeed, by (Cauchy-Schwartz) we have for any $f \geq 0$

$$\left(\frac{1}{|Q|} \int_Q f(y)\, dy\right)^2 w(Q) \leq [w]_{A_2} \int_Q f(y)^2\, w(y)dy,$$

and hence if $E \subset Q$, setting $f = \chi_E$,

$$\left(\frac{|E|}{|Q|}\right)^2 \leq [w]_{A_2} \frac{w(E)}{w(Q)}$$

and in particular, by definition of E_Q,

$$\left(\frac{|E_Q|}{|Q|}\right)^2 \leq [w]_{A_2} \frac{w(E_Q)}{w(Q)} \leq [w]_{A_2} \frac{w_Q}{w(Q)} |E_Q| \frac{1}{2[w]_{A_2}} = \frac{1}{2} \frac{|E_Q|}{|Q|},$$

from which the claim follows. In particular, this implies that

(5.3) $$|Q| \leq 2|Q \backslash E_Q| = 2|\{x \in Q : w(x) > \frac{1}{2[w]_{A_2}} w_Q\}|.$$

The second claim is the following

(5.4) $$w(\{x \in Q : w(x) > \lambda\}) \leq 2^{n+1}\lambda\, |\{x \in Q : w(x) > \frac{\lambda}{2[w]_{A_p}} w_Q\}| \qquad \lambda > w_Q.$$

Indeed, since $\lambda > w_Q$ to prove this claim we consider the standard (local) Calderón-Zygmund decomposition of w at level λ. Then there is a family of disjoint cubes $\{Q_i\}$ contained in Q satisfying

$$\lambda < w_{Q_i} \leq 2^n \lambda$$

for each i. Now, observe that except for a null set we have

$$\{x \in Q : w(x) > \lambda\} \subset \{x \in Q : M_Q^d w(x) > \lambda\} = \cup_i Q_i,$$

where M_Q^d is the dyadic maximal operator restricted to a cube Q. This together with (5.3) yields

$$w(\{x \in Q : w(x) > \lambda\}) \leq \sum_i w(Q_i)$$

$$\leq 2^n \lambda \sum_i |Q_i| \leq 2^{n+1}\lambda \sum_i |\{x \in Q_i : w(x) > \frac{1}{2[w]_{A_2}} w_{Q_i}\}|$$

$$\leq 2^{n+1}\lambda\, |\{x \in Q : w(x) > \frac{1}{2[w]_{A_2}} \lambda\}|$$

since $w_{Q_i} > \lambda$. This proves the second claim (5.4).

Now, combining

$$II = \frac{\delta}{|Q|} \int_{w_Q}^{\infty} \lambda^{\delta} w(\{x \in Q : w(x) > \lambda\}) \frac{d\lambda}{\lambda}$$

$$\leq \frac{2^{n+1}\delta}{|Q|} \int_{w_Q}^{\infty} \lambda^{\delta+1} |\{x \in Q : w(x) > \frac{1}{2[w]_{A_2}}\lambda\}| \frac{d\lambda}{\lambda}$$

$$\leq (2[w]_{A_2})^{1+\delta} 2^{n+1} \delta \frac{1}{|Q|} \int_{\frac{w_Q}{2[w]_{A_2}}}^{\infty} \lambda^{\delta+1} |\{x \in Q : w(x) > \lambda\}| \frac{d\lambda}{\lambda}$$

$$\leq (2[w]_{A_2})^{1+\delta} 2^{n+1} \frac{\delta}{1+\delta} \frac{1}{|Q|} \int_Q w^{1+\delta} \, dx.$$

Setting here $\delta = \frac{1}{2^{5+n}[w]_{A_2}}$, we obtain using that $t^{1/t} \leq 2$, $t \geq 1$

$$II \leq \frac{1}{2} \frac{1}{|Q|} \int_Q w^{\delta+1} dx$$

and finally

$$\frac{1}{|Q|} \int_Q w^{\delta+1} dx \leq 2(w_Q)^{\delta+1},$$

which proves (5.1).

$\square$

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