Revista de la
Unión Matemática Argentina
Volumen 37, 1991.

A POINTWISE ESTIMATE FOR THE KERNEL OF A PSEUDO-DIFFERENTIAL OPERATOR, WITH APPLICATIONS

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1. Introduction

Given a pseudo-differential operator L in the Hörmander class $L^m_{\rho,\delta}$, it is a classical result, (cf. [8]) that the distribution kernel $k(x,y)$ of L will be a C^∞ function away from the diagonal and will decay rapidly with all its derivatives as $|x-y| \longrightarrow \infty$, if $0<\rho\leq 1, 0\leq\delta<1$. Moreover, $k(x,y)$ will coincide with a C^j function in all $\mathbb{R}^n \times \mathbb{R}^n$, provided $m+n+j<0$. In [7], we completed the analysis, by proving a sharp estimate for $k(x,y)$, when $m+n+j\geq 0$. We used, however, a non standard partition of unity, which forced us to consider separately the cases $\rho = 1, 0<\rho<1$.

We give here a much simpler proof of that estimate, using a standard partition in dyadic rings. The rest of the paper is devoted to show several new applications of the estimates, to other integral estimates, weak type (p,q), etc. Specially, in Section 6 we combine this estimates with a pointwise estimate for a modified sharp maximal function, to obtain L^p weighted estimates with A_∞ weights for a class of oscillatory integrals closely related to those studied by D. Phong and E. Stein. Particularly, we have in this class pseudo-differential operators of order $\leq -(n+1)(1-\rho)$, which generalizes the weighted estimates obtained in [14].

The organization of the paper is as follows. In Section 2, we precise the definition of the class $L^m_{\rho,\delta}$ and state some classical properties for the distribution kernel. In Section 3 we complete the analysis by proving a sharp estimate when $m+n+j\geq 0$ for some j. In Section 4 an integral estimate in the spirit of the one considered in [17] is proved. This generalizes [15], Lemma 2.1. In Section 5, we prove a weak type $(1,q), q>1$ estimate in $L^m_{\rho,\delta}, m<-n(1-\rho)$, by estimating the operator in terms of a fractional integral I_α. Finally, Section 6 is mainly devoted to weighted estimates for a class of oscillatory integrals.

The notation used in this paper is the standard in the subject.

2. The class $L^m_{\rho,\delta}$

Following [3], we will denote by $S^m_{\rho,\delta}, m \in \mathbb{R}, 0\leq \rho,\delta\leq 1$, the class of $C^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ functions $p(x,\xi)$ such that

$$|D_x^\alpha D_\xi^\beta p(x,\xi)| \leq C(\alpha,\beta)(1+|\xi|)^{m-\rho|\beta|+\delta|\alpha|}$$

for every $\alpha, \beta \in \mathbb{N}^n$.

Given $p \in S^m_{\rho,\delta}$, we define the pseudo-differential operator L in the Hörmander class $L^m_{\rho,\delta}$ (cf. [8]), as

$$Lf(x) = \int e^{ix\cdot\xi} p(x,\xi)\hat{f}(\xi)d\xi, \quad f \in C_0^\infty(\mathbb{R}^n).$$

The operator L is uniquely defined by p, which is called for that reason the symbol of L.

L defines a linear and continuous operator from $C_0^\infty(\mathbb{R}^n)$ into $C^\infty(\mathbb{R}^n)$. Let $k(x,y) \in D'(\mathbb{R}^n \times \mathbb{R}^n)$ be its distribution kernel. It is defined by

$$(k, f \otimes g) = \int e^{ix\cdot\xi} p(x,\xi)\hat{f}(\xi)g(x)dxd\xi. \tag{2.1}$$

The following theorem summarizes several classical properties. We refer to [8] for proofs.

Theorem 2.1 *(cf. [8]). Let $L \in L^m_{\rho,\delta}, m \in \mathbb{R}, 0 \leq \delta, \rho \leq 1$, be a pseudo-differential operator with symbol $p(x,\xi)$ and let $k(x,y)$ be the distribution kernel of L. Assume that $0 < \rho$ and $\delta < 1$. Then,*

a). (Pseudo-local property). The distribution $k(x,y)$ coincides with a C^∞ function outside the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$. Moreover, given $\alpha, \beta \in \mathbb{N}^n$, there exists $N_0 \in \mathbb{N}$ such that for each $N \geq N_0$,

$$\sup_{x \neq y} |x-y|^N |D_x^\alpha D_y^\beta k(x,y)| < \infty$$

b). Suppose that the symbol $p(x,\xi)$ has compact support in ξ uniformly with respect to x. Then, the distribution $k(x,y)$ is a C^∞ function in $\mathbb{R}^n \times \mathbb{R}^n$ and given $\alpha, \beta \in \mathbb{N}^n, N \in \mathbb{N}$, we have

$$\sup_{\mathbb{R}^n \times \mathbb{R}^n} |x-y|^N |D_x^\alpha D_y^\beta k(x,y)| < \infty$$

c). Assume that $m+n+M < 0$, for some $M \in \mathbb{N}$. Then, the distribution $k(x,y)$ is a bounded continuous function with bounded continuous derivatives of order $\leq M$.

The case $m+n+M = 0$ for some $M \in \mathbb{N}$ is not considered in [8]. When $m+n+N = 0$ a logarithmic estimate can be proved and when $m+n+M > 0$, we obtain the sharp pointwise estimate we referred to in the introduction. Both cases can be obtained simultaneously as it will be proved in the next section.

Under the hypothesis of Theorem 2.1, the distribution kernel $k(x,y)$ can be written as a well defined oscillatory integral,

$$k(x,y)=\frac{1}{(2\pi)^n}\int e^{i(x-y)\cdot\xi}p(x,\xi)d\xi$$

if $x\neq y$.

This representation is obtained from (2.1) by repeated integrations by parts.

3. The pointwise estimate for the kernel

Theorem 3.1: *Let $L\in L^m_{\rho,\delta}, m\in\mathbb{R}, 0\leq\delta<1,\ 0<\rho\leq 1$ be a pseudo-differential operator with symbol $p(x,\xi)$ and let $k(x,y)$ be the distribution kernel of L. Then,*

a) if $m+n+M=0$ for some $M\in\mathbb{N}$, there exists $C>0$ such that

$$\sup_{|\alpha+\beta|=M}|D^\alpha_x D^\beta_y k(x,y)|\leq C|log|x-y||,\quad x\neq y.$$

b) if $m+n+M>0$ for some $M\in\mathbb{N}$, there exists $C>0$ such that

$$\sup_{|\alpha+\beta|=M}|D^\alpha_x D^\beta_y k(x,y)|\leq C|x-y|^{-(m+n+M)/\rho}\quad x\neq y. \tag{3.1}$$

Proof: We first observe that if $k(x,y)$ is the kernel of a pseudo-differential operator in $L^m_{\rho,\delta}$, it follows that $D^\alpha_x D^\beta_y k(x,y)$ is the kernel of a pseudo-differential operator of order $\leq m+|\alpha|+|\beta|$. Hence, it is enough to prove the above inequalities with $\alpha=\beta=0$.

Also, according to Theorem 2.1, parts a) and b), it sufices to estimate $k(x,y)$ for $0<|x-y|<1$ assuming that the symbol $p(x,\xi)$ vanishes for $|\xi|\leq 1$.

Now, let $\varphi\in C^\infty_0(\mathbb{R})$ be such that $\varphi>0, \operatorname{supp}(\varphi)\subset[1/2,1], \int_0^\infty\frac{\varphi(t)}{t}dt=1$.

Then, we can write

$$k(x,y)=\frac{1}{(2\pi)^n}\int_1^\infty\int_{\{|\xi|\geq 1\}}e^{i(x-y)\cdot\xi}p(x,\xi)\varphi\left(\frac{|\xi|}{t}\right)d\xi\frac{dt}{t}$$

let

$$k(x,y,t)=\int_{\{|\xi|\geq 1\}}e^{i(x-y)\cdot\xi}p(x,\xi)\varphi\left(\frac{|\xi|}{t}\right)d\xi.$$

Given $\beta\in\mathbb{N}^n$, we can write

$$(x-y)^\beta k(x,y,t)=\sum_{\alpha\leq\beta}C(\alpha,\beta)\int_{\{|\xi|\geq 1\}}e^{i(x-y)\cdot\xi}D^\alpha_\xi p(x,\xi)D^{\alpha-\beta}_\xi\varphi\left(\frac{|\xi|}{t}\right)d\xi.$$

The function $\xi \longrightarrow \varphi\left(\frac{|\xi|}{t}\right)$ belongs to $S^0_{1,0}$. Moreover, for each $\gamma \in I\!N^n$, we have the estimate

$$|D^\gamma_\xi \varphi\left(\frac{|\xi|}{t}\right)| \leq C(\gamma) t^{-|\gamma|}(1+|\xi|)^{-|\gamma|}.$$

Thus,

$$|(x-y)^\beta k(x,y,t)| \leq \sum_{\alpha \leq \beta} C(\alpha,\beta) \int_{\{|\xi| \geq 1\}} (1+|\xi|)^{m-\rho|\alpha|-|\alpha-\beta|} \cdot \chi_{supp_\xi \varphi(|\xi|/t)}(\xi) t^{-|\alpha-\beta|} d\xi,$$

where χ_A denotes the characteristic function of the set A. Since $|\xi| \geq 1$ implies $t \geq 1$, we can estimate the expression above by $C(\beta) t^{m+n-\rho|\beta|}$.

Thus, we obtain

$$|x-y|^N t^{\rho N} |k(x,y,t)| \geq C(N) t^{m+n},$$

when $|\beta| = N$. Or,

$$|k(x,y,t)| \leq C(N) \frac{t^{m+n}}{1+|x-y|^N t^{\rho N}}. \tag{3.2}$$

Thus,

$$|k(x,y)| \leq C(N) \int_1^\infty \frac{t^{m+n-1}}{1+(|x-y|t^\rho)^N} dt.$$

If $N \geq \left[\frac{m+n}{\rho}\right]+1$, the integral converges. We can write

$$|k(x,y)| \leq \frac{C}{|x-y|^{(m+n)/\rho}} \int_{|x-y|}^\infty \frac{s^{\frac{(m+n)}{\rho}-1}}{1+s^N} ds.$$

Thus, if $|x-y| \neq 0$ we have $|k(x,y)| \leq C|log|x-y||$ when $m+n = 0$ and $|k(x,y)| \leq C|x-y|^{(m+n)/\rho}$ when $m+n > 0$. This completes the proof of the theorem.

It can be shown that these estimates are sharp, (cf. [7]).

If $m \leq -(n+1)(1-\rho)$, we obtain from (3.1) the estimates,

$$|k(x,y)| \leq C|x-y|^{-n+\frac{1}{\rho}-1}, \qquad \rho \neq \frac{1}{n+1},$$
$$|\nabla_{x,y} k(x,y)| \leq C|x-y|^{-n-1}.$$

Since we know that k is rapidly decreasing as $|x-y| \longrightarrow \infty$, we deduce that the operators in the class $L^{-(n+1)(1-\rho)}_{\rho,\delta}$ are associated to standard kernels in the sense of R. Coifman and Y. Meyer, (cf. [10]). Moreover, these kernels are bounded by integrable convolution kernels, when $0 < \rho < 1$. This observation gives the L^p continuity result, $1 < p < \infty$, (cf. [11]).

If $m \leq -n(1-\rho)$, the estimates we obtain from (3.1) are

$$|k(x,y)| \leq C|x-y|^{-n}$$
$$|\nabla_{x,y} k(x,y)| \leq C|x-y|^{-n-1/\rho}. \tag{3.3}$$

Thus, operators in the class $L^{-n(1-\rho)}_{\rho,\delta}$ are associated to weakly strongly singular kernels, (cf. [12]), that is to say, kernels more singular at the diagonal than standard kernels, but not too singular to prevent the operators to be continuous in $L^p, 1<p<\infty$.

C. Fefferman has proved, (cf. [13]), that pseudo-differential operators in the class $L^{-(n/2)(1-\rho)}_{\rho,\delta}$ $0 \leq \delta < \rho \leq 1$ are bounded in L^p, for $1<p<\infty$, but this is not longer true if $m > -(n/2)(1-\rho)$. Thus, $-(n/2)(1-\rho)$ is the maximum order for which a pseudo-differential operator can be considered weakly strongly singular. For this class the estimates on the kernel will be

$$|k(x,y) \leq C|x-y|^{-n(\frac{1}{\rho}+1)/2}$$
$$|\nabla_{x,y} k(x,y)| \leq C|x-y|^{-n[(\frac{1}{\rho}+1)/2]-1/\rho}.$$

These are the weakly strongly estimates (3.3) only when $\rho = 1$. However, it is possible to prove for these kernels integral conditions resembling those considered by J.L. Rubio de Francia, F. Ruiz and J.L. Torrea, (cf. [7]).

This will be shown in the next section, using the pointwise estimate already obtained for the kernel $k(x,y)$.

4. An integral estimate for the kernel

Theorem 4.1: *Let $L \in L^m_{\rho,\delta}, m \geq -(n+1)(1-\rho), m+n+1 > 0$. Then, given $1<p<\infty, \exists \ \{d_j\} \in l^1, 0<\theta< 1$, such that:*

If $0<r<1$,

$$\sup_{|x-z|<r} \left(\int_{2^j r^\theta < |y-z| < 2^{j+1} r^\theta} (|k(x,y)-k(z,y)|^p + |k(y,x)-k(y,z)|^p) dy \right)^{1/p} \leq \tag{4.1}$$
$$\leq C \frac{d_j}{|B(z,2^j r^\theta)|^{1-1/p}}.$$

If $r \geq 1$,

$$\sup_{|x-z|<r}\left(\int_{2^j r<|y-z|<2^{j+1} r}(|k(x,y)-k(z,y)|^p+|k(y,x)-k(y,z)|^p)dy\right)^{1/p} \leq \tag{4.2}$$
$$\leq C\frac{d_j}{|B(z,2^j r^\theta)|^{1-1/p}}.$$

Proof: Using (3.1) with $M=1$, we obtain by the mean value theorem that

$$|k(x,y)-k(z,y)|+|k(y,x)-k(y,z)| \leq C\frac{|x-z|}{|y-z|^{(m+n+1)/\rho}}, \tag{4.3}$$

if $2|x-z|<|y-z|$.

Let us first prove (4.1). It suffices to consider one of the terms in the left hand side. Using (4.3), we can write

$$\left(\int_{2^j r^\theta<|y-z|<2^{j+1} r^\theta}|k(x,y)-k(z,y)|^p dy\right)^{1/p} \leq$$
$$C\left(\int_{2^j r^\theta<|y-z|<2^{j+1} r^\theta}\frac{|x-z|^p}{|y-z|^{(m+n+1)p/\rho}}dy\right)^{1/p} \leq$$
$$C\ r\left(\int_{2^j r^\theta}^{\infty} t^{n-1-(p/\rho)(m+n+1)}dt\right)^{1/p}.$$

Since $-(n+1)(1-\rho)>-n-1+\frac{\rho}{p}n$ for $\rho>0, 1<p$, we have that $n-\frac{p}{\rho}(m+n+1)<0$. Thus, the integral above converges to $C\ r(2^j r^\theta)^{[n-(p/\rho)(m+n+1)]/p}$. Since $-(n+1)(1-\rho)>-n(1-\rho)-1$, the condition $m\geq-(n+1)(1-\rho)$ implies that the exponent of 2^j is <0. On the other hand, we want the exponent of r to be ≥ 0. Or, $1-\theta\left(\frac{m+n+1}{\rho}-n\right)\geq 0$. This is equivalent to the condition

$$\theta \leq \frac{1}{\frac{m+n+1}{\rho}-n}.$$

Let us now consider (4.2). As before, it is enough to look at the first term in the left hand side. We obtain

$$\left(\int_{2^j r<|y-z|<2^{j+1} r}|k(x,y)-k(z,y)|^p dy\right)^{1/p} \leq$$
$$C\left(\int_{2^j r<|y-z|<2^{j+1} r}\frac{|x-z|^p}{|y-z|^{(pdy}}\right)^{1/p} \leq$$

$$C(2^j r)^{-n(1-1/p)}\cdot r(2^j r)^{n(1-1/p)+(n-p/\rho(m+n+1))}/p.$$

Once again, the exponent of 2^j is $n-\frac{m+n+1}{\rho}<0$. This time we want the exponent of r to be ≤ 0, which it is, since $m\geq -(n+1)(1-\rho)$.

Finally, we have showed that (4.1) and (4.2) hold with

$$d_j = C(2^j)^{n-\frac{m+n+1}{\rho}}$$

$$0<\theta\leq min\left\{1, \frac{1}{\frac{m+n+1}{\rho}-n}\right\}. \tag{4.4}$$

This completes the proof of Theorem 4.1. Let us observe that when $m=-(n+1)(1-\rho)$, we can choose in (4.4) $\theta=1$. When $m=-n(1-\rho)$ (4.4) holds for $\theta=\rho$. Finally, when $m=-\frac{n}{2}(1-\rho)$, we can choose $\theta=\frac{2\rho}{2+n(1-\rho)}$. In [15], S. Chanillo and A. Torchinsky proved an estimate similar to (4.1) with $m=-\frac{n}{2}(1-\rho)$ and $p=2$ and asked whether a similar estimate would hold for $p>2$.

5. A weak type estimate in the class $L^m_{\rho,\delta}$

We will now obtain as an application of Theorem 3.1, that operators in $L^m_{\rho,\delta}$, $-n<m<-n(1-\rho)$ can be estimated, pointwisely, in terms of a fractional integral I_α. This estimate will immediately lead to weak type estimates.

Theorem 5.1: *Let $L\in L^m_{\rho,\delta}$, $-n<m<-n(1-\rho)$, $0<\rho\leq 1, 0\leq\delta<1$. Then, there exists $C>0$ such that for each $f\in C_0^\infty(I\!R^n), x\in I\!R^n$,*

$$|Lf(x)|\leq C\; I_\alpha(|f|)(x), \quad \alpha=n-\frac{m+n}{\rho}.$$

Proof: According to Theorem 2.1 b), it suffices to assume that the symbol $p(x,\xi)$ vanishes for $|\xi|\leq 1$.

Using the same notation as in Theorem 3.1, we have

$$\begin{aligned} Lf(x) &= \int_{I\!R^n} e^{ix\cdot\xi}p(x,\xi)\int_1^\infty \varphi(|\xi|/t)\frac{dt}{t}\hat{f}(\xi)d\xi = \\ &= \lim_{M\to\infty}\int_1^M\int_{I\!R^n} e^{ix\cdot\xi}p(x,\xi)\varphi(|\xi|/t)\hat{f}(\xi)\frac{dt}{t}\,d\xi = \\ &= \lim_{M\to\infty}\frac{1}{(2\pi)^n}\int_{I\!R^n}\int_1^M\int_{I\!R^n} e^{i(x-y)\cdot\xi}p(x,\xi)\varphi(|\xi|/t)d\xi\frac{dt}{t}f(y)dy. \end{aligned}$$

For M fixed, let

$$k_M(x,y)=\frac{1}{(2\pi)^n}\int_1^M\int_{I\!R^n} e^{i(x-y)\cdot\xi}p(x,\xi)\varphi(|\xi|/t)d\xi\frac{dt}{t}.$$

From the inequality (3.2), there exists $C>0$ depending on M such that

$$|k_M(x,y)|\leq C\int_1^M\frac{t^{m+n-1}}{1+(|x-y|t^\rho)^N}dt\leq$$
$$\leq C\int_1^\infty\frac{t^{m+n-1}}{1+(|x-y|t^\rho)^N}dt.$$

Thus,

$$\left|\int_{I\!R^n}k_M(x,y)f(y)dy)\right|=\left|\sum_{j\in\mathbb{Z}}\int_{2^{j+1}>|x-y|>2^j}k_M(x,y)f(y)dy\right|\leq$$
$$C\sum_{j\in\mathbb{Z}}\int_{2^{j+1}>|x-y|>2^j}\frac{|f(y)|}{|x-y|^{(m+n)/\rho}}dy=$$
$$C\int_{I\!R^n}\frac{|f(y)|}{|x-y|^{n-\alpha}}dy=C\ I_\alpha(|f|)(x),\ \ \alpha=n-\frac{m+n}{\rho}$$

Finally, $Lf(x)=\lim_{M\to\infty}\int k_M(x,y)f(y)dy$. Thus,

$$|Lf(x)|\leq C\ I_\alpha(|f|)(x).$$

This completes the proof of the theorem.

Corollary 5.2: *Let $L\in L^m_{\rho,\delta}, -n<m<-n(1-\rho), 0<\rho\leq 1, 0\leq\delta<1$. Then L is of weak type $(1,\frac{1}{\theta})$, for some $0<\theta<1$.*

Proof: This result is an immediate consequence of the fact that I_α is of weak type $(1,q)$, where $\frac{1}{q}=1-\frac{\alpha}{n}$, (cf. [9], p.120).

Thus, $\theta=\frac{m+n}{n\rho}$.

This completes the proof of the corollary.

6. L^p weighted estimates for a a class of oscillatory integrals

It is true that, for most operators in Harmonic Analysis, boundedness on

$L^p(I\!R^n)$, $1<p<\infty$, will imply boundedness on $L^p(I\!R^n;w), 1<p<\infty, w\in A_p$.

However, there is a remarkable theorem, due to R. Coifman, (cf. [1]), which states that for a classical singular integral operator,

$$Tf(x)=p.v\int k(x-y)\ f(y)\ dy$$

a deeper result holds. Namely,

$$(6.1) \qquad \int_{\mathbb{R}^n} |Tf(y)|^p w(y)dy \leq c \int_{\mathbb{R}^n} [Mf(y)]^p w(y)dy, \quad f \in C_0^\infty(\mathbb{R}^n),$$

where $0<p<\infty, w \in A_\infty, C = C(p,w)>0$ and where M denotes de Hardy-Littlewood maximal operator.

Coifman's proof of (6.1) is based on a difficult good λ inequality involving the maximal operator T^*, defined as

$$T^*f(x) = \sup_{\epsilon>0} \left| \int_{|y-x|>\epsilon} k(x,y)f(y)dy \right|, \quad f \in C_0^\infty(\mathbb{R}^n),$$

and the operator M.

We describe in [2] a different proof of (6.1), under conditions that allow for consideration of a wider class of operators. Our approach combines the following two ingredients.

First, we prove a pointwise estimate. Indeed, we show that there exist $0 < s < 1, C = C(s)>0$, such that

$$(6.2) \qquad M_s^\#(Tf)(x) \leq C\ Mf(x), \quad x \in \mathbb{R}^n, \ f \in C_0^\infty(\mathbb{R}^n),$$

where $M_s^\#$ is the s-sharp maximal operator, defined as

$$(6.3) \qquad M_s^\#(g) = [M^\#(|g|^s)]^{1/s},$$

$M^\#$ being the sharp maximal operator of C. Fefferman and E. Stein, (cf. [3]),

$$(6.4) \qquad M^\#(g)(x) = \sup_{B(x)} \inf_{c\in\mathbb{C}} \left(\frac{1}{|B|} \int_B |g(y)-c|dy \right)$$

where $\sup_{B(x)}$ means that the supremum is taken over all the balls centered at x and $|B|$, as usual, is the volume of $B = B(x)$.

Second, we use an estimate due to C. Fefferman and E. Stein, (cf. [3]). Namely,

$$(6.5) \qquad \int_{\mathbb{R}^n} [Mf(y)]^p w(y)dy \leq C \int_{\mathbb{R}^n} [M^\# f(y)]^p w(y)dy, \ f \in C_0^\infty(\mathbb{R}^n)$$

where $0<p<\infty, w \in A_\infty, C = C(p,w)>0$.

A simple proof of (6.5) can be found in [4], p.42.

Then, in order to conclude (6.1), we combine (6.2) and (6.5) in the following way.

$$\begin{aligned}\int_{\mathbb{R}^n} |Tf(y)|^p w(y)dy \leq & \int_{\mathbb{R}^n} [M(|Tf|^s)(y)]^{p/s} w(y)dy \leq \\ & \leq C\int_{\mathbb{R}^n} [M^{\#}(|Tf|^s)(y)]^{p/s} w(y)dy = \\ & = C\int_{\mathbb{R}^n} [M_s^{\#}(Tf)(y)]^p w(y)dy \leq \\ & \leq C\int_{\mathbb{R}^n} [M(f)(y)]^p w(y)dy.\end{aligned}$$

Coifman's result could be extended in two directions. One, allowing T to be a more general operator. The other, allowing w to belong to a larger class of weights. It is clear that estimate (6.2) will deal with the first generalization, while (6.3) will have to do with the second. In this respect there are some extensions of the class A_∞ for which (6.3) still holds, (cf. [5]). However, the focus of [2] is on proving the estimate (6.2) and extending it to operators other than Calderón-Zygmund operators. Among them, it is mentioned in [2] a class of oscillatory integrals related to those studied by D. Phong and E. Stein, (cf. [6]).

Our purpose now is to prove the estimate (6.2) for these operators, thus obtaining weighted L^p estimates for them, with A_∞ weights.

We will first precise the class of operators under consideration and show some properties.

Given a distribution $K(x,y) \in D'(\mathbb{R}^n \times \mathbb{R}^n)$ and given a real bilinear form $E(x,y)$, we define the operator

$$(L_E f, g) = (k(x,y), e^{iE(x,y)} f \otimes g), \text{ for } f, g \in C_0^\infty(\mathbb{R}^n). \tag{6.6}$$

When $k(x,y)$ is the distribution kernel of a Calderón-Zygmund operator, (6.6) coincides with the class studied by D. Phong and E. Stein.

When $k(x,y)$ is the distribution kernel of a pseudo-differential operator, an alternative description can be given. Indeed, given a symbol $p(x,\xi)$ in the Hörmander class $S^m_{\rho,\delta}, m \in \mathbb{R}, 0 \leq \rho, \delta \leq 1$ and given a real bilinear form $E(x,y)$, the integral

$$\int e^{ix.\xi} p(x,\xi) \mathcal{F}_y[e^{iE(x,y)} f(y)](\xi) d\xi,$$

where $f \in C_0^\infty(\mathbb{R}^n)$ and $\mathcal{F}_y$ denotes the Fourier transform in the variable y, is well defined pointwise, since $\mathcal{F}_y[e^{iE(x,y)} f(y)](\xi)$ decays rapidly as a function of ξ, for each x fixed.

Lemma 6.1: *Assume that* $0<\rho\leq 1, 0\leq\delta<1$ *and let* $k(x,y)$ *be the distribution kernel of a pseudo-differential operator* L *with symbol* $p(x,\xi)$. *Then, if* $f\in C_0^\infty(\mathbb{R}^n)$ *and if* $x\notin supp(f)$, *we have,*

$$\int e^{iE(x,y)}k(x,y)\,f(y)\,dy=\int e^{ix.\xi}p(x,\xi)\mathcal{F}_y[e^{iE(x,y)}\,f(y)](\xi)d\xi. \tag{6.7}$$

Proof: Let $\eta\in C_0^\infty(\mathbb{R}^n)$ be such that $\mathrm{supp}(\eta)\subset\{|\xi|\leq 2\}$ and $\eta(\xi)=1$ if $|\xi|\leq 1$.

We can write the right hand side of (6.7) as

$$\int e^{ix.\xi}p(x,\xi)\eta(\xi)\mathcal{F}_y(e^{iE(x,y)}\,f(y))(\xi)d\xi+\int e^{ix.\xi}p(x,\xi)(1-\eta(\xi))\mathcal{F}_y(e^{iE(x,y)}\,f(y))(\xi)d\xi.$$

Let us consider the first term.

$$\frac{1}{(2\pi)^n}\int e^{ix.\xi}p(x,\xi)\eta(\xi)\int e^{-iy.\xi}e^{iE(x,y)}f(y)dy\;d\xi=$$
$$=\int\left[\frac{1}{(2\pi)^n}\int e^{i(x-y).\xi}p(x,\xi)\eta(\xi)d\xi\right]e^{iE(x,y)}f(y)dy\;d\xi.$$

The expression between brackets defines a function, $k_1(x,y)$. Integrating by parts we can see that $k_1\in C^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ and each derivative decays rapidly as $|x-y|\longrightarrow\infty$. Let us now consider the second term.

As in Theorem 5.1, we have

$$\int e^{ix.\xi}p(x,\xi)(1-\eta(\xi))\mathcal{F}_y[e^{iE(x,y)}\,f(y)](\xi)d\xi$$

$$=\lim_{M\to\infty}\int_{\mathbb{R}^n}\int_1^M\left[\tfrac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{i(x-y).\xi}p(x,\xi)(1-\eta(\xi))\varphi(|\xi|/t)d\xi\tfrac{dt}{t}e^{iE(x,y)}f(y)dy\right].$$

$$k_M(x,y)=\frac{1}{(2\pi)^n}\int_1^M\int_{\mathbb{R}^n}e^{i(x-y).\xi}p(x,\xi)\varphi(|\xi|/t)d\xi\frac{dt}{t}.$$

Again integrating by parts, we can prove that $\{k_M\}$ converges in $C^\infty(\mathbb{R}^n\times\mathbb{R}^n\backslash$ diagonal) to a function $k_2(x,y)$.

Thus, if $|x-y|>0$, we deduce that the distribution kernel of L coincides with $k_1(x,y)+k_2(x,y)$ and that (6.7) holds.

This completes the proof of the lemma.

Theorem 6.2: *Let* $p(x,\xi)\in S^m_{\rho,\delta}, -n<m<-n(1-\rho), 0<\rho\leq 1, 0\leq\delta<1$ *and let* E *be a real bilinear form.*

Then, the operator L_E *given by (6.7) is of weak type* $\left(1,\frac{n\rho}{m+n}\right)$.

Proof: According to the proof of Corollary 5.2 it suffices to show that $L_E f(x)$ can be estimated in terms of a fractional integral $I_\alpha|f|(x)$.

And to prove this pointwise estimate, we will follow the proof of Theorem 5.1. With the notation used in this theorem, we can write,

$$L_E f(x) = \lim_{M\to\infty} \int_{\mathbb{R}^n} e^{iE(x,y)} k_M(x,y) f(y)\, dy.$$

Now,

$$\left|\int_{\mathbb{R}^n} e^{iE(x,y)} k_M(x,y) f(y)\, dy\right| = \left|\sum_{j\in\mathbb{Z}} \int_{2^{j+1}>|x-y|>2^j} e^{iE(x,y)} k_M(x,y) f(y)\, dy\right| \leq$$

$$\leq \sum_{j\in\mathbb{Z}} \int_{2^{j+1}>|x-y|>2^j} |k_M(x,y) f(y)|\, dy \leq C \sum_{j\in\mathbb{Z}} \int_{2^{j+1}>|x-y|>2^j} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dy =$$

$$= C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy = C\, I_\alpha(|f|)(x), \text{ if } \alpha = n - \frac{n+m}{\rho}.$$

Thus

$$|L_E f(x)| \leq C\, I_\alpha(|f|)(x).$$

This completes the proof of the theorem.

It is clear that the same result will be true for any oscillatory factor $e^{iS(x,y)}$, where S is a real function.

We are now ready for the main result.

Theorem 6.3: *Let $k(x,y)$ be the distribution kernel of a Calderon-Zygmund operator and let $E(x,y)$ be a real bilinear form. Assume that k is rapidly decreasing as $|x-y| \longrightarrow \infty$. Thus, there exist $0<s<1$, $C>0$ such that*

$$M_s^{\#}(L_E(f))(x_0) \leq C\, M(f)(x_0)$$

for every $f \in C_0^\infty(\mathbb{R}^n)$, $x_0 \in \mathbb{R}^n$.

Proof: According to (6.3) and (6.4) it suffices to find $0<s<1$ and $C>0$ such that for each $E = E(x_0)$ and for some $c \in \mathbb{C}$,

$$\left(\frac{1}{|B|}\int_B ||L_E f(y)|^s - |C|^s| dy\right)^{1/s} \leq C\, Mf(x_0). \tag{6.8}$$

Let $B = B(x_0, r)$ be a ball centered at x_0 with radius r.

Given $f \in C_0^\infty(\mathbb{R}^n)$, we write

$$\begin{aligned} f =& f\chi_{B(x_0,2r)} + f\chi_{B^c(x_0,2r)\cap B^c(x_0,\frac{2}{r})} + \\ &+ f\chi_{B^c(x_0,2r)\cap B^c(x_0,\frac{2}{r})} = f_1 + f_2 + f_3, \end{aligned}$$

where χ denotes a characteristic function and B^c is the complement of B.

Let $c = c_B = \int_{\mathbb{R}^n} e^{iE(x_0,y-x_0)} k(x_0,y) f_2(y)\, dy$

Replacing in the left hand side of (6.8), and observing that $|e^{-iE(x,y)}| = 1$ for every x, y we get

$$\begin{aligned} &\left(\frac{1}{|B|}\int_B \left||e^{-iE(x,x_0)}(L_E f_1 + L_E f_3)(x) + e^{-iE(x,x_0)} L_E f_2(x)|^s - |c_B|^s\right| dx\right)^{1/s} \leq \\ &\leq \left(\frac{1}{|B|}\int_B |(L_E f_1 + L_E f_3)(x)|^s dx + \frac{1}{|B|}\int_B |e^{-iE(x,x_0)} L_E f_2(x) - c_B|^s dx\right)^{1/s} \leq \\ (6.9)\quad &\leq C\left(\frac{1}{|B|}\int_B |L_E f_1|^s dx\right)^{1/s} + C\left(\frac{1}{|B|}\int_B |L_E f_3|^s dx\right)^{1/s} + \\ &+ C\left(\frac{1}{|B|}\int_B \left|\int_{\mathbb{R}^n} e^{-iE(x,x_0)} e^{iE(x,y)} k(x,y) f_2(y) dy - \right.\right. \\ &\left.\left. - \int e^{iE(x_0,y-x_0)} k(x_0,y)\, f_2(y)\, dy\right|^s dx\right)^{1/s}. \end{aligned}$$

Let us consider each term separately.

For the first term in (6.9), we observe that the operator L_E is of weak type $(1,1)$, (cf. [6]). Thus, using the Kolmogorov condition, (cf. [13]), we have the estimate,

$$\left(\frac{1}{|B|}\int_B |L_E f_1|^s dx\right)^{1/s} \leq C\frac{1}{|B|}\int |f_1| dx \leq C\, Mf(x_0).$$

Let us consider the second term in (6.9).

We observe that in $supp(f_3)$, we have $|y - x_0| > 2r$ and also $|y - x_0| > \frac{2}{r}$. Moreover, $|x - x_0| < r$.

Thus, if $0 < r < 1$, $|y-x| \geq |y-x_0| - |x-x_0| > \frac{2}{r} - r > 1$. And if $r \geq 1$, $|y-x| > 2r - r = r \geq 1$.

In any case, we can use the fact that the kernel $k(x,y)$ is rapidly decreasing as $|x-y| \longrightarrow \infty$.

Thus,

$$
\begin{aligned}
|L_E f_3(x)| &\leq C\int_{|y-x_0|>2r}\frac{|f(y)|}{|y-x_0|^{n+1}}dy \leq \\
(6.10)\qquad &\leq C\sum_{j=1}^{\infty}\int_{2^j r<|y-x_0|<2^{j+1}r}\frac{|f(y)|}{|y-x_0|^{n+1}}dy \leq \\
&\leq C\sum_{j=1}^{\infty}(2^j r)^{-n-1}\int_{|y-x_0|<2^{j+1}r}|f(y)|dy \leq C\, Mf(x_0).
\end{aligned}
$$

Finally, let us consider the third term in (6.9).

$$
\begin{aligned}
&\Bigg|\int_{2r<|y-x_0|<2/r} e^{iE(x,y-x_0)}k(x,y)f(y)\,dy - \\
&-\int_{2r<|y-x_0|<2/r} e^{iE(x_0,y-x_0)}k(x_0,y)f(y)\,dy\Bigg| \leq \\
(6.11)\qquad &\int_{2r<|y-x_0|<2/r} |e^{iE(x-x_0,y-x_0)}k(x,y)-k(x_0,y)|\;|f(y)|\;dy \leq \\
&\leq \int_{2r<|y-x_0|<2/r} |e^{iE(x-x_0,y-x_0)}-1||k(x,y)||f(y)|dy + \\
&+\int_{2r<|y-x_0|<2/r} |k(x,y)-k(x_0,y)|\;|f(y)|\;dy.
\end{aligned}
$$

The first term in (6.11) can be estimated as

$$
\begin{aligned}
&C\int_{2r<|y-x_0|<2/r}\frac{|x-x_0||y-x_0|}{|y-x_0|}|f(y)|dy \leq Cr\int_{|y-x_0|<2/r}\frac{|f(y)|}{|y-x_0|^{n-1}}dy = \\
&Cr\sum_{j=0}^{\infty}\int_{2^{-j}/r<|y-x_0|<2^{-j+1}/r}\frac{|f(y)|}{|y-x_0|^{n-1}}dy \leq Cr\sum_{j=0}^{\infty}(2^{-j}/r)^{-n+1} \\
&\int_{|y-x_0|<2^{-j+1}/r}|f(y)|dy \leq C\cdot Mf(x_0).
\end{aligned}
$$

Let us consider the second term in (6.11). Since $2|x-x_0| < 2r < |y-x_0|$, we can estimate it by,

$$
C\int_{2r<|y-x_0|}\frac{|x-x_0|}{|y-x_0|^{n+1}}|f(y)|dy.
$$

As in (6.10), this integral is $\leq C\; Mf(x_0)$.

This completes the proof of the theorem. The proof of Theorem 6.3 uses several ideas from [6]. Particularly, the decomposition of the function f. However, our assumption that the kernel $k(x,y)$ is rapidly decreasing as $|x-y| \longrightarrow \infty$ allows us to work with any real bilinear form, not necessarily nondegenerate. The natural setting in which this hypothesis on k holds is the class $L^m_{\rho,\delta}$. Thus, we obtain from Theorem 6.3 the following consequence.

Corollary 6.4: *Let $L \in L^m_{\rho,\delta}, m \leq -(n+1)(1-\rho), 0 \leq \delta < 1, 0 < \rho \leq 1$ be a pseudo-differential operator with distribution kernel $k(x,y)$ and let $E(x,y)$ be a real bilinear form. Then, there exist $C > 0, 0 < s < 1$ such that*

$$M^{\#}_s(L_E f)(x_0) \leq C\ Mf(x_0)$$

for every $x_0 \in I\!R^n$, $f \in C^\infty_0(I\!R^n)$.

Proof: If suffices to observe that the estimates (3.1) imply in this case that L is a Calderon-Zygmund operator.

When $E = 0$, we obtain

$$M^{\#}_s(Lf)(x_0) \leq C\ Mf(x_0)$$

from which weighted L^p estimates with A_∞ weights are obtained. This generalizes [14].

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