Maximal Functions and the Control of Weighted Inequalities for the Fractional Integral Operator

MARÍA J. CARRO, CARLOS PÉREZ, FERNANDO SORIA & JAVIER SORIA

ABSTRACT. We study weak-type $(1,1)$ weighted inequalities for the fractional integral operator I_α. We show that the fractional maximal operator M_α controls these inequalities when the weight is radially decreasing. However, we exhibit some counterexamples which show that M_α is not appropriate for this control on general weights. We do provide, nevertheless, some positive results related to this problem by considering other suitable maximal functions.

1. INTRODUCTION

The purpose of this paper is to study certain weighted estimates related to the fractional operator, defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy,$$

and to the fractional maximal operator

$$M_\alpha f(x) = \sup_{x\in Q} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)|\,dy,$$

where $0<\alpha<n$. To be more precise, the problem we want to analyze is the following:

Given a weight w (i.e., a positive, locally integrable function), find the best ("smallest") weight W such that

$$w(\{x \in \mathbb{R}^n : |I_\alpha f(x)| > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)|\, W(x)\, dx, \tag{1.1}$$

for every $\lambda > 0$ and for suitable functions f.

Similar results have been considered for other kind of operators like the Hardy-Littlewood maximal function ([5]), the fractional maximal operator (see (1.4)), the Hilbert transform and Singular Integral operators ([2], [13]), the Kakeya maximal function ([10]), etc.

We recall that, for $p > 1$, the weights (w, W) for which the weak-type (p, p) version of (1.1) holds, were characterized by E. Sawyer ([15]). The characterization is as follows: the inequality

$$w(\{x \in \mathbb{R}^n : |I_\alpha f(x)| > \lambda\}) \leq \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |f(x)|^p W(x)\, dx$$

holds if and only if there is a constant C such that for every cube Q

$$\int_Q (I_\alpha(\chi_Q w)(x))^{p'} W(x)^{1-p'}\, dx \leq C \int_Q w(x)\, dx.$$

It is known that (1.1) is equivalent to checking the inequality on finite linear combinations of Dirac deltas (see [7] and [9]). In particular, taking in (1.1) $f = \delta_{x_0}$ (the Dirac mass at x_0), we obtain that $I_\alpha f(x) = |x - x_0|^{\alpha - n}$, and hence, for regular weights W,

$$\begin{aligned} w(\{x \in \mathbb{R}^n : I_\alpha \delta_{x_0}(x) > \lambda\}) &= w(\{x \in \mathbb{R}^n : |x - x_0|^{\alpha-n} > \lambda\}) \\ &= \int_{\{|x-x_0| < \lambda^{1/(\alpha-n)}\}} w(x)\, dx \\ &\leq \frac{C}{\lambda} \int_{\mathbb{R}^n} W(x)\, d\delta_{x_0}(x) = \frac{C}{\lambda} W(x_0), \end{aligned}$$

and consequently, a necessary condition for (1.1) to hold is that

$$M_\alpha w(x) \leq C W(x), \quad \text{a.e.} \tag{1.2}$$

In view of (1.2), it seems natural to consider whether the extreme case works; that is,

$$w(\{x \in \mathbb{R}^n : |I_\alpha f(x)| > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)|\, M_\alpha w(x)\, dx, \tag{1.3}$$

where C is independent of λ, f and, possibly, of w too. This question was raised by the second author during the Spring School on Analysis: *Function Spaces and Their Applications*, held at Paseky nad Jizerou, Czech Republic, in 1999.

An indication that this might be true can be found in the statement of Theorem 4.2, which says that there exists a constant C, depending only on dimension, so that for every $\lambda > 0$, all measurable functions f and all radial decreasing weights w, (1.3) holds.

The main result of this paper (Theorem 2.1) shows that (1.3) is false, however, for general weights w and for any $\alpha \in (0, n)$. (A first result in this direction, for certain values of α and n, was given by Jan Malý [8].) This should be compared with the following well known estimate, which follows as a consequence of classical covering lemmas ([5]):

$$w(\{x \in \mathbb{R}^n : M_\alpha f(x) > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)| M_\alpha w(x)\, dx. \tag{1.4}$$

These results naturally suggested the following problem: to determine which other maximal operators T may replace M_α so that inequality (1.1) holds with $W = Tw$. Theorems 3.1 and 4.2 below are particular solutions to this question. The paper is organized as follows. We present in Section 2 the counterexamples to inequality (1.3). The remaining part of the paper is devoted to the study of the aforementioned question. In Section 3 we show that a good substitute for M_α is given by the composition of M_α with a generalized Hardy-Littlewood maximal function associated to logarithmic Orlicz norms. Finally, in Section 4 we prove Theorem 4.2 for radial weights as a consequence of a more general approach, which also produces a different kind of "admissible" mappings $w \to Tw = W$ for general weights.

2. The Counterexamples

We will prove in this section that (1.3) does not hold in general. The precise result is the following.

Theorem 2.1. *Given $0 < \alpha < n$, there exists a weight w such that for every finite constant C, one can find $\lambda > 0$ and a function f so that*

$$w(\{x \in \mathbb{R}^n : |I_\alpha f(x)| > \lambda\}) > \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)| M_\alpha(w)(x)\, dx. \tag{2.1}$$

Proof. We will construct initially the weight w depending on the value of C. A simple argument by iteration will allow us to remove later that constraint. The proof is based on the discretization results of [7] and [9], and the main idea, for general $\alpha \in (0, n)$, is to choose the densities specially located on a suitable Cantor set. We also present a second example in the case $\alpha \in \mathbb{N}$.

Let $0 < \delta < 1$ be the solution to the equation $2^\alpha (1-\delta)^{n-\alpha} = 1$, and consider the following Cantor set (see [4]):

Let E_0 be the unit cube in $\mathbb{R}^n$, and delete all but the 2^n corner cubes $\{Q_k^1\}$, of side $(1-\delta)/2$ to obtain E_1. Continue in this way, at the Nth stage replacing each cube of E_{N-1} by the 2^n corner cubes $\{Q_k^N\}$, of side $((1-\delta)/2)^N$ to get E_N. Thus E_N contains 2^{nN} cubes of volume $((1-\delta)/2)^{nN}$, and hence $|E_N| = (1-\delta)^{nN}$. (If $E = \bigcap_N E_N$, then it can be shown that E has Hausdorff dimension $n-\alpha$.) For each $k, \ell \in \{1, \dots, 2^{nN}\}$, we say that Q_k^N and Q_ℓ^N are j-relatives, $j = 1, \dots, N$, if j is the smallest index for which there exists a (unique) cube Q_m^{N-j} such that $Q_k^N, Q_\ell^N \subset Q_m^{N-j}$, where $Q_m^0 = E_0$. Thus Q_k^N has (2^n-1) 1-relatives, and $(2^n-1)2^{(j-1)n}$ j-relatives, $j = 2, \dots, N$. Moreover, if Q_k^N and Q_ℓ^N are j-relatives, $x_k \in Q_k^N$ and $x_\ell \in Q_\ell^N$, then it is easy to see that

$$|x_k - x_\ell| \approx \left(\frac{1-\delta}{2}\right)^{N-j}. \tag{2.2}$$

Now, in order to prove (2.1), we take $w = \chi_{E_N}$, and by the result of [7] and [9], it suffices to consider a finite sum of Dirac deltas, instead of a function f. For this, we choose x_k to be the center of the cube Q_k^N, and $f = \sum_{k=1}^{2^{nN}} \delta_{x_k}$. Given $x \in E_N$ we find a lower bound for $I_\alpha(f)(x)$ as follows: if $x \in Q_{k_0}^N$,

$$\begin{aligned} I_\alpha(f)(x) &= \sum_{k=1}^{2^{nN}} \frac{1}{|x-x_k|^{n-\alpha}} \\ &\geq \sum_{k\neq k_0} \frac{1}{|x-x_k|^{n-\alpha}} \approx \sum_{j=1}^{N} \frac{(2^n-1)2^{n(j-1)}}{((1-\delta)/2)^{(N-j)(n-\alpha)}} \\ &\approx \left(\frac{2}{1-\delta}\right)^{N(n-\alpha)} \sum_{j=1}^{N} \frac{2^{j\alpha}}{(1-\delta)^{-j(n-\alpha)}} \approx N\left(\frac{2}{1-\delta}\right)^{N(n-\alpha)}. \end{aligned}$$

Therefore, if we take $\lambda_N \approx N(2/(1-\delta))^{N(n-\alpha)}$, we have

$$w(\{I_\alpha(f) > \lambda_N\}) = |E_N| = (1-\delta)^{nN}.$$

On the other hand, if we show that

$$M_\alpha \chi_{E_N}(x_k) \approx \left(\frac{2}{1-\delta}\right)^{-N\alpha}, \tag{2.3}$$

then (2.1) follows by taking $N \gg C$, since

$$\frac{1}{\lambda_N} \sum_{k=1}^{2^{nN}} M_\alpha \chi_{E_N}(x_k) \approx \frac{1}{N}\left(\frac{2}{1-\delta}\right)^{N(\alpha-n)} 2^{nN} \left(\frac{2}{1-\delta}\right)^{-N\alpha} = \frac{|E_N|}{N}.$$

To prove (2.3), fix x_k and assume $x_k \in Q_m^j$, $j = 0, \dots, N$. If $j = 0$,

$$\frac{|Q_m^0 \cap E_N|}{|Q_m^0|^{1-\alpha/n}} = |E_N| = \left(\frac{2}{1-\delta}\right)^{-N\alpha}.$$

If $j = 1, \dots, N$, then $Q_m^j \cap E_N$ is equal to the union of all the $2^{n(N-j)}$ cubes $Q_\ell^N \subset Q_m^j$, and hence

$$\frac{|Q_m^j \cap E_N|}{|Q_m^j|^{1-\alpha/n}} = \frac{((1-\delta)/2)^{nN} 2^{n(N-j)}}{((1-\delta)/2)^{nj(1-\alpha/n)}} = \left(\frac{2}{1-\delta}\right)^{-N\alpha}.$$

Now, if $x_k \in Q_k^N$, take any cube Q centered at the point x_k, and let $j \in \{1, \dots, N\}$ be the largest index for which Q has nontrivial intersection with a cube Q_ℓ^N which is a j-relative of Q_k^N (if Q meets only Q_k^N, the estimate is trivial). Then by (2.2),

$$|Q| \geq C\left(\frac{1-\delta}{2}\right)^{n(N-j)},$$

and $Q \cap E_N$ contains only cubes Q_ℓ^N which are ℓ-relatives of Q_k^N, $\ell \in \{1, \dots, j\}$. Since there are at most 2^{nj} of such cubes, we get

$$\frac{|Q \cap E_N|}{|Q|^{1-\alpha/n}} \leq C \frac{2^{nj}((1-\delta)/2)^{nN}}{((1-\delta)/2)^{n(N-j)(1-\alpha/n)}} \approx \left(\frac{2}{1-\delta}\right)^{-N\alpha},$$

which proves (2.3).

A second example for $\alpha \in \mathbb{N}$***:*** When α is a positive integer, the proof simplifies considerably as can be seen in the following construction. Given $n \geq 2$, let us take $\alpha \in \{1, \dots, n-1\}$. For large $N \in \mathbb{N}$, define the rectangle in $\mathbb{R}^n$

$$R = \overbrace{(0,N] \times \cdots \times (0,N]}^{n-\alpha \text{ times}} \times \overbrace{(0,1] \times \cdots \times (0,1]}^{\alpha \text{ times}},$$

and consider the net $\{x_k\}_k$ of points with integral coordinates inside R. Let $\mu = \sum_k \delta_{x_k}$. Then, the total mass of μ and the volume of R coincide and equal

$$\|\mu\| = |R| = N^{n-\alpha}.$$

It is easy to see that for $x \in R$ one has

$$\sum_{k \geq 1} \frac{1}{|x - x_k|^{n-\alpha}} \geq c \sum_{m=1}^{N} m^{n-\alpha-1} \frac{1}{|m|^{n-\alpha}} \approx \log N,$$

where c denotes a small constant depending only on dimension, so that

$$R \subset \{x : I_\alpha\mu(x) > c\log N\}.$$

Also, if we set $w = \chi_R$, then $M_\alpha w(x_k) \approx 1$. Hence,

$$w(\{x : I_\alpha\mu(x) > c\log N\}) = |R| = N^{n-\alpha},$$

whereas $\sum_k M_\alpha w(x_k) \approx \|\mu\| = N^{n-\alpha}$. This shows, as in the first example, that

$$\sup\left\{\frac{\lambda w(\{x : I_\alpha\mu(x) > \lambda\})}{\int M_\alpha w(x)\,d\mu(x)}\right\} = \infty,$$

the supremum taken over all $\lambda > 0$, all linear combinations of Dirac deltas μ and all weights w.

Let us now show how we can iterate the above examples in order to construct a weight w for which inequality (1.3) fails. We will only consider the second example, the construction for the first example being similar.

Given $j \in \mathbb{N}$, set $N = 2^{2^j}$ and let R be the rectangle constructed above for such N. Write $R_j = R - z_j$, where $z_j \in \mathbb{R}$ will be chosen later. Let $\{x_k^j\}_k$ be the net of points with integral coordinates inside R_j and $\mu_j = \sum_k \delta_{x_k^j}$. In this way $R_j \subset \{x : I_\alpha\mu_j(x) > c2^j\}$. Put $w_j = \chi_{R_j}$ and

$$d\mu = \sum_{j\in\mathbb{N}} c_j\,d\mu_j;\; w = \sum_{j\in\mathbb{N}} c_j w_j, \quad \text{with } c_j = \frac{1}{j|R_j|^{1/2}}.$$

Choose finally the sequence $\{z_j\}$ sufficiently separated so that the rectangles $\{R_j\}$ are disjoint and, moreover, we have $M_\alpha w(x_k^j) \approx M_\alpha(c_j w_j)(x_k^j) \approx c_j$ for each j.

We now have for every $m \in \mathbb{N}$

$$w(\{x : I_\alpha\mu(x) > c\,c_m 2^m\}) \geq w(R_m) = c_m|R_m|,$$

whereas

$$\int_{\mathbb{R}^n} M_\alpha w(x)\,d\mu(x) = \sum_{k,j} c_j M_\alpha w(x_k^j) \approx \sum_j c_j^2\|\mu_j\| = \sum_j \frac{1}{j^2} < \infty.$$

If inequality (1.3) were true, we should have

$$c_m|R_m| \leq \frac{C}{c_m 2^m},$$

for a certain finite constant C independent of m. But this is clearly false, for the above would imply $2^m \leq Cm^2$, $\forall m \in \mathbb{N}$. This finishes the proof of Theorem 2.1. □

3. Logarithmic Maximal Functions

We are now going to consider alternative solutions to (1.1) in terms of weights W which are obtained by means of certain maximal operators defined in [11]. In order to do that, we introduce some additional notation and give several definitions (see [1]).

We will write, as usual, the weak-L^p norm of a function, with respect to the measure $w(x)\,dx$ as:

$$\|f\|_{L^{p,\infty}(w)} = \sup_{t>0} t\,w(\{x \in \mathbb{R}^n : |f(x)| > t\})^{1/p}.$$

If B is a Young function, the mean Luxemburg norm of a measurable function f on a cube Q is defined by

$$\|f\|_{B,Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q B\left(\frac{|f|}{\lambda}\right)dx \le 1\right\}. \tag{3.1}$$

When $B(t) = t\log(e+t)^\beta$, this norm is also denoted by $\|\cdot\|_{L(\log L)^\beta,Q}$. When $\beta = 0$ we have the usual average $|f|_Q = (1/|Q|)\int_Q |f|$. We define the maximal function

$$M_{L(\log L)^\alpha} f(x) = \sup_{Q\ni x} \|f\|_{L(\log L)^\alpha,Q}.$$

For $\alpha = 0$ we get the standard Hardy-Littlewood maximal function Mf.

More generally, for any B we define

$$M_B f(x) = \sup_{Q\ni x} \|f\|_{B,Q}.$$

The main result in this section is contained in the following theorem. Here, M^2 denotes the composition of M with itself.

Theorem 3.1. *Let* $0 < \alpha < n$ *and* $\delta > 0$. *Then*

$$\|I_\alpha f\|_{L^{1,\infty}(w)} \le C\int_{\mathbb{R}^n} |f(x)| M_\alpha(M_{L(\log L)^\delta}(w))(x)\,dx, \tag{3.2}$$

and in particular,

$$\|I_\alpha f\|_{L^{1,\infty}(w)} \le C\int_{\mathbb{R}^n} |f(x)| M_\alpha(M^2(w))(x)\,dx. \tag{3.3}$$

That is, (1.1) holds with $W = M_\alpha(M^2 w)$.

We now make the observation that (3.2) is not as sharp as

$$\|Tf\|_{L^{1,\infty}(w)} \le C\int_{\mathbb{R}^n} |f(x)| M_{L(\log L)^\delta}(w)(x)\,dx,$$

derived in [13], where T is any classical Calderón-Zygmund singular integral operator. Indeed, observe that $M_\alpha(M_{L(\log L)^\delta}(w)) \geq M_\alpha(M_L(w)) = M_\alpha(M(w))$. Hence, when $\alpha = 0$ we simply have M^2, which is bigger than $M_{L(\log L)^\delta}(w)$. Probably, (3.2) holds for the weight $M_\alpha(M(w))$

Theorem 3.1 is an immediate consequence of the next theorem, using also (1.4) and the fact that $M_{L(\log L)^\delta}(w) \leq CM^2(w)$, for $0 \leq \delta \leq 1$ ([13]).

Theorem 3.2. *Let* $0 < \alpha < n$ *and* $\delta > 0$. *Then*

$$\|I_\alpha f\|_{L^{1,\infty}(w)} \leq C\|M_\alpha f\|_{L^{1,\infty}(M_{L(\log L)^\delta}(w))}. \tag{3.4}$$

The method used to prove this theorem combines ideas from [11] and [12]. We begin by recalling the following decomposition lemma.

Lemma 3.3 ([11, Lemma 3.1])**.** *Let f and g be L^∞ positive functions with compact support, and let μ be a nonnegative measure finite on compact sets. Let $a > 2^n$, then there exist a family of cubes $Q_{k,j}$ and a family of pairwise disjoint subsets $\{E_{k,j}\}$, $E_{k,j} \subset Q_{k,j}$, with*

$$|Q_{k,j}| < \frac{1}{1 - 2^n/a}|E_{k,j}|, \tag{3.5}$$

for all k, j, and such that

$$\int_{\mathbb{R}^n} I_\alpha f(x)g(x)\,d\mu(x) \leq C\sum_{k,j}\frac{|Q_{k,j}|^{\alpha/n}}{|Q_{k,j}|}\int_{3Q_{k,j}} f(y)\,dy\,\frac{1}{|Q_{k,j}|}\int_{Q_{k,j}} g(y)\,d\mu(y)|E_{k,j}|. \tag{3.6}$$

We will also make use of the so called RH_∞ condition, which we now recall.

Definition 3.4. A weight v satisfies the RH_∞ condition if there is a constant $C > 0$ such that for each cube Q

$$\operatorname*{ess\,sup}_{x\in Q} v(x) \leq \frac{C}{|Q|}\int_Q v\,dx.$$

It is very easy to check that $RH_\infty \subset A_\infty$ (see [3]).

Lemma 3.5. *Let v be a weight satisfying the RH_∞ condition. Then, there is a constant C such that for any weight w and all positive f,*

$$\int_{\mathbb{R}^n} I_\alpha f(x)w(x)v(x)\,dx \leq C\int_{\mathbb{R}^n} M_\alpha(f)(x)Mw(x)v(x)\,dx. \tag{3.7}$$

Proof. We start with inequality (3.6) with g replaced by w, and $d\mu$ replaced by $v(x)\,dx$:

$$\begin{aligned}\int_{\mathbb{R}^n} I_\alpha f(x) w(x)\, v(x)\, dx \\ &\le C \sum_{k,j} \frac{|Q_{k,j}|^{\alpha/n}}{|Q_{k,j}|} \int_{3Q_{k,j}} f(y)\, dy \int_{Q_{k,j}} w(y) v(y)\, dy \\ &\le C \sum_{k,j} \frac{|Q_{k,j}|^{\alpha/n}}{|Q_{k,j}|} \int_{3Q_{k,j}} f(y)\, dy \int_{Q_{k,j}} w(y)\, dy \operatorname*{ess\,sup}_{Q_{k,j}} v \\ &\le C \sum_{k,j} \frac{|Q_{k,j}|^{\alpha/n}}{|Q_{k,j}|} \int_{3Q_{k,j}} f(y)\, dy \frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} w(y)\, dy v(Q_{k,j}).\end{aligned}$$

Since $v \in A_\infty$ and by the properties of the sets $\{E_{k,j}\}$, we have $v(Q_{k,j}) \le Cv(E_{k,j})$ for each k, j. Combining this with the fact that the family $\{E_{k,j}\}$ is formed by pairwise disjoint subsets, with $E_{k,j} \subset Q_{k,j}$, we continue with

$$\begin{aligned}\int_{\mathbb{R}^n} I_\alpha f(x) w(x) v(x)\, dx \\ &\le C \sum_{k,j} \frac{|Q_{k,j}|^{\alpha/n}}{|Q_{k,j}|} \int_{3Q_{k,j}} f(y)\, dy \frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} w(y)\, dy v(E_{k,j}) \\ &\le C \sum_{k,j} \int_{E^k_j} M_\alpha(f)(x)\, Mw(x)\, v(x)\, dx \\ &\le C \int_{\mathbb{R}^n} M_\alpha(f)(x)\, Mw(x)\, v(x)\, dx.\end{aligned}$$ ◻

The second lemma gives interesting examples of RH_∞ weights.

Lemma 3.6 ([3]). *Let g be any function such that Mg is finite a.e. Then $(Mg)^{-\alpha} \in RH_\infty$, $\alpha > 0$.*

Theorem 3.7. *If $0 < p \le 1$, then*

$$\int_{\mathbb{R}^n} |I_\alpha f(x)|^p w(x)\, dx \le C \int_{\mathbb{R}^n} (M_\alpha(f)(x))^p Mw(x)\, dx. \tag{3.8}$$

Proof. To prove (3.8) we will use the appropriate duality for the spaces L^p, $p < 1$: if $f \ge 0$,

$$\|f\|_p = \inf\left\{ \int f u^{-1} : \|u^{-1}\|_{p'} = 1 \right\} = \int f u^{-1},$$

for some $u \geq 0$ such that $\|u^{-1}\|_{p'} = 1$, where $p' = p/(p-1) < 0$. This follows from the "reverse" Hölder's inequality:

$$\int fg \geq \|f\|_p \, \|g\|_{p'},$$

which is a consequence of the usual Hölder's inequality.

We start the proof by choosing a nonnegative g, with $\|g^{-1}\|_{L^{p'}(Mw)} = 1$, and such that

$$\begin{aligned}\|M_\alpha f\|_{L^p(Mw)} &= \int_{\mathbb{R}^n} M_\alpha(f)(x)\frac{Mw(x)}{g(x)}\,dx \\ &\geq \int_{\mathbb{R}^n} M_\alpha(f)(x)\frac{Mw(x)}{M(g^\delta)^{1/\delta}(x)}\,dx,\end{aligned}$$

where we have used the Lebesgue differentiation theorem for any $\delta > 0$. We apply both Lemma 3.5 and Lemma 3.6 to the weight $M(g^\delta)^{-1/\delta}$, to continue with

$$\begin{aligned}\|M_\alpha f\|_{L^p(Mw)} &\geq \int_{\mathbb{R}^n} I_\alpha(f)(x)\frac{w(x)}{M(g^\delta)(x)^{1/\delta}}\,dx \\ &\geq \|I_\alpha(f)\|_{L^p(w)}\,\|M(g^\delta)^{-1/\delta}\|_{L^{p'}(w)},\end{aligned}$$

and everything is reduced to prove

$$\|M(g^\delta)^{-1/\delta}\|_{L^{p'}(w)} \geq \|g^{-1}\|_{L^{p'}(Mw)}.$$

Since $p' < 0$, this is equivalent to saying that

$$\int_{\mathbb{R}^n} M(g^\delta)^{-p'/\delta}(x)w(x)\,dx \leq \int_{\mathbb{R}^n} g^{-p'}(x)Mw(x)\,dx.$$

But, if we choose $0 < \delta < p/(1-p)$, we have that $-p'/\delta > 1$ and this follows from the classical weighted norm inequality of C. Fefferman and E. Stein ([5]):

$$\int_{\mathbb{R}^n} (Mf)^p(x)w(x)\,dx \leq \int_{\mathbb{R}^n} |f(x)|^p Mw(x)\,dx, \quad p > 1. \qquad \square$$

Proof of Theorem 3.2. By standard density arguments, we may assume that both f and the weight w are bounded, nonnegative functions and with compact support.

Raising the quantity $\|I_\alpha f\|_{L^{1,\infty}(w)}$ to the power $1/p$, with $p > 1$ (which will be chosen at the end of the proof), then

$$\begin{aligned}||I_\alpha f||_{L^{1,\infty}(w)}^{1/p} &= \|(I_\alpha f)^{1/p}\|_{L^{p,\infty}(w)} \\ &= \sup_{g\in L^{p',1}(w),\ \|g\|_{L^{p',1}(w)}=1} \int_{\mathbb{R}^n} (I_\alpha f(x))^{1/p} g(x)w(x)\,dx.\end{aligned}$$

The last equality follows since $L^{p',1}(w)$ and $L^{p,\infty}(w)$ are associate spaces (this equality can also be proved as a consequence of Kolmogorov's identity). Fixing one of these g's we use (3.8) to continue with

$$\begin{aligned}||I_\alpha f||_{L^{1,\infty}(w)}^{1/p} &\le C\int_{\mathbb{R}^n}(M_\alpha f(x))^{1/p}\,M(gw)(x)\,dx\\ &= C\int_{\mathbb{R}^n}(M_\alpha f(x))^{1/p}\frac{M(gw)(x)}{\tilde{M}w(x)}\tilde{M}w(x)\,dx,\end{aligned}$$

where $\tilde{M}$ is an appropriate (maximal type) operator to be chosen soon. We continue with Hölder's inequality for Lorentz spaces (the underlying measure is now $\tilde{M}w(x)\,dx$):

$$\begin{aligned}||I_\alpha f||_{L^{1,\infty}(w)}^{1/p} &\le C\|(M_\alpha f)^{1/p}\|_{L^{p,\infty}(\tilde{M}w)}\left\|\frac{M(gw)}{\tilde{M}w}\right\|_{L^{p',1}(\tilde{M}w)}\\ &= C||M_\alpha f||_{L^{1,\infty}(\tilde{M}w)}^{1/p}\left\|\frac{M(gw)}{\tilde{M}w}\right\|_{L^{p',1}(\tilde{M}w)}.\end{aligned}$$

To conclude we just need to show that

$$\left\|\frac{M(gw)}{\tilde{M}w}\right\|_{L^{p',1}(\tilde{M}w)} \le c\|g\|_{L^{p',1}(w)},$$

or equivalently

$$S : L^{p',1}(w) \to L^{p',1}(\tilde{M}w), \quad \text{where } Sf = \frac{M(fw)}{\tilde{M}(w)}. \tag{3.9}$$

To do this we choose $\tilde{M}$ pointwise bigger than M; that is, such that $Mw \le \tilde{M}w$ for each w. With this choice we trivially have

$$S : L^{\infty}(w) \to L^{\infty}(\tilde{M}w).$$

Therefore by the Marcinkiewicz's interpolation theorem for Lorentz spaces due to R. Hunt ([1]) it will be enough to show that for some $\varepsilon > 0$

$$S : L^{(p+\varepsilon)'}(w) \to L^{(p+\varepsilon)'}(\tilde{M}w), \tag{3.10}$$

which amounts to proving

$$\begin{aligned}&\int_{\mathbb{R}^n}(M(wf)(y))^{(p+\varepsilon)'}(\tilde{M}w(y))^{1-(p+\varepsilon)'}\,dy\\ &\qquad\le C\int_{\mathbb{R}^n}(f(y))^{(p+\varepsilon)'}w(y)\,dy,\end{aligned} \tag{3.11}$$

for any $f \geq 0$ bounded and with compact support. But this result follows from [12]: indeed it is shown there that, for $r > 1$ and $\eta > 0$,

$$\int_{\mathbb{R}^n} (M(f)(y))^{r'} (M_{L(\log L)^{r-1+\eta}}(w)(y))^{1-r'}\, dy \leq C \int_{\mathbb{R}^n} (f(y))^{r'} w(y)^{1-r'}\, dy. \tag{3.12}$$

We finally choose the appropriate parameters and weight. Let $r = p + \varepsilon$, $\eta = \varepsilon$, and pick the weight

$$\tilde{M}w = M_{L(\log L)^{p-1+2\varepsilon}}(w).$$

This shows that for any $p > 1$ and $\varepsilon > 0$,

$$\|I_\alpha f\|_{L^{1,\infty}(w)} \leq c\|M_\alpha f\|_{L^{1,\infty}(M_{L(\log L)^{p-1+2\varepsilon}}(w))}.$$

We conclude the proof of (3.4) by choosing $p = 1 + \delta - 2\varepsilon$ and ε such that $0 < 2\varepsilon < \delta$. □

4. LOCAL AND GLOBAL PARTS. RADIAL WEIGHTS

The purpose of this section is to prove Theorem 4.2 below. In fact, we are a going to show a more general result from which the theorem easily follows.

Following arguments from [16] we shall construct a collection of operators T such that (1.1) holds for any weight w, with $W = Tw$.

Let $k \in \mathbb{Z}$ and set

$$J_k = \{y \in \mathbb{R}^n : 2^k \leq |y| < 2^{k+1}\},$$

and

$$J_k^* = \{y \in \mathbb{R}^n : 2^{k-1} \leq |y| < 2^{k+2}\}.$$

Let f be a positive function and, for each $k \in \mathbb{Z}$, let us write

$$f = f_{k,0} + f_{k,1}, \quad f_{k,0} = f\chi_{J_k^*}.$$

Then,

$$\begin{aligned} I_\alpha f(x) &= \sum_{k\in\mathbb{Z}} (I_\alpha f(x))\chi_{J_k}(x) \\ &\leq \sum_{k\in\mathbb{Z}} (I_\alpha f_{k,0}(x))\chi_{J_k}(x) + \sum_{k\in\mathbb{Z}} (I_\alpha f_{k,1}(x))\chi_{J_k}(x) \\ &= I_\alpha^0 f(x) + I_\alpha^1 f(x). \end{aligned}$$

I_α^0 is called the local part and I_α^1 the global part of the operator I_α.

Clearly,

$$
\begin{aligned}
&w(\{x \in \mathbb{R}^n : I_\alpha f(x) > \lambda\}) \le \\
&\le w\left(\left\{x : I_\alpha^0 f(x) > \frac{\lambda}{2}\right\}\right) + w\left(\left\{x : I_\alpha^1 f(x) > \frac{\lambda}{2}\right\}\right) \\
&= \sum_{k\in\mathbb{Z}} w\left(\left\{x \in J_k : I_\alpha f_{k,0}(x) > \frac{\lambda}{2}\right\}\right) + \sum_{k\in\mathbb{Z}} w\left(\left\{x \in J_k : I_\alpha f_{k,1}(x) > \frac{\lambda}{2}\right\}\right) \\
&= L(w) + G(w).
\end{aligned}
$$

We shall estimate $L(w)$ and $G(w)$ separately.

The global part: To estimate the global part, we observe that if $x \in J_k$ and $y \notin J_k^*$, then $|x - y| \approx |x| + |y|$, and hence, for every $x \in J_k$,

$$
I_\alpha f_{k,1}(x) \le C \int_{\mathbb{R}^n} \frac{f(y)}{(|x| + |y|)^{n-\alpha}}\, dy \le C M_\alpha f(x) + C \int_{\{|y|>|x|\}} \frac{f(y)}{|y|^{n-\alpha}}\, dy.
$$

Therefore,

$$
\begin{aligned}
G(w) \le\ & w\left(\left\{x \in \mathbb{R}^n : M_\alpha f(x) > \frac{\lambda}{2C}\right\}\right) \\
&+ w\left(\left\{x \in \mathbb{R}^n : \int_{\{|y|>|x|\}} \frac{f(y)}{|y|^{n-\alpha}}\, dy > \frac{\lambda}{2C}\right\}\right) := \bar{G}(w).
\end{aligned}
$$

Using (1.4) in the first term and Chebyshev's inequality in the second, we get that

$$
G(w) \le \frac{C}{\lambda} \int_{\mathbb{R}^n} f(y) M_\alpha w(y)\, dy.
$$

This shows that estimate (1.3) does hold for general weights w, if we replace I_α by its global part I_α^1.

The local part: To estimate this part we use that I_α is self adjoint:

$$
\begin{aligned}
L(w) &\le \frac{2}{\lambda} \sum_{k\in\mathbb{Z}} \int_{J_k} I_\alpha f_{k,0}(x) w(x)\, dx \\
&= \frac{2}{\lambda} \sum_{k\in\mathbb{Z}} \int_{J_k^*} f(y) I_\alpha(w\chi_{J_k})(y)\, dy \\
&= \frac{2}{\lambda} \int_{\mathbb{R}^n} f(y) T(w)(y)\, dy,
\end{aligned}
$$

where

$$
T(w)(y) = \sum_{k\in\mathbb{Z}} \int_{J_k} \frac{w(x)}{|x - y|^{n-\alpha}}\, dx \chi_{J_k^*}(y). \tag{4.1}
$$

Now, observe that if $x \in J_k$ and $y \in J_k^*$, then $|x-y| \le 2^{k+3}$, and hence the inner integral can be estimated by

$$\int_{|x-y|\le 2^{k+3}} \frac{w(x)}{|x-y|^{n-\alpha}}\,dx \le \sum_{j=-\infty}^{k+3} \frac{1}{2^{j(n-\alpha)}} \int_{|x-y|\le 2^j} w(x)\,dx := S_k.$$

Let $(h(j))_{j\in\mathbb{Z}}$ be any positive sequence such that, for every $k \in \mathbb{Z}$,

$$H(k) = \sum_{j=-\infty}^{k+3} \frac{1}{h(j)} < \infty. \tag{4.2}$$

Then,

$$S_k \le H(k) \sup_{j\in\mathbb{Z}} \frac{h(j)}{2^{j(n-\alpha)}} \int_{\substack{x\in J_k \\ |x-y|\le 2^j}} w(x)\,dx,$$

and if we denote by

$$M_{\alpha,h} f(x) = \sup_{j\in\mathbb{Z}} \frac{h(j)}{2^{j(n-\alpha)}} \int_{|x-y|\le 2^j} f(y)\,dy,$$

we obtain

$$T(w)(y) \le \sum_{k\in\mathbb{Z}} H(k) M_{\alpha,h}(w\chi_{J_k})(y)\chi_{J_k^*}(y).$$

Thus, if $y \in J_k^*$,

$$M_{\alpha,h}(w\chi_{J_k})(y) \le \sup_{j\in\mathbb{Z}} \frac{h(j)}{2^{j(n-\alpha)}} \int_{\substack{|x-y|\le 2^j \\ |y|/4\le|x|<4|y|}} w(x)\,dx := C_{\alpha h}(w)(y). \tag{4.3}$$

Finally, combining these estimates we have that

$$L(w) \le \frac{C}{\lambda} \int_{\mathbb{R}^n} f(y)\bar{H}(y)C_{\alpha,h}(w)(y)\,dy,$$

where

$$\bar{H}(y) = \sum_{k\in\mathbb{Z}} H(k)\chi_{J_k^*}(y).$$

Therefore, we have obtained the following result.

Theorem 4.1. *With the above definitions and notations, for every positive sequence h satisfying (4.2), we have that*

$$w(\{y \in \mathbb{R}^n : |I_\alpha f(y)| > \lambda\}) \le \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(y)| W(y)\,dy$$

holds, with

$$W(y) = M_\alpha w(y) + \bar{H}(y) C_{\alpha,h}(w)(y). \tag{4.4}$$

Examples. We present here several interesting consequences of the above theorem.

(i) If $h(j) = 2^{-\alpha j}$, then $M_{\alpha,h} \le CM$, where M is the Hardy-Littlewood maximal operator, and we get that (1.1) holds for

$$W(y) = M_\alpha w(y) + |y|^\alpha M(w)(y).$$

Now, if w is a radial decreasing weight, then

$$M(w)(y) \approx \frac{1}{|y|^n} \int_{|x|\le|y|} w(x)\,dx,$$

and, therefore, $|y|^\alpha Mw(y) \le M_\alpha w(y)$. A corollary of this gives the following result announced in the introduction.

Theorem 4.2. *There is a constant C such that if w is a radial decreasing weight, then*

$$w(\{x \in \mathbb{R}^n : |I_\alpha f(x)| > \lambda\}) \le \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)| M_\alpha(w)(x)\,dx. \tag{4.5}$$

Remark 4.3. It should be mentioned that reference [14] contains a particular form of Theorem 4.1, closely related to the result above, Theorem 4.2. We are thankful to the referee for pointing out to us this reference. We want to stress, nonetheless, that Theorem 4.2 as well as some of the arguments of this Section 4, are implicitly described and based on ideas from the paper [16], where the case of "radially decreasing weights" for singular integrals is considered.

A similar argument to that in the proof of Theorem 4.1, shows that in the case $h(j) = 2^{-\alpha j}$ we are considering, one can also see that (1.1) holds for

$$W(y) = M_\alpha w(y) + M(w|x|^\alpha)(y),$$

as well as for

$$W(y) = \sup_{r>0} \frac{\max(r,|y|)^\alpha}{r^n} \int_{|y-x|\le r} w(x)\,dx.$$

(ii) If $0 < \beta < \alpha$ and we take $h(t) = 2^{(\beta-\alpha)t}$, then

$$M_{\alpha,h} w(y) \le \frac{1}{\alpha-\beta} |y|^{\alpha-\beta} C_\beta w(y),$$

where

$$C_\beta w(y) = \sup_{r>0} \frac{1}{r^{n-\beta}} \int_{\substack{|x-y|\le r \\ |y|/4\le|x|<4|y|}} w(x)\,dx,$$

and, hence, we get that (1.1) holds for

$$W(y) = M_\alpha w(y) + \frac{1}{\alpha-\beta} |y|^{\alpha-\beta} C_\beta w(y).$$

(iii) If we take an increasing h defined on $[0,\infty)$, such that

$$\sum_{j=0}^{\infty}\frac{1}{h(j)}<\infty,$$

then in (4.1) we obtain

$$\begin{aligned}
T(w)(y) &\le \sum_{k\in\mathbb{Z}}\Bigg(\sum_{j=-\infty}^{k+3}\frac{1}{2^{j(n-\alpha)}}\int_{|x-y|\le 2^j}w(x)\,dx\Bigg)\chi_{J_k^*}(y)\\
&= \sum_{k\in\mathbb{Z}}\Bigg(\sum_{j=-\infty}^{k+3}\frac{h(k+3-j)}{h(k+3-j)2^{j(n-\alpha)}}\int_{|x-y|\le 2^j}w(x)\,dx\Bigg)\chi_{J_k^*}(y)\\
&\le \sum_{k\in\mathbb{Z}}\Bigg(\sup_{j\le k+3}\frac{h(k+3-j)}{2^{j(n-\alpha)}}\int_{|x-y|\le 2^j}w(x)\,dx\Bigg)\chi_{J_k^*}(y)\sum_{j=0}^{\infty}\frac{1}{h(j)}\\
&= C\sum_{k\in\mathbb{Z}}\Bigg(\sup_{j\in\mathbb{Z}}\frac{h((\log 2)^{-1}\log^+(2^{k+3}/2^j))}{2^{j(n-\alpha)}}\int_{|x-y|\le 2^j}w(x)\,dx\Bigg)\chi_{J_k^*}(y)\\
&\le C\sum_{k\in\mathbb{Z}}\Bigg(\sup_{j\in\mathbb{Z}}\frac{h((\log 2)^{-1}\log^+(16|y|/2^j))}{2^{j(n-\alpha)}}\int_{|x-y|\le 2^j}w(x)\,dx\Bigg)\chi_{J_k^*}(y)\\
&\le C\sup_{j\in\mathbb{Z}}\frac{h((\log 2)^{-1}\log^+(16|y|/2^j))}{2^{j(n-\alpha)}}\int_{|x-y|\le 2^j}w(x)\,dx.
\end{aligned}$$

In particular, if we define, for $t>0$,

$$h(t)=(1+t)(1+\log(1+t))^{1+\delta},$$

then we get that (1.1) holds with

$$W(y)=\sup_{r>0}\frac{(1+\log^+(|y|/r))(1+\log(1+\log^+(|y|/r)))^{1+\delta}}{r^{n-\alpha}}\times\int_{|x-y|\le r}w(x)\,dx.$$

(iv) Finally, we can also prove that, for every $0<\beta<\alpha$, (1.1) holds for

$$W(y)=\frac{1}{\alpha-\beta}C_\beta w(y)+M_\alpha((1+\log^+|\cdot|)w(\cdot))(y).$$

To see this, we observe that

$$\begin{aligned}
Tw(y) &\le \Bigg(\sum_{k\le 0}+\sum_{k>0}\Bigg)\Bigg(\sum_{j=-\infty}^{k+3}\frac{1}{2^{j(n-\alpha)}}\int_{|x-y|\approx 2^j}w(x)\chi_{J_k}(x)\,dx\Bigg)\chi_{J_k^*}(y)\\
&= I+II.
\end{aligned}$$

Let us take the following sequence $\{h(j)\}_{j\in\mathbb{Z}}$

$$h(j)=\begin{cases}2^{-j(\alpha-\beta)}, & j<0,\\ 1, & j\ge 0.\end{cases}$$

Then, we have

$$\begin{aligned}I&\le\sum_{k\le 0}\frac{2^{k(\alpha-\beta)}}{\alpha-\beta}\sup_j\left(\frac{1}{2^{j(n-\beta)}}\int_{|x-y|\approx 2^j}w(x)\chi_{J_k}(x)\,dx\right)\chi_{J_k^*}(y)\\&\le\frac{1}{\alpha-\beta}C_\beta w(y).\end{aligned}$$

On the other hand, II can be estimated as follows

$$\begin{aligned}II&\le\sum_{k=1}^{\infty}\sum_{j=-\infty}^{0}2^{j(\alpha-\beta)}C_\beta w(y)\chi_{J_k^*}(y)+\sum_{k=1}^{\infty}(k+3)C_\alpha w(y)\chi_{J_k^*}(y)\\&\approx\frac{1}{\alpha-\beta}C_\beta w(y)+C_\alpha((1+\log^+|\cdot|)w(\cdot))(y),\end{aligned}$$

where we have used that if $x\in J_k$, with $k\ge 1$, then $k+3\approx 1+\log|x|$.

Acknowledgment. We are grateful to one of the referees for pointing out reference [14].

References

[1] Colin Bennett and Robert Sharpley, *Interpolation of Operators*, Pure and Applied Mathematics, vol. 129, Academic Press Inc., Boston, MA, 1988, ISBN 0-12-088730-4. MR928802 (89e:46001)

[2] A. Cordoba and C. Fefferman, *A weighted norm inequality for singular integrals*, Studia Math. **57** (1976), 97–101. MR0420115 (54 #8132)

[3] David Cruz-Uribe and C.J. Neugebauer, *The structure of the reverse Hölder classes*, Trans. Amer. Math. Soc. **347** (1995), 2941–2960. MR1308005 (95m:42026)

[4] K.J. Falconer, *The Geometry of Fractal Sets*, Cambridge Tracts in Mathematics, vol. 85, Cambridge University Press, Cambridge, 1986, ISBN 0-521-25694-1, 0-521-33705-4. MR867284 (88d:28001)

[5] C. Fefferman and E.M. Stein, *Some maximal inequalities*, Amer. J. Math. **93** (1971), 107–115. MR0284802 (44 #2026)

[6] José García-Cuerva and José L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland Mathematics Studies, vol. 116, North-Holland Publishing Co., Amsterdam, 1985, ISBN 0-444-87804-1. MR807149 (87d:42023)

[7] Miguel de Guzmán, *Real Variable Methods in Fourier Analysis*, North-Holland Mathematics Studies, vol. 46, North-Holland Publishing Co., Amsterdam, 1981, ISBN 0-444-86124-6. MR596037 (83j:42019)

[8] JAN MALÝ, *Personal communication to the second author*.

[9] M. TRINIDAD MENÁRGUEZ and FERNANDO SORIA, *Weak type* $(1,1)$ *inequalities of maximal convolution operators*, Rend. Circ. Mat. Palermo (2) **41** (1992), 342–352. MR1230582 (94i:42025)

[10] DETLEF MÜLLER and FERNANDO SORIA, *A double-weight* L^2*-inequality for the Kakeya maximal function*, J. Fourier Anal. Appl. (1995), 467–478. MR1364903 (96k:42026)

[11] CARLOS PÉREZ, *Sharp* L^p*-weighted Sobolev inequalities*, Ann. Inst. Fourier (Grenoble) **45** (1995), 809–824. MR1340954 (96m:42032) (English, with English and French summaries)

[12] ———, *On sufficient conditions for the boundedness of the Hardy-Littlewood maximal operator between weighted* L^p*-spaces with different weights*, Proc. London Math. Soc. (3) **71** (1995), 135–157. MR1327936 (96k:42023)

[13] ———, *Weighted norm inequalities for singular integral operators*, J. London Math. Soc. (2) **49** (1994), 296–308. MR1260114 (94m:42037)

[14] YVES RAKOTONDRATSIMBA, *Weighted inequalities of weak type for the fractional integral operator*, Z. Anal. Anwendungen **17** (1998), 115–134. MR1616068 (99j:42029)

[15] ERIC SAWYER, *A two weight weak type inequality for fractional integrals*, Trans. Amer. Math. Soc. **281** (1984), 339–345. MR719674 (85j:26010)

[16] FERNANDO SORIA and GUIDO WEISS, *A remark on singular integrals and power weights*, Indiana Univ. Math. J. **43** (1994), 187–204, http://dx.doi.org/10.1512/iumj.1994.43.43009. MR1275458 (95g:42028)

MARÍA J. CARRO, JAVIER SORIA:
Departament de Matemàtica Aplicada i Anàlisi
Universitat de Barcelona
08071 Barcelona, Spain .
E-MAIL: carro@ub.edu
E-MAIL: soria@ub.edu

CARLOS PÉREZ:
Departamento de Análisis Matemático
Universidad de Sevilla
41080 Sevilla, Spain .
E-MAIL: carlosperez@us.es

FERNANDO SORIA:
Departamento de Matemáticas
Universidad Autónoma de Madrid
28049 Madrid, Spain .
E-MAIL: fernando.soria@uam.es

KEY WORDS AND PHRASES: weighted inequalities, maximal operator, fractional integral
2000 MATHEMATICS SUBJECT CLASSIFICATION: 42B25, 26A33
Received: *August 17th, 2004; revised: May 5th, 2005.*