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A New Characterization of the Muckenhoupt A_p Weights Through an Extension of the Lorentz-Shimogaki Theorem

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ABSTRACT. Given any quasi-Banach function space X over $\mathbb{R}^n$ it is defined an index α_X that coincides with the upper Boyd index $\bar{\alpha}_X$ when the space X is rearrangement-invariant. This new index is defined by means of the local maximal operator $m_\lambda f$. It is shown then that the Hardy-Littlewood maximal operator M is bounded on X if and only if $\alpha_X < 1$ providing an extension of the classical theorem of Lorentz and Shimogaki for rearrangement-invariant X.

As an application it is shown a new characterization of the Muckenhoupt A_p class of weights: $u \in A_p$ if and only if for any $\varepsilon > 0$ there is a constant c such that for any cube Q and any measurable subset $E \subset Q$,

$$\frac{|E|}{|Q|}\log^{\varepsilon}\left(\frac{|Q|}{|E|}\right) \le c\left(\frac{u(E)}{u(Q)}\right)^{1/p}.$$

The case $\varepsilon = 0$ is false corresponding to the class $A_{p,1}$.

Other applications are given, in particular within the context of the variable L^p spaces.

1. INTRODUCTION

The main purpose of this paper is to provide a new way of defining the upper Boyd index for general function quasi-Banach spaces X over $\mathbb{R}^n$ with respect to the Lebesgue measure but *not* necessarily rearrangement-invariant. To do this we first investigate the question on the boundedness of the Hardy-Littlewood maximal operator M on X. This problem is characterized in Theorem 1.2 below.

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We give several applications of our main result. The most interesting is a *new* characterization of the A_p class of weights. This characterization is best understood if we compare it with the $A_{p,1}$ class of weights. Recall that a weight u satisfies the $A_{p,1}$ condition if there is a constant c such that for any cube Q and any measurable subset $E \subset Q$,

$$\frac{|E|}{|Q|} \le c \left(\frac{u(E)}{u(Q)} \right)^{1/p}, \tag{1.1}$$

where, as usual, $u(E) = \int_E u(x)\,\mathrm{d}x$. This class of weights is interesting because it characterizes the weights for which M is of restricted weak type (p,p), namely,

$$\sup_{t>0} t^p u\{x : M\chi_E(x) > t\} \le c u(E),$$

which, as it is well known, is equivalent to $M : L_u^{p,1} \to L_u^{p,\infty}$. Observe that $A_p \subset A_{p,1}$, although the inclusion is proper (see [7, 16] and also [17, 28]). We will show that $u \in A_p$ if and only if condition (1.1) is slightly "bumped" by multiplying the left-hand side by $\psi(|Q|/|E|)$ where ψ is so that $\lim_{t\to\infty} \psi(t) = +\infty$. As an special case we show that $u \in A_p$ if and only if for any $\varepsilon > 0$ there is a constant c such that for any cube Q and any subset $E \subset Q$,

$$\frac{|E|}{|Q|} \log^{\varepsilon}\left(\frac{|Q|}{|E|} \right) \le c \left(\frac{u(E)}{u(Q)} \right)^{1/p}.$$

The case $\varepsilon = 0$ is false corresponding to the class $A_{p,1}$. See Theorem 2.4 for the precise and more general statement of the result.

In the classical setting, if X is any rearrangement-invariant Banach function space, then the well-known result due to Lorentz [19] and Shimogaki [25] about the boundedness of M on X is formulated in terms of the upper Boyd index $\bar{\alpha}a_X$. It establishes that M is bounded on X if and only if $\bar{\alpha}a_X < 1$. This result was extended by Montgomery-Smith [20] to the case of any rearrangement-invariant quasi-Banach function space. In both cases, the key ingredient of the proofs was the fact that $(Mf)^*(t)$, the non-increasing rearrangement of Mf, is pointwise equivalent to $(1/t)\int_0^t f^*(\tau)\,\mathrm{d}\tau$. Then, since X is rearrangement-invariant, the problem is reduced to the study of the boundedness of the Hardy operator. Thus, the rearrangement-invariance of X is crucial in this approach.

However, in Analysis there are lots of important spaces that are not rearrangement-invariant in general. Examples include weighted Lebesgue, Lorentz or Orlicz spaces, Musielak-Orlicz spaces. For some particular spaces different criteria of the boundedness of M are well known. The aim of this paper is to provide a unified approach to the study of the boundedness of M on any quasi-Banach function space within spirit of the Lorentz-Shimogaki theorem.

To pursue this direction we introduce a generalized definition of the upper Boyd index. In this new approach, the main role is played by the so-called *local maximal operator* $m_\lambda f$ defined for any measurable function f by

$$m_\lambda f(x) = \sup_{Q \ni x} (f\chi_Q)^*(\lambda|Q|) \quad (0 < \lambda < 1),$$

where the supremum is taken over all cubes Q containing x, and f^* denotes the non-increasing rearrangement of f.

We give the following generalization of the upper Boyd index.

Definition 1.1. For any quasi-Banach function space X over $\mathbb{R}^n$, we define the non-increasing function Φ_X on $(0,1)$ as the operator norm of m_λ on X, namely,

$$\Phi_X(\lambda) = \|m_\lambda\|_X = \sup_{\|f\|_X \le 1} \|m_\lambda f\|_X \quad (0 < \lambda < 1).$$

We define the generalized upper Boyd index as

$$\alpha_X = \lim_{\lambda \to 0} \frac{\log \Phi_X(\lambda)}{\log(1/\lambda)}. \tag{1.2}$$

We observe that $\Phi_X(\lambda) \ge 1$, $0 < \lambda < 1$, since $|f| \le m_\lambda f$ a.e. (see [18, Lemma 6]). We will show below that the limit defining α_X exists because either $\Phi_X \equiv \infty$ or Φ_X is equivalent to a finite submultiplicative function on $(0,1]$, when $\Phi_X(\lambda_0) < \infty$, for some small positive λ_0.

Our main result is the following theorem, which can be regarded as an extension of the Lorentz-Shimogaki theorem.

Theorem 1.2. *Let $X(\mathbb{R}^n)$ be any quasi-Banach function space. The following statements are equivalent:*

(i) *M is bounded on X;*
(ii) *$\alpha_X < 1$;*
(iii) *$\Phi_X \in L^1(0,1)$;*
(iv) *$\lim_{\lambda \to 0} \lambda \Phi_X(\lambda) = 0$.*

Moreover, if the space X is rearrangement-invariant, then α_X coincides with the upper Boyd index of X, $\bar{\alpha}_X$.

As a consequence of this result, we can show that if X satisfies any of the condition of the theorem, then X has a certain kind of self-improving property. Indeed, if we let $M_r f(x) = M(|f|^r)(x)^{1/r}$, $0 < r < \infty$, for many particular spaces X, it has been observed that the boundedness of M on X implies the boundedness of M_r on X for some $r > 1$. This property is well known for weighted Lebesgue spaces [6, 21], and also for Lorentz spaces [2, 5] and for variable L^p spaces [10]. However, each case requires its own proof. For instance, in the case of weighted

Lebesgue spaces, it is easy to see that this is equivalent to the fact that the A_p condition of Muckenhoupt implies the $A_{p-\varepsilon}$ condition for some $\varepsilon > 0$. Theorem 1.2 implies easily that such a phenomenon occurs for *any* X, namely, we have the following result.

Corollary 1.3. *Let $X(\mathbb{R}^n)$ be any quasi-Banach function space. Then M is bounded on X if and only if M_r is bounded on X for some $r > 1$.*

The paper is organized as follows. In Section 2 we state some applications of Theorem 1.2 to weighted Lebesgue and Lorentz spaces, and also to variable L^p spaces. The results of Section 2 reveal a new approach to many previous well known results, as well as yielding some new results. The remaining sections of the paper provide the proofs of the results stated in Sections 1 and 2. Section 3 contains some preliminary information, Section 4 gives the proofs of Theorem 1.2 and Corollary 1.3, and, finally, Section 5 provides proofs of the results about applications stated in Section 2.

2. APPLICATIONS

2.1. *Weighted Lebesgue and Lorentz spaces* As usual, by a weight we mean any non-negative locally integrable function. Given a weight u, we denote by L^p_u, $p > 0$, the space of all measurable f for which

$$\|f\|_{L^p_u} \equiv \left(\int_{\mathbb{R}^n} |f(x)|^p\, u(x)\, dx\right)^{1/p} < \infty.$$

We say that a weight u satisfies the A_p, $1 < p < \infty$, condition if

$$\sup_Q \left(\frac{1}{|Q|}\int_Q u(x)\, dx\right)\left(\frac{1}{|Q|}\int_Q u(x)^{-1/(p-1)}\, dx\right)^{p-1} < \infty.$$

Given a locally integrable function f on $\mathbb{R}^n$, the Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|}\int_Q |f(y)|\, dy,$$

where the supremum is taken over all cubes Q containing x.

The following fundamental theorem was proved in [21] (see also [11, Chapter 4] or [27, Chapter 5]).

Theorem 2.1 (Muckenhoupt [21])**.** *Let $1 < p < \infty$. Then M is bounded on L^p_u if and only if $u \in A_p$.*

As we mentioned in the Introduction, $u \in A_{p,1}$ is equivalent to the condition that M be of restricted weak type (p,p) with respect to u. Thus, $A_p \subset A_{p,1}$; on the other hand, by the Stein-Weiss interpolation theorem [3, p. 233], $A_{p,1} \subset A_{p+\varepsilon}$ for any $\varepsilon > 0$. From this and from the fact that A_p implies $A_{p-\varepsilon}$ for some $\varepsilon > 0$ (see [6]), one can deduce the following well-known proposition, which is implicitly contained in [7, 16].

Proposition 2.2. *Let $1 < p < \infty$. Then $u \in A_p$ if and only if there exists $1 < q < p$ such that $u \in A_{q,1}$.*

Observe that $A_p \not\equiv A_{p,1}$; for example, $u(x) = |x|^{n(p-1)}$ satisfies $A_{p,1}$ but not A_p.

Our Theorem 1.2 yields several characterizations of the A_p condition. One of them is equivalent to Proposition 2.2; however its proof is much simpler, without the use of interpolation and the property $A_p \Rightarrow A_{p-\varepsilon}$. Another characterization seems to be new; it gives us a new and clearer way of formulating the difference between A_p and $A_{p,1}$.

First of all, we calculate the generalized upper Boyd index of L^p_u. Given a weight u, we define the function ν_u defined by

$$\nu_u(\lambda) = \inf_Q \inf_{E\subset Q:|E|=\lambda|Q|} \frac{u(E)}{u(Q)} \quad (0 < \lambda < 1).$$

Theorem 2.3. *For any $p > 0$ we have*

$$\Phi_{L^p_u}(\lambda) \asymp \frac{1}{\nu_u^{1/p}(\lambda)} \tag{2.1}$$

and

$$\alpha_{L^p_u} = \frac{1}{p} \lim_{\lambda\to 0} \frac{\log(1/\nu_u(\lambda))}{\log(1/\lambda)}. \tag{2.2}$$

Denote by $\mathcal{A}$ the class of increasing functions ψ on $[1,\infty)$ such that $\lim_{t\to\infty} \psi(t) = +\infty$, and $\psi(t) = O(t^\varepsilon)$ for any $\varepsilon > 0$. We have the following application of Theorem 1.2.

Theorem 2.4. *Let $1 < p < \infty$. Given a weight u, the following statements are equivalent.*

(i) *M is bounded on L^p_u;*
(ii) *$\lim_{\lambda\to 0} \nu_u(\lambda)/\lambda^p = +\infty$;*
(iii) *$\lim_{\lambda\to 0} \log(1/\nu_u(\lambda))/\log(1/\lambda) < p$;*
(iv) *if $\psi \in \mathcal{A}$, then for any cube Q and any subset $E \subset Q$,*

$$\frac{|E|}{|Q|}\psi\left(\frac{|Q|}{|E|}\right) \le c\left(\frac{u(E)}{u(Q)}\right)^{1/p}. \tag{2.3}$$

Some interesting examples to which the theorem can be applied are $\psi(t) = \log^{\varepsilon} t$ or $\psi(t) = \log\log^{\varepsilon}(e + t)$, $t \geq 1$, $\varepsilon > 0$, but the theorem is false for such functions when $\varepsilon = 0$.

The proof of this theorem *completely bypasses* the A_p condition. In particular, it avoids the use of the well known "reverse Hölder" property of the A_p class of weights (see [11] for several proofs of the classical theorem of Muckenhoupt). However, it is interesting to stress the following corollary.

Corollary 2.5. *Let $1 < p < \infty$. The A_p condition is equivalent to any of the conditions above.*

We remark that it is easy to show that condition (iii) is equivalent to Proposition 2.2. Conditions (ii) and (iv) seem to be new and (iv) reveals that the difference between the A_p and $A_{p,1}$ conditions is precisely the presence or absence of a factor $\psi(|Q|/|E|)$, where ψ is an arbitrary slowly increasing function. We also remark that the proofs of both Theorems 2.3 and 2.4 use only *basic* properties of M such as the weak type and the reverse weak type inequalities. Also, using only these basic properties, it is shown directly in next proposition that the A_p condition implies (2.3).

Proposition 2.6. *Let $1 < p < \infty$; then the A_p condition implies (2.3) with $\psi(t) = \log^{1-1/p}(1 + t)$.*

Consider now the weighted Lorentz spaces $\Lambda_u^p(w)$. Let u and w be weights defined on $\mathbb{R}^n$ and $\mathbb{R}_+$ respectively. The space $\Lambda_u^p(w), p > 0$, consists of all measurable f for which

$$\|f\|_{\Lambda_u^p(w)} = \left(\int_0^\infty (f_u^*(t))^p w(t)\,\mathrm{d}t \right)^{1/p} < \infty,$$

where $f_u^*(t)$ is the non-increasing rearrangement of f with respect to u defined by

$$f_u^*(t) = \inf\left\{\alpha > 0 : u\{x \in \mathbb{R}^n : |f(x)| > \alpha\} \leq t\right\} \quad (0 < t < u(\mathbb{R}^n)).$$

A full characterization of the boundedness of M on $\Lambda_u^p(w)$ for arbitrary u and w was obtained recently by Carro, Raposo, and Soria [5]; we also refer to [5] for a complete account of related results in this area. Here we mention only that in the case $w = 1$ and $p > 1$, $\Lambda_u^p(w)$ becomes the standard L_u^p spaces, and by Muckenhoupt's theorem, u must satisfy the A_p condition. On the other hand, the case of $\Lambda^p(w)$ (i.e., when $u = 1$) was characterized by Ariño and Muckenhoupt [2]; in this case w must satisfy the so-called B_p condition.

Denote $W(t) = \displaystyle\int_0^t w(s)\,\mathrm{d}s$. In [5, Theorem 3.3.5], among others, the following characterization was obtained.

Theorem 2.7 ([5]). *Let $0 < p < \infty$. M is bounded on $\Lambda_u^p(w)$ if and only if there exists $q < p$ such that for some constant c and for every finite family of cubes and sets $(Q_j, E_j)_j$ with $E_j \subset Q_j$.*

$$\frac{W(u(\bigcup_j Q_j))}{W(u(\bigcup_j E_j))} \leq c \max_j \left(\frac{|Q_j|}{|E_j|}\right)^q. \tag{2.4}$$

It is also mentioned in [5] that for a wide class of w, for instance for $w(t) = t^\alpha$, $\alpha > -1$, (2.4) is equivalent to the same condition but with a unique Q and $E \subset Q$. Thus, Theorem 2.7 represents a generalized version of Proposition 2.2.

We show that Theorems 2.3 and 2.4 and their proofs can be generalized with minor changes to the spaces $\Lambda_u^p(w)$. To be more precise, given weights u and w, we associate the function $\nu_{u,w}$ defined by

$$\nu_{u,w}(\lambda) = \inf_{\{Q_j\}} \inf_{\{E_j\}:E_j\subset Q_j,\ \min_j |E_j|/|Q_j|=\lambda} \frac{W(u(\bigcup_j E_j))}{W(u(\bigcup_j Q_j))},$$

where the infimum is taken over all finite families of cubes $\{Q_j\}$ and over all families of sets $\{E_j\}$ such that $E_j \subset Q_j$ with $\min_j |E_j|/|Q_j| = \lambda$.

Theorem 2.8. *For any $p > 0$ we have*

$$\Phi_{\Lambda_u^p(w)}(\lambda) \asymp \frac{1}{\nu_{u,w}^{1/p}(\lambda)} \tag{2.5}$$

and

$$\alpha_{\Lambda_u^p(w)} = \frac{1}{p} \lim_{\lambda\to 0} \frac{\log(1/\nu_{u,w}(\lambda))}{\log(1/\lambda)}. \tag{2.6}$$

Theorem 2.9. *Let $0 < p < \infty$. Given weights u and w, the following statements are equivalent.*

(i) *M is bounded on $\Lambda_u^p(w)$;*

(ii) $\displaystyle\lim_{\lambda\to 0} \frac{\nu_{u,w}(\lambda)}{\lambda^p} = +\infty$;

(iii) $\displaystyle\lim_{\lambda\to 0} \frac{\log(1/\nu_{u,w}(\lambda))}{\log(1/\lambda)} < p$;

(iv) *if $\psi \in \mathcal{A}$, then for any finite family of cubes $\{Q_j\}$ and any family of sets $\{E_j\}$ with $E_j \subset Q_j$,*

$$\min_j \frac{|E_j|}{|Q_j|}\, \psi\left(\frac{|Q_j|}{|E_j|}\right) \leq c \left(\frac{W(u(\bigcup_j E_j))}{W(u(\bigcup_j Q_j))}\right)^{1/p}.$$

Exactly as in Theorem 2.4, item (iii) here is a reformulation of Theorem 2.7 but with a different proof; items (ii) and (iv) are new.

2.2. *Variable L^p spaces* Let $p : \mathbb{R}^n \to [1, \infty)$ be a measurable function. Denote by $L^{p(\cdot)}(\mathbb{R}^n)$ the space of all measurable f on $\mathbb{R}^n$ such that for some $\lambda > 0$,

$$\int_{\mathbb{R}^n} \left| \frac{f(x)}{\lambda} \right|^{p(x)} \mathrm{d}x < \infty,$$

with norm

$$\|f\|_{L^{p(\cdot)}} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left| \frac{f(x)}{\lambda} \right|^{p(x)} \mathrm{d}x \le 1 \right\}.$$

The spaces $L^{p(\cdot)}(\mathbb{R}^n)$ are a special case of Musielak-Orlicz spaces (cf. [22]). The behavior of some classical operators in harmonic analysis on $L^{p(\cdot)}(\mathbb{R}^n)$ has been intensively investigated in recent years.

Denote by $\mathcal{P}(\mathbb{R}^n)$ the class of all measurable functions p for which M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. Given any measurable function p, let

$$p_- = \operatorname{ess\,inf}_{x \in \mathbb{R}^n} p(x) \quad \text{and} \quad p_+ = \operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(x).$$

Assume that $p_- > 1$ and $p_+ < \infty$. It has been proved by Diening [9] that if p satisfies the following uniform continuity condition:

$$|p(x) - p(y)| \le \frac{c}{\log(1/|x - y|)}, \quad |x - y| < \tfrac{1}{2}, \tag{2.7}$$

and if p is a constant outside some large ball, then $p \in \mathcal{P}(\mathbb{R}^n)$. After that, the second condition on p has been improved independently by Cruz-Uribe, Fiorenza, and Neugebauer [8] and Nekvinda [23]. It is shown in [8] that if p satisfies (2.7) and

$$|p(x) - p_\infty| \le \frac{c}{\log(e + |x|)} \tag{2.8}$$

for some $p_\infty > 1$, then $p \in \mathcal{P}(\mathbb{R}^n)$. In [23], the boundedness of M is deduced from (2.7) and from an integral condition more general than (2.8): there exist constants c, p_∞ such that $0 < c < 1$, $p_\infty > 1$, and

$$\int_{\mathbb{R}^n} |p(x) - p_\infty| c^{1/|p(x) - p_\infty|} \, \mathrm{d}x < \infty. \tag{2.9}$$

We make several remarks about (2.9). First, since p is bounded, it is clear that (2.9) concerns the behavior of p at infinity. Next, (2.9) can be stated in a simpler way. Indeed, for any c_1 one can take $c_2 < c_1$ and a constant k depending on c_1, c_2 such that $c_2^{1/x} \le kxc_1^{1/x}$ for any $x \ge 0$. Therefore, (2.9) is equivalent to saying that there exist α, p_∞ such that $0 < \alpha < 1$, $p_\infty > 1$, and

$$\int_{\mathbb{R}^n} \alpha^{1/|p(x)-p_\infty|}\,\mathrm{d}x < \infty. \tag{2.10}$$

It is easy to see that (2.8) implies (2.10) with $\alpha < e^{-nc}$.

Using Theorem 1.2, we give a different approach to Nekvinda's theorem. We note that in the following result the requirement $p_- > 1$ is replaced by $p_- > 0$.

Theorem 2.10. *Let p be a bounded positive function with $p_- > 0$ satisfying (2.7) and (2.10) for some $\alpha \in (0,1)$ and $p_\infty > 0$. Then*

$$\|m_\lambda f\|_{L^{p(\cdot)}} \le \frac{c}{\lambda^{1/p_-}}\|f\|_{L^{p(\cdot)}} \quad (0 < \lambda < 1),$$

where c depends only on p and n.

Since (2.10) necessarily implies $p_\infty \ge p_-$, in the case $p_- > 1$ the conditions of Theorem 2.10 coincide with the conditions of Nekvinda's theorem. In this case Theorem 2.10 clearly yields $\alpha_{L^{p(\cdot)}} \le 1/p_- < 1$, and thus, by Theorem 1.2, the boundedness of M on $L^{p(\cdot)}$.

We should mention that when proving Theorem 2.10, we use a much simplified variant of Nekvinda's argument, from [23].

3. Preliminaries

3.1. *Local maximal operator* First of all, we recall that the non-increasing rearrangement of a measurable function f is defined by

$$f^*(t) = \inf\Big\{\alpha > 0 : \big|\{x \in \mathbb{R}^n : |f(x)| > \alpha\}\big| \le t\Big\} \quad (0 < t < \infty).$$

Recall that the local maximal operator $m_\lambda f$ is defined for any measurable function f by

$$m_\lambda f(x) = \sup_{Q \ni x}(f\chi_Q)^*(\lambda|Q|) \quad (0 < \lambda < 1),$$

where the supremum is taken over all cubes Q containing x. Then, it follows directly from the definitions that for any f and for all $x \in \mathbb{R}^n$,

$$m_\lambda f(x) > \alpha \iff M\chi_{\{|f|>\alpha\}}(x) > \lambda. \tag{3.1}$$

In particular, for any measurable set E,

$$m_\lambda(\chi_E)(x) = \chi_{\{M(\chi_E)>\lambda\}}(x). \tag{3.2}$$

We will use the following simple properties of m_λ:

$$(m_\lambda f(x))^\delta = m_\lambda(|f|^\delta)(x) \quad (\delta > 0), \tag{3.3}$$

$$m_\lambda(f+g)(x) \le m_{\lambda/2}f(x) + m_{\lambda/2}g(x). \tag{3.4}$$

They follow immediately from the corresponding properties of the rearrangements (see [3, p. 41]).

Lemma 3.1. *Let* $0 < \lambda < 1$. *For any measurable function* f,

$$f^*(2^n\lambda t) \le (m_\lambda f)^*(t) \le f^*(\lambda t/3^n) \quad (t > 0), \tag{3.5}$$

and

$$m_{2^n\lambda\xi}f(x) \le m_\xi(m_\lambda f)(x) \quad (x \in \mathbb{R}^n,\ \xi < \frac{1}{2^n}). \tag{3.6}$$

Proof. By (3.1) and the reverse weak type $(1,1)$ inequality for the maximal function [26],

$$\Big|\{x \in Q : m_\lambda f(x) > \alpha\}\Big| \ge \frac{1}{2^n\lambda}\Big|\{x \in Q : |f(x)| > \alpha\}\Big|,$$

whenever $|\{x \in Q : |f(x)| > \alpha\}| \le \lambda|Q|$. Therefore, setting $\alpha = ((m_\lambda f)\chi_Q)^*(t)$ we obtain

$$(f\chi_Q)^*(2^n\lambda t) \le ((m_\lambda f)\chi_Q)^*(t). \tag{3.7}$$

In particular, when $Q = \mathbb{R}^n$, (3.7) gives the left-hand inequality in (3.5). On the other hand, putting in (3.7) $t = \xi|Q|$, we immediately get (3.6).

Similarly, by (3.1) and the weak type $(1,1)$ property of M,

$$|\{x : m_\lambda f(x) > \alpha\}| \le \frac{3^n}{\lambda}\Big|\{x : |f(x)| > \alpha\}\Big|,$$

which is equivalent to the right-hand inequality in (3.5). ∎

3.2. *Quasi-Banach function spaces.* Let $\mathcal{M}_0$ be the set of all real-valued measurable functions on $\mathbb{R}^n$. A quasi-Banach function space X over $\mathbb{R}^n$ is a subspace of $\mathcal{M}_0$ equipped with a complete quasi-norm $\|\cdot\|_X$ such that:

- $\|f\|_X = 0 \iff f = 0$ a.e., $\|\alpha f\|_X = |\alpha|\,\|f\|_X$, $\|f+g\|_X \le c(\|f\|_X + \|g\|_X)$;
- $|f| \le |g|$ a.e. $\Rightarrow \|f\|_X \le \|g\|_X$;
- $|E| < \infty \Rightarrow \|\chi_E\|_X < \infty$;
- $0 \le f_k \uparrow f$ a.e. $\Rightarrow \|f_k\|_X \uparrow \|f\|_X$.

We will essentially use a version of the Aoki-Rolewicz theorem (see [1, 24] or [15, p. 3]), which asserts that for any $f_1, \dots, f_k$ one has

$$\Big\| \sum_{i=1}^{k} f_i \Big\|_X \le 4^{1/\rho} \Big(\sum_{i=1}^{k} \|f_i\|_X^\rho \Big)^{1/\rho}, \tag{3.8}$$

where $0 < \rho \le 1$ is given by $c = 2^{1/\rho - 1}$ where c is the "quasi-norm" constant.

We recall that two functions f and g from $\mathcal{M}_0$ are said to be equimeasurable if they have the same distribution function. A function space X is said to be rearrangement-invariant ('r-i' from now on) if $\|f\|_X = \|g\|_X$ for every pair of equimeasurable functions f and g.

3.3. The upper Boyd index. Originally, the notion of the upper Boyd index was given for r-i Banach function spaces (see [4] and [3, p. 149]). We refer to [3, Chapter 1] for a complete account concerning these spaces.

We briefly recall how the upper Boyd index is defined in [3, 4]. Indeed, the Luxemburg representation theorem [3, p. 62] says that for any r-i Banach function space X over $\mathbb{R}^n$ there is a r-i Banach function space $\bar{X}$ over $(0, \infty)$ such that $\|f\|_X = \|f^*\|_{\bar{X}}$. Given any function φ on $(0, \infty)$ the dilation operator E_t, $0 < t < \infty$, is defined by $E_t(\varphi)(s) = \varphi(st)$, $0 < s < \infty$; we denote by $h_X(t)$ the operator norm of $E_{1/t}$ from $\bar{X}$ to $\bar{X}$, that is,

$$h_X(t) = \sup_{\|\varphi\|_{\bar{X}} \le 1} \|E_{1/t}(\varphi)\|_{\bar{X}} \quad (t > 0).$$

Finally, the upper Boyd index $\bar{\alpha}_X$ of X is defined by

$$\bar{\alpha}_X = \inf_{1<t<\infty} \frac{\log h_X(t)}{\log t} = \lim_{t \to \infty} \frac{\log h_X(t)}{\log t}. \tag{3.9}$$

Observe that the function $h_X(t)$ can be defined more naturally, without the use of the space $\bar{X}$. Namely, given the dilation operator D_a defined on $\mathbb{R}^n$ by $D_a f(x) = f(ax), a > 0$, it is easy to see that $h_X(t)$ is the operator norm of $D_{(1/t)^{1/n}}$ from X to X. Indeed, given any φ on $(0, \infty)$, one can consider the function $A_\varphi(x) = \varphi^*(v_n |x|^n)$ on $\mathbb{R}^n$, where v_n is the volume of the unit ball. Then $(A_\varphi)^*(t) = \varphi^*(t)$ (we emphasize that here on the left-hand side the rearrangement is "n-dimensional", while on the right-hand side it is "one-dimensional"). Also, we use that $(D_a f)^*(t) = f^*(a^n t)$. Thus, for any function φ on $(0, \infty)$

$$\|E_{1/t}(\varphi)\|_{\bar{X}} = \|(E_{1/t}(\varphi))^*\|_{\bar{X}} = \|(D_{(1/t)^{1/n}} A_\varphi)^*\|_{\bar{X}} = \|D_{(1/t)^{1/n}} A_\varphi\|_X.$$

On the other hand,

$$\|D_{(1/t)^{1/n}} f\|_X = \|E_{1/t} f^*\|_{\bar{X}}.$$

From the last two identities we easily have that

$$h_X(t) = \sup_{\|f\|_X \le 1} \|D_{(1/t)^{1/n}} f\|_X \quad (t > 0). \tag{3.10}$$

Consider now the case of the quasi-Banach r-i space X. Curiously enough, we were not able to find in the literature the precise definition of Boyd indices of X over $\mathbb{R}^n$; for definitions given in the one-dimensional case we refer to [12, 14, 20]. Given any quasi-Banach r-i space $X(\mathbb{R}^n)$, we define its upper Boyd index $\bar{\alpha}_X$ by equality (3.9), where the function h_X is defined by (3.10).

3.4. *Submultiplicative functions.* Recall that the generalized upper index given in Definition 1.1 is given in terms of

$$\Phi_X(\lambda) = \|m_\lambda\|_X = \sup_{\|f\|_X \le 1} \|m_\lambda f\|_X \quad (0 < \lambda < 1).$$

In this section we show that this function is essentially equivalent to a submultiplicative function. We first give some properties of this class of functions that will be used in the proof of Theorem 1.2.

Let E be any subset of $\mathbb{R}_+$ such that $E \cdot E \subset E$. A non-negative function φ on E is said to be submultiplicative if

$$\varphi(\xi\lambda) \le \varphi(\xi)\varphi(\lambda) \quad (\lambda, \xi \in E).$$

Proposition 3.2 ([3, p. 147])**.** *Let ψ be any non-decreasing submultiplicative function on $[1, \infty)$ with $\psi(1) = 1$. Then*

$$\int_1^\infty \psi(t) \frac{dt}{t^2} < \infty \iff \bar{\alpha}(\psi) < 1,$$

where

$$\bar{\alpha}(\psi) = \lim_{t \to \infty} \frac{\log \psi(t)}{\log t} = \inf_{t > 1} \frac{\log \psi(t)}{\log t}.$$

Proposition 3.3. *Let φ be any non-increasing submultiplicative function on $(0, 1]$ with $\varphi(1) = 1$. Then*

$$\int_0^1 \varphi(\lambda)\, d\lambda < \infty \iff \bar{\alpha}(\varphi) < 1 \iff \lim_{\lambda \to 0} \lambda\varphi(\lambda) = 0$$

where

$$\bar{\alpha}(\varphi) = \lim_{\lambda \to 0} \frac{\log \varphi(\lambda)}{\log(1/\lambda)} = \inf_{\lambda < 1} \frac{\log \varphi(\lambda)}{\log(1/\lambda)}.$$

The first equivalence follows from the previous proposition, and the second one is trivial.

The following lemma shows that except for the trivial case $\Phi_X \equiv \infty$, Φ_X is equivalent to a finite non-increasing submultiplicative function near the origin. This is enough to give meaning to the limit in Definition 1.1 since, by Proposition 3.3, the limit defining α_X exists:

$$\alpha_X = \lim_{\lambda\to 0} \frac{\log \Phi_X(\lambda)}{\log(1/\lambda)}.$$

Lemma 3.4. *Let X be any quasi-Banach function space. If $\Phi_X(\lambda_0) < \infty$, for some $\lambda_0 \in (0, 1/4^n]$ then there is a non-increasing, submultiplicative on $(0,1]$ function $\tilde{\Phi}_X$ such that $\tilde{\Phi}_X(1) = 1$, and*

$$c\Phi_X(\lambda) \le \tilde{\Phi}_X(\lambda) \le \Phi_X(\lambda) \quad (0 < \lambda < 1), \tag{3.11}$$

where c depends only on X.

Proof. It follows from (3.6) that

$$\|m_{2^n\lambda\xi} f\|_X \le \|m_\xi(m_\lambda f)\|_X \le \Phi_X(\xi)\|m_\lambda f\|_X \le \Phi_X(\xi)\Phi_X(\lambda)\|f\|_X,$$

and thus,

$$\Phi_X(2^n\lambda\xi) \le \Phi_X(\xi)\Phi_X(\lambda) \quad (\lambda < 1,\ \xi < \frac{1}{2^n}). \tag{3.12}$$

Set now

$$\tilde{\Phi}_X(\lambda) = \sup_{0<\xi<1} \frac{\Phi_X(\xi\lambda)}{\Phi_X(\xi)} \quad (0 < \lambda \le 1).$$

It is clear that $\tilde{\Phi}_X$ is submultiplicative on $(0,1]$ and $\tilde{\Phi}_X(1) = 1$. Next, $\tilde{\Phi}_X$ is non-increasing because Φ_X is so. Also, due to the fact that Φ_X is non-increasing, the left-hand inequality in (3.11) holds trivially with $c_1 = 1/\Phi_X(1-)$. Further, it follows from (3.12) that

$$\frac{\Phi_X(\xi\lambda)}{\Phi_X(\xi)} \le \frac{\Phi_X(\xi/2^n)}{\Phi_X(\xi)} \Phi(\lambda) \le \Phi_X(1/4^n)\Phi(\lambda),$$

which proves the right-hand inequality in (3.11) with $c_2 = \Phi_X(1/4^n)$. Observe that c_2 is finite since $\Phi_X(\lambda_0) < \infty$, $0 < \lambda_0 \le 1/4^n$. ◻

4. Proof of the Main Results

Denote $M^2 f = MMf$. We start with the following simple lemma.

Lemma 4.1. *There is a constant* c *such that for any* $0 < \lambda < 1$ *and measurable function* f

$$m_\lambda f(x) \le \frac{c}{\lambda \log(1/\lambda)} M^2 f(x) \quad (x \in \mathbb{R}^n). \tag{4.1}$$

Proof. Set

$$f^{**}(t) = t^{-1}\int_0^t f^*(\tau)\,\mathrm{d}\tau .$$

Then, using the well-known estimate $f^{**}(t) \le c(Mf)^*(t)$ [3, p. 122], we get

$$\begin{aligned}(f\chi_Q)^*(\lambda|Q|) &\le \frac{1}{\lambda|Q|}\int_0^{\lambda|Q|}(f\chi_Q)^*(\tau)\,\mathrm{d}\tau\\ &\le \frac{1}{\lambda\log(1/\lambda)|Q|}\int_0^{\lambda|Q|}(f\chi_Q)^*(\tau)\log\frac{|Q|}{\tau}\,\mathrm{d}\tau\\ &\le \frac{1}{\lambda\log(1/\lambda)|Q|}\int_0^{|Q|}(f\chi_Q)^{**}(\tau)\,\mathrm{d}\tau\\ &\le \frac{c}{\lambda\log(1/\lambda)|Q|}\int_0^{|Q|}(M(f\chi_Q))^*(\tau)\,\mathrm{d}\tau.\end{aligned}$$

Since for some geometric constant c,

$$M(f\chi_Q)(x) \le c\,\inf_Q Mf, \quad x \notin 2Q,$$

we have

$$\int_0^{|Q|}(M(f\chi_Q))^*(\tau)\,\mathrm{d}\tau \le c|Q|\,\inf_Q Mf + \int_{2Q} Mf \le c\int_{2Q} Mf,$$

which, along with the previous estimate, implies (4.1). ∎

Proof of Theorem 1.2. The equivalences (ii) $\iff$ (iii) $\iff$ (iv) follow from Lemma 3.4 combined with Propositions 3.3. We will show that (i) $\Rightarrow$ (iv) and (ii) $\Rightarrow$ (i).

If M is bounded on X, then M^2 is. Thus, by Lemma 4.1,

$$\|m_\lambda f\|_X \le \frac{c}{\lambda\log(1/\lambda)}\|f\|_X,$$

and hence,

$$\Phi_X(\lambda) \le \frac{c}{\lambda\log(1/\lambda)}.$$

Therefore $\lim_{\lambda\to 0}\lambda\Phi_X(\lambda) = 0$, which proves (i) $\Rightarrow$ (iv).

Assume now that (ii) holds. This means that there are constants $c > 0$ and $\delta < 1$ such that for any f,

$$\|m_\lambda f\|_X \le c\lambda^{-\delta}\|f\|_X. \tag{4.2}$$

We next observe that for any cube Q,

$$\frac{1}{|Q|}\int_Q |f| = \int_0^1 (f\chi_Q)^*(\lambda|Q|)\,\mathrm{d}\lambda,$$

and hence,

$$Mf(x) \le \int_0^1 m_\lambda f(x)\,\mathrm{d}\lambda \le \sum_{i=1}^{\infty} 2^{-i} m_{2^{-i}} f(x).$$

From this and from (3.8) along with (4.2), we obtain

$$\begin{aligned}\|Mf\|_X &\le \Big\|\sum_{i=1}^{\infty} 2^{-i} m_{2^{-i}} f\Big\|_X \le 4^{1/\rho}\Big(\sum_{=1}^{\infty}\|2^{-i}m_{2^{-i}}f\|_X^\rho\Big)^{1/\rho}\\ &\le c\Big(\sum_{i=1}^{\infty} 2^{-(1-\delta)\rho i}\Big)^{1/\rho}\|f\|_X \le c'\|f\|_X.\end{aligned}$$

This completes the proof of (ii) $\Rightarrow$ (i).

Let us show now that if the space X is r-i space, then $\alpha_X = \bar{\alpha}_X$. Consider the spherically symmetric rearrangement of f defined by

$$f^\star(x) = f^*(v_n|x|^n),$$

where v_n is the volume of the unit ball. Note that the functions f and $f^\star$ are equimeasurable. It follows from (3.5) that

$$(D_{(2^n\lambda)^{1/n}} f)^\star(x) \le (m_\lambda f)^\star(x) \le (D_{(\lambda/3^n)^{1/n}} f)^\star(x).$$

Therefore,

$$\|D_{(2^n\lambda)^{1/n}} f\|_X \le \|m_\lambda f\|_X \le \|D_{(\lambda/3^n)^{1/n}} f\|_X$$

and

$$h_X\Big(\frac{1}{2^n\lambda}\Big) \le \Phi_X(\lambda) \le h_X\Big(\frac{3^n}{\lambda}\Big).$$

From this and from the definitions (1.2) and (3.9), we readily obtain that $\alpha_X = \bar{\alpha}_X$. ◻

Proof of Corollary 1.3. The boundedness of M_r on X is equivalent to the boundedness of M on the space X_r with the norm

$$\|f\|_{X_r} = \| \, |f|^{1/r} \, \|_X^r.$$

From (3.3) we easily obtain that

$$\Phi_{X_r}(\lambda) = \Phi_X(\lambda)^r,$$

and therefore

$$\alpha_{X_r} = r\alpha_X.$$

By Theorem 1.2, $\alpha_X < 1$, and hence $\alpha_{X_r} < 1$ for some $r > 1$. Applying Theorem 1.2 again, we conclude that M is bounded on X_r. □

5. PROOFS RELATED TO THE APPLICATIONS

5.1. *Weighted Lebesgue and Lorentz spaces.* In order to prove Theorem 2.3, we will need the following two lemmas.

Lemma 5.1. *Let $p > 0$. We have*

$$\Phi_{L_u^p}(\lambda) = \left(\sup_E \frac{u\{x : M\chi E(x) > \lambda\}}{u(E)}\right)^{1/p} \quad (0 < \lambda < 1), \tag{5.1}$$

where the supremum is taken over all measurable sets E with $0 < u(E) < \infty$.

Proof. Denote the function on the right-hand side of (5.1) by $\psi_p(\lambda)$. It follows from the definition of $\Phi_{L_u^p}$ and from (3.2) that for any set E with $0 < u(E) < \infty$,

$$\Phi_{L_u^p}(\lambda) \geq \frac{\|m_\lambda(\chi_E)\|_{L_u^p}}{\|\chi_E\|_{L_u^p}} = \left(\frac{u\{x : M\chi_E(x) > \lambda\}}{u(E)}\right)^{1/p}.$$

Therefore, taking the supremum over all such E, we obtain

$$\psi_p(\lambda) \leq \Phi_{L_u^p}(\lambda). \tag{5.2}$$

On the other hand, by (3.1), for any measurable f we have

$$\begin{aligned} u\{x : m_\lambda f(x) > \alpha\} &= u\{x : M\chi_{\{|f|>\alpha\}}(x) > \lambda\} \\ &\leq \psi_p(\lambda)^p u\{x : |f(x)| > \alpha\} \quad (0 < \alpha < \infty). \end{aligned}$$

Multiplying this inequality by $p\alpha^{p-1}$ and then integrating with respect to $\alpha \in (0, \infty)$, we get

$$\|m_\lambda f\|_{L_u^p} \leq \psi_p(\lambda)\|f\|_{L_u^p}.$$

Thus, $\Phi_{L_u^p}(\lambda) \leq \psi_p(\lambda)$, which, along with (5.2), proves (5.1). □

Lemma 5.2. *Let* φ *be any non-increasing positive function on* $(0,1)$. *Given a weight* u, *the following statements are equivalent.*

(i) *There is a positive constant* c *such that for any measurable set* E,

$$u\{x : M\chi_E(x) > \lambda\} \le c\varphi(\lambda)u(E) \quad (0 < \lambda < 1); \tag{5.3}$$

(ii) *there is a positive constant* c *such that for any cube* Q *and any subset* $E \subset Q$ *with* $|E| < |Q|$,

$$\frac{1}{\varphi(|E|/|Q|)} \le c\frac{u(E)}{u(Q)}. \tag{5.4}$$

Proof. In the particular case of $\varphi(t) = t^p$, this lemma was proved in [16]. Almost the same proof works in a more general situation. We briefly outline the details.

Suppose that (5.3) holds. Let $E \subset Q$. Then, setting in (5.3) $\lambda = |E|/|Q|$, we easily get (5.4). Assume now that we have (5.4). Then u is doubling (i.e., there is a constant c such that $u(2Q) \le cu(Q)$ for any cube Q). Next, it follows from (5.4) that

$$M(\chi_E)(x) > \lambda \Rightarrow \frac{1}{\varphi(\lambda)} \le c\, M_u(\chi_E)(x),$$

where M_u is the weighted maximal function. Since u is doubling, M_u is of weak type $(1,1)$ with respect to u, and hence,

$$u\{x : M(\chi_E) > \lambda\} \le u\left\{x : M_u(\chi_E)(x) > \frac{1}{c\varphi(\lambda)}\right\} \le c'\,\varphi(\lambda)u(E),$$

proving (5.3). ◻

Proof of Theorem 2.3. We have to prove that

$$\Phi_{L^p_u}(\lambda) \asymp \frac{1}{\nu_u^{1/p}(\lambda)},$$

where we recall that

$$\nu_u(\lambda) = \inf_Q \inf_{E\subset Q:|E|=\lambda|Q|} \frac{u(E)}{u(Q)} \quad (0 < \lambda < 1).$$

It follows from Lemmas 5.1 and 5.2 that

$$\frac{1}{\Phi_{L^p_u}(|E|/|Q|)} \le c\left(\frac{u(E)}{u(Q)}\right)^{1/p} \quad (E \subset Q),$$

and therefore,

$$\frac{1}{\nu_u^{1/p}}(\lambda) \le c\,\Phi_{L_u^p}(\lambda). \tag{5.5}$$

On the other hand, by the definition of ν_u,

$$\nu_u\left(\frac{|E|}{|Q|}\right) \le \frac{u(E)}{u(Q)} \quad (E \subset Q).$$

It is clear also that $\varphi(\lambda) = 1/\nu_u(\lambda)$ is non-increasing. Hence, by Lemma 5.2,

$$u\{x : M(\chi_E)(x) > \lambda\} \le c\,\frac{u(E)}{\nu_u(\lambda)} \quad (0 < \lambda < 1).$$

From this and from Lemma 5.1 we obtain

$$\Phi_{L_u^p}(\lambda) \le \frac{c}{\nu_u^{1/p}(\lambda)},$$

which, along with (5.5), yields (2.1). Next, from (2.1) we trivially have (2.2). □

Proof of Theorem 2.4. The equivalences (i) $\iff$ (ii) $\iff$ (iii) follow from Theorems 1.2 and 2.3. To prove that (i)–(iii) are equivalent to (iv), we show that (iv) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv).

It follows from (2.3) that $\lambda\psi(1/\lambda) \le c\nu_u^{1/p}(\lambda)$ or, in other words,

$$\psi(1/\lambda)^p \le c\frac{\nu_u(\lambda)}{\lambda^p}.$$

Since $\lim_{t\to\infty}\psi(t) = +\infty$, we have that (iv) $\Rightarrow$ (ii).

It follows from (iii) that there is $\delta < 1$ such that $\lambda^\delta \le c\nu_u^{1/p}(\lambda)$. Therefore,

$$\left(\frac{|E|}{|Q|}\right)^\delta \le c\left(\frac{u(E)}{u(Q)}\right)^{1/p}$$

or, equivalently,

$$\frac{|E|}{|Q|}\left(\frac{|Q|}{|E|}\right)^{1-\delta} \le c\left(\frac{u(E)}{u(Q)}\right)^{1/p}.$$

Since $\psi(t) = O(t^\varepsilon)$ for all $\varepsilon > 0$, we have

$$\frac{|E|}{|Q|}\psi\left(\frac{|Q|}{|E|}\right) \le c\left(\frac{\omega(E)}{\omega(Q)}\right)^{1/p},$$

which completes the proof. □

Proof of Proposition 2.6. The proof is based on three well-known facts. The first one is that the A_p condition is trivially equivalent (see, e.g., [27, p. 195]) to that there exists $c > 0$ such that for any function f and any cube Q,

$$\frac{1}{|Q|}\int_Q |f|\,\mathrm{d}x \le c\left(\frac{1}{u(Q)}\int_Q |f|^p u\,\mathrm{d}x\right)^{1/p}. \tag{5.6}$$

The second fact is that any A_p weight is a doubling weight (this follows easily from (5.6)). The third fact is that for any doubling weight u one has (see, e.g., [13, p. 175] for the unweighted case; the proof readily works for any doubling weight)

$$\int_Q M_{u,Q}f(x)u(x)\,\mathrm{d}x \sim \int_Q |f(x)|\,\log\left(1+\frac{|f(x)|}{|f|_{Q,u}}\right)u(x)\,\mathrm{d}x, \tag{5.7}$$

where $M_{u,Q}f$ is the weighted maximal function supported in Q, and $f_{Q,u}$ is the weighted mean value of f over Q.

By (5.6) we have, for any Q and for all $x \in Q$, that

$$M_Q f(x) \le c(M_{u,Q}(|f|^p)(x))^{1/p}.$$

Therefore, setting in (5.6) $M_Q f$ instead of f, we get

$$\frac{1}{|Q|}\int_Q M_Q f\,\mathrm{d}x \le c\left(\frac{1}{u(Q)}\int_Q M_{u,Q}(|f|^p)u\,\mathrm{d}x\right)^{1/p}$$

Setting here $f = \chi_E$, where $E \subset Q$, and applying (5.7), we obtain

$$B\left(\frac{|E|}{|Q|}\right) \le c\left(B\left(\frac{u(E)}{u(Q)}\right)\right)^{1/p}, \tag{5.8}$$

where $B(t) = t\log(1+1/t)$. On the other hand, it is easy to see that

$$B^{-1}(B(t)^p) \sim t^p \log(1+1/t)^{p-1} \quad (0 < t < 1),$$

and therefore (5.8) implies

$$\left(\frac{|E|}{|Q|}\right)^p \log^{p-1}\left(1+\frac{|Q|}{|E|}\right) \le c\frac{u(E)}{u(Q)},$$

which proves (2.3) with $\psi(t) = \log^{1-1/p}(1+t)$. □

The proofs of Theorems 2.8 and 2.9 are almost identical to the proofs of Theorems 2.3 and 2.4. In order to prove Theorem 2.8, we will need two lemmas similar to Lemmas 5.1 and 5.2. We outline their proofs briefly. Then we give a brief proof of Theorem 2.8. We omit the proof of Theorem 2.9, since it follows from Theorem 2.8 exactly in the same way as Theorem 2.4 follows from Theorem 2.3.

Recall that $W(t) = \int_0^t w(s)\,ds$.

Lemma 5.3. *Let $p > 0$. We have*

$$\Phi_{\Lambda_u^p(w)}(\lambda) = \left(\sup_E \frac{W(u\{x : M\chi_E(x) > \lambda\})}{W(u(E))}\right)^{1/p} \quad (0 < \lambda < 1), \tag{5.9}$$

where the supremum is taken over all measurable sets E with $0 < u(E) < \infty$.

Proof. Denote the function on the right-hand side of (5.9) by $\psi_p(\lambda)$. Observe that $(\chi_E)_u^*(t) = \chi_{(0,u(E))}(t)$. From this and from (3.2) we easily obtain that

$$\psi_p(\lambda) \le \Phi_{\Lambda_u^p(w)}(\lambda).$$

On the other hand, by (3.1),

$$W(u\{x : m_\lambda f(x) > \alpha\}) \le \psi_p(\lambda)^p W(u\{x : |f(x)| > \alpha\}). \tag{5.10}$$

Since

$$W(u\{x : |f(x)| > \alpha\}) = \int_{\{t : f_u^*(t) > \alpha\}} w(t)\,dt,$$

we obtain from (5.10) that

$$\|m_\lambda f\|_{\Lambda_u^p(w)} \le \psi_p(\lambda)\|f\|_{\Lambda_u^p(w)}$$

Thus,

$$\Phi_{\Lambda_u^p(w)}(\lambda) \le \psi_p(\lambda),$$

which along with the opposite inequality proves (5.9). □

Lemma 5.4. *Let φ be any non-increasing positive function on $(0,1)$. The following statements are equivalent.*

(i) *There is a positive constant c such that for any measurable set E,*

$$W(u\{x : M\chi_E(x) > \lambda\}) \le c\varphi(\lambda)W(u(E)) \quad (0 < \lambda < 1); \tag{5.11}$$

(ii) *there is a positive constant c such that for any finite family of cubes $\{Q_j\}$ and any family of sets $\{E_j\}$ of positive measure with $E_j \subset Q_j$,*

$$\min_j \frac{1}{\varphi(|E_j|/|Q_j|)} \le c\,\frac{W(u(\bigcup_j E_j))}{W(u(\bigcup_j Q_j))}. \tag{5.12}$$

Proof. The proof of this lemma even simpler that the one of Lemma 5.2, since it follows directly from the definitions.

First we show that (i) $\Rightarrow$ (ii). Let $E_j \subset Q_j$, $j = 1, \dots, J$. Denote $E = \bigcup_j E_j$ and $\lambda = \min_j |E_j|/|Q_j|$. Then, clearly, $\bigcup_j Q_j \subset \{M\chi_E > \lambda\}$. From this and from (5.11) we obtain

$$W\Big(u\Big(\bigcup_j Q_j\Big)\Big) \le W(u\{M\chi_E > \lambda\}) \le c\varphi(\lambda)W(u(E))$$
$$\le c \max_j \varphi\left(\frac{|E_j|}{|Q_j|}\right) W(u(E)),$$

proving (5.12).

Conversely, for any compact set K there exists a finite family of cubes $\{Q_j\}$ such that $K \cap \{M\chi_E > \lambda\} \subset \bigcup_j Q_j$ and $|Q_j \cap E| > \lambda |Q_j|$. Hence, by (5.12),

$$W(u(K \cap \{M\chi_E > \lambda\})) \le W\Big(u\Big(\bigcup_j Q_j\Big)\Big)$$
$$\le \max_j \varphi\left(\frac{|E_j|}{|Q_j|}\right) W\Big(u\Big(\bigcup_j E \cap Q_j\Big)\Big)$$
$$\le \varphi(\lambda)W(u(E)).$$

From this, by a limiting argument, we get (5.11). ◻

Proof of Theorem 2.8. Recall that $\nu_{u,w}$ is defined by

$$\nu_{u,w}(\lambda) = \inf_{\{Q_j\}} \inf_{\{E_j\}:E_j \subset Q_j,\ \min_j |E_j|/|Q_j| = \lambda} \frac{W(u(\bigcup_j E_j))}{W(u(\bigcup_j Q_j))},$$

where the infimum is taken over all finite families of cubes $\{Q_j\}$ and over all families of sets $\{E_j\}$ such that $E_j \subset Q_j$ with $\min_j |E_j|/|Q_j| = \lambda$.

Now, by Lemmas 5.3 and 5.4,

$$\min_j \frac{1}{\Phi^p_{\Lambda^p_u(w)}(|E_j|/|Q_j|)} \le c \frac{W(u(\bigcup_j E_j))}{W(u(\bigcup_j Q_j))} \quad (E_j \subset Q_j),$$

and hence

$$\frac{1}{\Phi^p_{\Lambda^p_u(w)}(\lambda)} \le c\nu_{u,w}(\lambda). \tag{5.13}$$

Next, from the definition of $\nu_{u,w}$ we easily have

$$\min_j \nu_{u,w}\left(\frac{|E_j|}{|Q_j|}\right) \le \frac{W(u(\bigcup_j E_j))}{W(u(\bigcup_j Q_j))} \quad (E_j \subset Q_j).$$

From this and from Lemmas 5.3 and 5.4 we obtain the opposite estimate to (5.13), which proves the theorem. □

5.2. Variable L^p spaces.

Lemma 5.5. *Let r, q and φ be non-negative functions such that $r_- > 0$ and $0 \le \varphi \le 1$. Then for any constant $0 < \alpha < 1$ and for all $x \in \mathbb{R}^n$,*

$$\varphi(x)^{r(x)} \le \alpha^{1/|r(x)-q(x)|} + \left(\left(\frac{1}{\alpha}\right)^{1/r_-} + 1\right)\varphi(x)^{q(x)}.$$

Proof. Set $E_1 = \{x : r(x) < q(x)\}$ and

$$E_2 = \{x \in E_1 : \varphi(x) \le \alpha^{1/r(x)(q(x)-r(x))}\}, \quad E_3 = E_1 \setminus E_2.$$

Then

$$\begin{aligned}\varphi(x)^{r(x)} &= \varphi(x)^{r(x)}\chi_{E_2} + \left(\frac{1}{\varphi(x)}\right)^{q(x)-r(x)}\varphi(x)^{q(x)}\chi_{E_3} + \varphi(x)^{r(x)}\chi_{E_1^c}\\ &\le \alpha^{1/(q(x)-r(x))}\chi_{E_2} + \left(\frac{1}{\alpha}\right)^{1/r(x)}\varphi(x)^{q(x)}\chi_{E_3} + \varphi(x)^{q(x)}\chi_{E_1^c},\end{aligned}$$

proving the lemma. □

Proof of Theorem 2.10. By the definition of the $L^{p(\cdot)}$-norm, the statement of the theorem is equivalent to that there exists a constant $c > 0$ (not depending on f and λ) such that

$$\int_{\mathbb{R}^n} (\lambda^{1/p_-} m_\lambda f(x))^{p(x)}\,dx \le c \tag{5.14}$$

whenever $\int_{\mathbb{R}^n} |f(x)|^{p(x)}\,dx \le 1$. Fix such an f, and set $f_1 = f\chi_{\{|f|>1\}}$ and $f_2 = f - f_1$.

Let us show that for any x,

$$(\lambda^{1/p_-} m_{\lambda/2} f_1(x))^{p(x)} \le c\lambda m_{\lambda/2}(f_1^{p(\cdot)})(x), \tag{5.15}$$

$$(\lambda^{1/p_-} m_{\lambda/2} f_2(x))^{p(x)} \le c\left(\psi(x) + \lambda m_{\lambda/4}(f_2^{p(\cdot)})(x)\right), \tag{5.16}$$

with $\psi \in L^1$, where c and $\|\psi\|_{L^1}$ depend only on p and n. Assuming for a moment (5.15) and (5.16) to be true, we note that they easily imply (5.14). Indeed, the second inequality in (3.5) implies

$$\|m_\lambda f\|_{L^1} \le \frac{3^n}{\lambda}\|f\|_{L^1} \tag{5.17}$$

and since $\int_{\mathbb{R}^n} |f(x)|^{p(x)}\,\mathrm{d}x \le 1$, (5.17) shows that the L^1-norms of the right-hand sides of (5.15) and (5.16) are bounded by the constants depending only on p and n. Observing also that, by (3.4),

$$(\lambda^{1/p_-} m_\lambda f(x))^{p(x)} \le 2^{p_+}\Big((\lambda^{1/p_-} m_{\lambda/2} f_1(x))^{p(x)} + (\lambda^{1/p_-} m_{\lambda/2} f_2(x))^{p(x)}\Big),$$

we have that (5.15) and (5.16) imply (5.14).

To prove (5.15), fix an arbitrary cube Q containing x. We claim that

$$F(Q,x) \equiv \left(\lambda^{1/p_-} (f_1\chi_Q)^*\left(\lambda\frac{|Q|}{2}\right)\right)^{p(x)-p_-(Q)} \le c, \tag{5.18}$$

where $p_-(Q) = \operatorname{ess\,inf}_{x\in Q} p(x)$. Indeed, by Chebyshev's inequality,

$$\begin{aligned}\lambda^{1/p_-} (f_1\chi_Q)^*\left(\lambda\frac{|Q|}{2}\right) &\le \left(\frac{2}{|Q|}\right)^{1/p_-} \|f_1\|_{p_-}\\ &\le \left(\frac{2}{|Q|}\right)^{1/p_-} \left(\int_{\mathbb{R}^n} |f_1(x)|^{p(x)}\,\mathrm{d}x\right)^{1/p_-}\\ &\le \left(\frac{2}{|Q|}\right)^{1/p_-}.\end{aligned}$$

Hence, if $\operatorname{diam} Q \ge \frac{1}{2}$, we trivially obtain (5.18). If $\operatorname{diam} Q < \frac{1}{2}$, applying (2.7) yields

$$F(Q,x) \le \left(\frac{2}{|Q|}\right)^{c/p_- \log(1/\operatorname{diam} Q)} \le (2e\sqrt{n})^{nc/p_- \log 2},$$

proving (5.18). Now, it follows from (5.18) and (3.3) that

$$\begin{aligned}\left(\lambda^{1/p_-} (f_1\chi_Q)^*\left(\lambda\frac{|Q|}{2}\right)\right)^{p(x)} &\le c\left(\lambda^{1/p_-} (f_1\chi_Q)^*\left(\lambda\frac{|Q|}{2}\right)\right)^{p_-(Q)}\\ &= c\lambda^{p_-(Q)/p_-} (f_1^{p_-(Q)}\chi_Q)^*\left(\lambda\frac{|Q|}{2}\right)\\ &\le c\lambda m_{\lambda/2}(f_1^{p(\cdot)})(x),\end{aligned}$$

proving (5.15).

To prove (5.16), we apply Lemma 5.5 twice along with (3.3) and (3.4) which gives

$$
\begin{aligned}
&(\lambda^{1/p_-} m_{\lambda/2} f_2(x))^{p(x)} \\
&\quad \le \alpha^{1/|p(x)-p_\infty|} + ((1/\alpha)^{1/p_-} + 1)(\lambda^{1/p_-} m_{\lambda/2} f_2(x))^{p_\infty} \\
&\quad \le \alpha^{1/|p(x)-p_\infty|} + ((1/\alpha)^{1/p_-} + 1)(\lambda m_{\lambda/2}(|f_2|^{p_\infty})(x)) \\
&\quad \le \alpha^{1/|p(x)-p_\infty|} + ((1/\alpha)^{1/p_-} + 1)\Big(\lambda m_{\lambda/4}(\alpha^{1/|p(\cdot)-p_\infty|})(x) \\
&\qquad\qquad + ((1/\alpha)^{1/p_\infty} + 1)\lambda m_{\lambda/4}(|f_2|^{p(\cdot)})(x)\Big)
\end{aligned}
$$

where we have used that $p_\infty \ge p_-$. This proves (5.16) with

$$\psi(x) = \alpha^{1/|p(x)-p_\infty|} + \lambda m_{\lambda/4}(\alpha^{1/|p(\cdot)-p_\infty|})(x).$$

It remains to note that, by (5.17) and (2.10), $\|\psi\|_{L^1}$ depends only on p and n.

Thus, we have proved (5.15) and (5.16) which proves the theorem. □

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