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ON FRACTIONAL BROWNIAN MOTIONS AND RANDOM DYNAMICAL SYSTEMS

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Abstract

In this paper we consider a class of nonlinear stochastic partial differential equations (SPDEs) driven by a fractional Brownian motion with the Hurst parameter bigger than 1/2. We show that these SPDEs generate random dynamical systems.

Key words: *Fractional Brownian motions, Random dynamical systems, Stochastic differential equations.*

AMS subject classifications: *60H15, 37H10, 60H05.*

1 Introduction

A central mathematical object in Stochastics and Stochastic Processes is the Ito integral. It plays an important role in many areas of pure and applied mathematics including mathematical finance, population dynamics, fluid dynamics, statistics, signal processing, control, particle systems, to name a few. The integrator of such an integral is often chosen to be the Brownian motion (the Wiener process) or its semimartingale generalizations. These random functions are of unbounded total variation, so that their Stieltjes integrals do not exist. Special properties of the integrators and the integrands are necessary to generalize the definition of the Stieltjes integral to the Ito integral, and enable the definition of solutions of differential equations driven by Brownian motion.

A property of paramount importance to this effect for Brownian motion is the independence of its increments. To move beyond integrals and processes constructed using this property is one of the most important tasks in the theory of Stochastics. We are most interested in using the fractional Brownian motion (fBm) process B^H where $H \in (0,1)$ is fixed. It is a type of stochastic process which deviates significantly from Brownian motion and semimartingales.

As a centered Gaussian process, it is characterized by the stationarity of its increments and a medium- or long-memory property which is in sharp contrast with martingales and Markov processes. It also exhibits power scaling and path regularity properties with Holder parameter H, which are very distinct from Brownian motion (note that the Brownian motion is included in this family of models when considering $H = 1/2$). Fractional Brownian motion has become a popular choice of late for applications where classical processes cannot model these non-trivial properties; for instance long memory, which is also known as persistence, and corresponds to the case $H \in (1/2, 1)$, is of fundamental importance for financial data and in internet traffic, see [12], [16] . Fractional Brownian motion is also a good candidate to model random long time influences in climate systems, see [15].

Ever since the pioneering works of Zähle [17], Decreusefond and Üstünel [5], and Lyons [11], the main thrust has been to understand how to perform stochastic integration with respect to fBm in a way which is consistent with some properties of the classical Ito theory for Brownian motion. In the case of higher regularity ($H > 1/2$), simple trajectorial methods, labelled as pathwise, can be used which make it easy to translate one integration theory into another, as fractional derivatives allow a pathwise estimate of the integrals in terms of integrand and integrator using special norms. Pathwise integrals historically gave the first cases where adequate solutions to stochastic differential equations (SDEs) were established, e.g. Nualart and Rascanu [14]; infinite-dimensional equations have been treated with the same success as finite-dimensional ones, e.g. Nualart and Maslowski [13], Garrido-Atienza *et al.* [6].

In this paper, we aim to investigate the equations' asymptotics. There are two theories dealing with the asymptotic qualitative behavior for general SDEs: the theory of random dynamical systems (RDS) and the theory of existence and uniqueness of invariant measures for the associated Markov semigroup. However, similarly to fBm itself, equations driven by fBm do not generate a Markov process; this precludes the study of invariant measures using classical tools for fBm-driven systems. This motivates our plan to concentrate on the study of fBm-driven SDEs as RDS.

The theory of RDS, developed by L. Arnold and coworkers, see [1], can be used to describe the asymptotical and qualitative behavior of systems of random and stochastic differential/difference equation in terms of stability, Lyapunov exponents, invariant manifolds, and attractors.

As we have said, considering fBm instead of Brownian motion has some advantages because of the nice properties that the fBm enjoys and the Brownian motion does not. Another crucial advantage is the following: for many Brownian-driven SPDEs with non–trivial diffusion coefficients, it is not known if these equations generate a RDS. The reason is that usually stochastic differential equations are only defined almost surely where the exceptional set may depend on ω since this exceptional set is related to the definition of an Ito integral which is defined as a limit of random variables in probability. And such a family of exceptional sets does not allow to use the theory of RDS. But we can overcome such exceptional sets dealing with SPDEs driven by a fBm with $H > 1/2$, provided the stochastic integrals are interpreted in the pathwise sense.

2 Preliminaries on random dynamical systems

In this section we review some basic concepts and results on random dynamical systems that will be used later.

In the next definition, we introduce a system that models the evolution of a noise.

Definition 1 *A metric dynamical system* $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{T}})$ *with two-sided time* $\mathbb{T}$ *(which is* $\mathbb{R}$ *in the continuous case and* $\mathbb{Z}$ *in the discrete one) consists of a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and a family of transformations* $\{\theta_t\}_{t\in\mathbb{T}}$ *such that:*
1. It is a one-parameter group, i.e.

$$\theta_0 = \mathrm{id}_\Omega, \quad \theta_{t+s} = \theta_t\theta_s, \forall t, s \in \mathbb{T},$$

2. $(t,\omega) \in \mathbb{T}\times\Omega \to \theta_t\omega$ *is measurable,*
3. $\mathbb{P}$ *is invariant with respect to* θ*, i.e.,* $\theta_t\mathbb{P} = \mathbb{P}$*, for all* $t \in \mathbb{T}$*, which means that* $\mathbb{P}(\theta_t A) = \mathbb{P}(A)$*, for all* $A \in \mathcal{F}$ *and all* $t \in \mathbb{T}$*.*
4. $\mathbb{P}$ *is ergodic with respect to* θ*, i.e, for any* $\{\theta_t\}_{t\in\mathbb{T}}$*-invariant set* $B \in \mathcal{F}$*, which means that* $\theta_t B = B$ *for all* $t \in \mathbb{T}$*, we have either* $\mathbb{P}(B) = 0$ *or* $\mathbb{P}(B) = 1$*.*

We now introduce a couple of examples of metric dynamical systems. Let $V = (V, \|\cdot\|, (\cdot,\cdot))$ be a separable Hilbert space.

Consider first the Brownian motion. We choose for Ω the set of continuous functions $C_0^V = C_0(\mathbb{R}, V)$ on $\mathbb{R}$ with values in V which are zero at zero. On this set we introduce the compact open topology given by the uniform convergence on compact intervals in $\mathbb{R}$. The Borel–σ–algebra over this space is denoted by $\mathcal{B}(C_0^V)$. $\mathbb{P}_{\frac{1}{2}}$ is the Wiener measure. The existence of such a canonical process $(C_0^V, \mathcal{B}(C_0^V), \mathbb{P}_{\frac{1}{2}})$ follows by Kolmogorov's theorem about the existence of a continuous modification of a process, see Bauer [2]. The flow θ is given by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \qquad \omega \in \Omega \tag{1}$$

which is called the Wiener shift. The Wiener shift is measurable, see Arnold [1] Page 544, because C_0^V is separable and $(t,\omega) \mapsto \theta_t\omega$ is continuous. We emphasize that this metric dynamical system is ergodic, see Boxler [3].

Now let us introduce the fractional Brownian motion. Given $H \in (0,1)$, a continuous centered Gaussian process $\beta^H(t)$, $t \in \mathbb{R}$, with the covariance function

$$\mathbb{E}\beta^H(t)\beta^H(s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \qquad t,\, s \in \mathbb{R}$$

is called a *two–sided one-dimensional fractional Brownian motion* (fBm), and H is the *Hurst parameter.*

Assume that Q is a bounded and symmetric linear operator on V which is of trace class, i.e., there exist a complete orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ in V and a sequence of nonnegative numbers $\{\lambda_i\}_{i\in\mathbb{N}}$ such that $trQ = \sum_{i=1}^{\infty}\lambda_i < \infty$ and $Qe_i = \lambda_i e_i$, $i \in \mathbb{N}$. A continuous V*-valued fractional Brownian motion* B^H with incremental covariance operator Q and Hurst parameter H is defined by

$$B^H(t) = \sum_{i=1}^{\infty}\sqrt{\lambda_i}e_i\beta_i^H(t), \qquad t \in \mathbb{R}$$

where $\{\beta_i^H(t)\}_{i\in\mathbb{N}}$ is a sequence of stochastically independent one-dimensional fBm. Notice that the above series is convergent in $L^2(\Omega,\mathcal{F},\mathbb{P})$ since $\sum_{i=1}^{\infty}\lambda_i < \infty$ and $\mathbb{E}(\beta_i^H(t))^2 = |t|^{2H}$ for $t \in \mathbb{R}$.

Remark 1 $B^{1/2}$ *is the Brownian motion.*

Using the definition of B^H, Kolmogorov's theorem ensures that B^H has a continuous version. Thus we can consider the canonical interpretation of an fBm: let $\Omega = C_0(\mathbb{R}, V)$, equipped again with the compact open topology. Let $\mathcal{F}$ be the associated Borel-σ-algebra and $\mathbb{P}_H$ the distribution of the fBm B^H, and $\{\theta_t\}_{t\in\mathbb{R}}$ be the flow of Wiener shifts defined by (1). Then the quadruple $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is a metric dynamical system which is ergodic, see [9]. Furthermore,

$$B^H(\cdot,\omega) = \omega(\cdot), \quad B^H(\cdot,\theta_r\omega) = B^H(\cdot+r,\omega) - B^H(r,\omega) = \omega(\cdot+r) - \omega(r). \tag{2}$$

We now introduce the concept of random dynamical systems that is used to describe the dynamics of systems under the influence of a noise.

Definition 2 *A random dynamical system (RDS) with one-sided time* $\mathbb{T}^+$ *and phase space* V *is a pair consisting of the metric dynamical system* $(\Omega,\mathcal{F},\mathbb{P},\theta)$ *and a mapping* $\varphi : \mathbb{T}^+ \times \Omega \times V \to V$ *which is* $(\mathcal{B}(\mathbb{T}^+) \otimes \mathcal{F} \otimes \mathcal{B}(V), \mathcal{B}(V))$*-measurable and satisfies the cocycle property*

$$\varphi(t,\theta_\tau\omega,\cdot) \circ \varphi(\tau,\omega,\cdot) = \varphi(t+\tau,\omega,\cdot), \quad \textit{for } t,\, \tau \in \mathbb{T}^+,\, \omega \in \Omega,$$
$$\varphi(0,\omega,\cdot) = \mathrm{id}_V.$$

A typical example of cocycle mapping is the solution operator of finite or infinite dimensional differential equations with random coefficients satisfying particular regularity assumptions. Another example is the solution operator of finite dimensional Ito-equations. As we announced in the Introduction, for infinite dimensional Ito-equations with non-trivial diffusion coefficients this problem is rather unsolved.

Notice that the cocycle property is the generalization of the semigroup property; in fact, if we deleted all ω-dependence in the cocycle property we would just get the semigroup property.

We want to stress that we have required the MDS to be defined on two-sided time $\mathbb{T}$, while the RDS is only required to be defined on one-sided time $\mathbb{T}^+$. The reason is that we cannot expect the mapping φ to be defined on $\mathbb{T}$, since it is given, for instance, by the solution operator of a SPDE, which is not invertible in general. However, we can consider expressions of the following type: $\varphi(t, \theta_{-t}\omega, x)$, for $x \in V$, $\omega \in \Omega$, $t \in \mathbb{T}^+$, expressions that play a crucial role when analyzing the existence of random fixed points or random attractors associated to the RDS φ, see [8].

As we have mentioned, the purpose of this paper is to show that an infinite dimensional stochastic differential equation driven by an fBm with general diffusion coefficients generates a random dynamical system.

3 Main results

In this section we first introduce some basic concepts and results on fractional calculus and stochastic integrals with respect to the fBm β^H and B^H.

For $T > 0$, let $W^{\alpha,1}(0,T;V)$ be the space of measurable functions $f : [0,T] \to V$ such that

$$|f|_\alpha = \int_0^T \left(\frac{\|f(s)\|}{s^\alpha} + \int_0^s \frac{\|f(s) - f(\zeta)\|}{(s-\zeta)^{\alpha+1}} d\zeta \right) ds < \infty,$$

where $1 - H < \alpha < \frac{1}{2}$ is fixed, so we need to consider from now on $H \in (1/2, 1)$.

Following Zähle [17], for $f \in W^{\alpha,1}(0,T;V)$ we define the stochastic integral as the generalized Stieltjes integral

$$\int_0^T f d\beta^H = (-1)^\alpha \int_0^T D^\alpha_{0+} f(s) D^{1-\alpha}_{T-} \beta^H_{T-}(s) ds, \tag{3}$$

$$\int_s^t f d\beta^H = \int_0^T f \mathbf{1}_{(s,t)} d\beta^H, \qquad \text{for } 0 \le s < t \le T,$$

where, in general, for $0 \le a < b \le T$, $\beta^H_{b-}(s) := \beta^H(s) - \beta^H(b)$, and for $a < t < b$ the Weyl derivatives are given by

$$D^\alpha_{a+} f(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(t-a)^\alpha} + \alpha \int_a^t \frac{f(t) - f(\zeta)}{(t-\zeta)^{\alpha+1}} d\zeta \right),$$

$$D^{1-\alpha}_{b-} \beta^H_{b-}(t) = \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} \left(\frac{\beta^H(t) - \beta^H(b)}{(b-t)^{1-\alpha}} + (1-\alpha) \int_t^b \frac{\beta^H(t) - \beta^H(\zeta)}{(\zeta - t)^{2-\alpha}} d\zeta \right),$$

where Γ denotes the Gamma function. It can be proved (see, for instance, Nualart and Răşcanu [14], Decreusefond and Üstünel [5], Zähle [17]) that the stochastic integral (3) exists.

Now we define the stochastic integral with respect to the infinite dimensional fBm B^H. Let $L(V)$ denote the space of linear bounded operators on V and let $G : \Omega \times [0,T] \to L(V)$ be an operator such that $G(\omega, \cdot)e_i \in W^{\alpha,1}(0,T;V)$ for each $i \in \mathbb{N}$ and $\omega \in \Omega$. We define

$$\int_0^T G d\omega = \sum_{i=1}^\infty \int_0^T G(s) Q^{1/2} e_i d\beta^H_i(s) = \sum_{i=1}^\infty \sqrt{\lambda_i} \int_0^T G(s) e_i d\beta^H_i(s), \tag{4}$$

where the convergence of the sums in (4) is understood in V.

The following result establish that when making a change of variable in the stochastic integral, we not only have to shift the integration interval and the variable but also the path of the fBm (for the proof, see [6]).

Lemma 1 *For $a, b, r \in \mathbb{R}$, assuming that both integrals are well-defined,*

$$\int_a^b G(s)d\omega(s) = \int_{a-r}^{b-r} G(s+r)d\theta_r\omega(s).$$

Consider now the following stochastic evolution equation in V

$$\begin{cases} du(t) = (Au(t) + F(u(t)))dt + G(u(t))d\omega(t), \\ u(0) = u_0 \in V \end{cases} \tag{5}$$

where ω denotes the infinite dimensional fBm B^H (see (2)).

Assume that A is the infinitesimal generator of an analytic semigroup $S(\cdot)$, and that $F : V \to V$ is Lipschitz continuous with Lipschitz constant L_F, and $G : V \to L(V)$ and $G' : V \to L(V, L(V))$ are Lipschitz continuous in the following senses:

$$\sup_{i \in \mathbb{N}} \|G(v_1)e_i - G(v_2)e_i\| \le L_G\|v_1 - v_2\|, \tag{6}$$

$$\sup_{i \in \mathbb{N}} \|G'(v_1)e_i - G'(v_2)e_i\|_{L(V)} \le L'_G\|v_1 - v_2\|, \tag{7}$$

where $\{e_i\}_{i\in\mathbb{N}}$ is the complete orthonormal basis in V introduced in Section 2.

The solution of (5) on $[0,T]$ is a V-valued process u whose paths are for every $\omega \in \Omega$ elements of $W^{\alpha,1}(0,T;V)$, for an $\alpha \in (1-H, \frac{1}{2})$, and

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds + \int_0^t S(t-s)G(u(s))d\omega, \qquad t \in [0,T], \tag{8}$$

where the stochastic integral has to be understood according to (4).

For such an $\alpha \in (1-H, \frac{1}{2})$, denote by $W_{\xi,\sigma}^{\alpha,\infty}(0,T;V)$ the Banach space of measurable functions $x : [0,T] \to V$ such that

$$\|x\|_{\alpha,\xi,\sigma} = \sup_{t\in[0,T]} e^{-\sigma t}\left(\|x(t)\| + t^\xi \int_0^t \frac{\|x(t) - x(r)\|}{(t-r)^{1+\alpha}}dr\right) < \infty$$

for $\sigma \ge 1$, and $\xi \in [\alpha, 1-\alpha)$. The role of the factor t^ξ is crucial when proving the following existence theorem, which proof can be found in [6].

Theorem 2 *Let $\alpha \in (1-H, \frac{1}{2})$, $\sigma \ge 1$ and $\xi \in [\alpha, 1-\alpha)$. Assume F is Lipschitz continuous, and that G and G' satisfy (6) and (7). Then, for each initial point $u_0 \in V$ there exists a unique solution to equation (8) with its paths in $W_{\xi,\sigma}^{\alpha,\infty}(0,T;V)$. In addition, the mapping $\Phi : V \to W_{\xi,\sigma}^{\alpha,\infty}(0,T;V)$ given by $\Phi : u_0 \mapsto u$ is continuous for $\omega \in \Omega$.*

Theorem 3 *The solution u of (8) defines a random dynamical system $\varphi : \mathbb{R}^+ \times \Omega \times V \to V$, given by*

$$\varphi(t,\omega,u_0) = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds + \int_0^t S(t-s)G(u(s))d\omega.$$

Proof. The measurability follows by [4] Lemma III.14.

Trivially $\varphi(0,\omega,x) = u_0$. Let us check then the cocycle property: for t, $\tau \in \mathbb{R}^+$, $\omega \in \Omega$ and $u_0 \in V$, we have

$$\begin{aligned}\varphi(t+\tau,\omega,u_0) &= S(t+\tau)u_0 + \int_0^{t+\tau} S(t+\tau-s)F(u(s))ds \\ &+ \int_0^{t+\tau} S(t+\tau-s)G(u(s))d\omega(s) \\ &= S(t)\left(S(\tau)u_0 + \int_0^{\tau} S(\tau-s)F(u(s))ds + \int_0^{\tau} S(\tau-s)G(u(s))d\omega(s)\right) \\ &+ \int_{\tau}^{t+\tau} S(t+\tau-s)F(u(s))ds + \int_{\tau}^{t+\tau} S(t+\tau-s)G(u(s))d\omega(s).\end{aligned}$$

Making the change of variable $s-\tau = r$, applying Lemma 1,

$$\int_{\tau}^{t+\tau} S(t+\tau-s)G(u(s))d\omega(s) = \int_0^t S(t-r)G(u(r+\tau))d\theta_\tau\omega(r),$$

and then, setting $y(s) = u(s+\tau)$, for $s \in [0,t]$,

$$\begin{aligned}\varphi(t+\tau,\omega,u_0) &= S(t)y(0) + \int_0^t S(t-r)F(y(r))dr + \int_0^t S(t-r)G(y(r))d\theta_\tau\omega(r) \\ &= \varphi(t,\theta_\tau\omega,\cdot)\circ\varphi(\tau,\omega,u_0).\end{aligned}$$

□

Proving that our stochastic equation (8) generates a RDS is the starting point to analyze its asymptotic behavior. One possibility, which is a key concept describing the dynamics of RDS generated by fBm-driven SDEs, is the so-called global attractor, which is an invariant compact random set attracting other bounded random sets. The essential dynamics take place in a neighborhood of the attractor (see [8]). Another option to discuss the stability of fBm-driven SDEs is to study the existence of stable and unstable manifolds and Lyapunov exponents, see [10] and [7]. Such smooth manifolds are invariant under the dynamics of the systems, and on them, the states are attracted or repelled by a steady state.

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