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THE LAGRANGE–CHARPIT METHOD*

MANUEL DELGADO†

Abstract. We give a rigorous description of the Lagrange–Charpit method used to find a complete integral of a nonlinear p.d.e. adapted for a university course in differential equations.

Key words. integral surface, complete integral, Pfaff's equation

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1. Introduction. The concepts of the complete integral and the Lagrange–Charpit method are topics which appear with some frequency in texts which study nonlinear p.d.e.s in a classical way. There are some which do not use them; thus [3] and [5] describe only the method of characteristics. But the method of characteristics provides the integral surface solution of the Cauchy problem with uniqueness of solution; so, for problems without uniqueness, transversality or compatibility (see [5]) will not be true.

References [1] and [7] introduce the concept of the complete integral and indicate its utility for Cauchy's problem but don't give a method to find it. Other texts describe the method without a detailed discussion of the hypotheses or proof which justifies it (see [8] and [9]).

The most exhaustive study of this matter is in [4], but this book is not suitable for a modern course on differential equations. The objective of this note is to describe how to reach the most important results with a reasonable amount of work.

We find an interesting historical review of concepts of solution for nonlinear p.d.e.s of first order in [2] and a modern view of the method of Lagrange–Charpit from the point of view of the geometrical theory of p.d.e.s in [6].

2. The complete integral. It is well known that if a monoparametric family of integral surfaces of a p.d.e. of first order admits a real envelope, then this envelope is also an integral surface of the p.d.e. Moreover, it is easy to see that the p.d.e. satisfied by the functions implicitly defined through expressions of the form

$$h(\phi^1(x,y,z),\phi^2(x,y,z)) = 0, \quad \phi^1,\phi^2 \text{ given}, \quad h \in C^2(\mathbb{R}^2) \text{ arbitrary}$$

is a quasi-linear p.d.e. of first order.

The idea of Lagrange is as follows: the solution of a nonlinear p.d.e. can't be of the previous form, but if we know that a biparametric family of integral surfaces and both parameters are connected through an arbitrary regular function and the resulting uniparametric family has a real envelope, then this envelope must also be an integral surface of the equation.

Let us consider the nonlinear p.d.e. of the first order

$$F(x,y,z,p,q) = 0 \tag{1}$$

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†Differential Equations and Numerical Analysis Department, University of Seville, C/ Tarfia s/n, 41012 Seville, Spain (delgado@numer.us.es).

being $F : \tilde{U} \times \tilde{V} \subset \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}$, $\tilde{U}, \tilde{V}$ both open, $F \in C^2(\tilde{U} \times \tilde{V})$, and $F_p^2 + F_q^2 \neq 0$ with $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$.

DEFINITION 1. *A complete integral is a biparametric family*

$$\Phi(x, y, z, a, b) \quad \textit{with} \quad \Phi \in C^2(U \times \Lambda), \quad U \subset \mathbb{R}^3,\ \Lambda \subset \mathbb{R}^2 \ \textit{both open} \tag{2}$$

such that

$$\operatorname{rank} \begin{pmatrix} \Phi_a & \Phi_{xa} & \Phi_{ya} & \Phi_{za} \\ \Phi_b & \Phi_{xb} & \Phi_{yb} & \Phi_{zb} \end{pmatrix} = 2 \tag{3}$$

and such that for every $(a, b) \in \Lambda$ the expression

$$\Phi(x, y, z, a, b) = 0$$

determines one or several integral surfaces $z = \varphi(x, y)$ of (1).

Hypothesis (3) is sufficient to ensure that the constants a and b are independent.

THEOREM 1. *Let us consider the p.d.e.* (1). *Let*

$$z = \varphi(x, y, a, b), \quad \varphi \in C^2(G \times \Lambda), \quad G \subset \mathbb{R}^2 \ \textit{open}$$

be a family of integral surfaces determined by the complete integral $\Phi(x, y, z, a, b) = 0$.

Let $(x_0, y_0) \in G$ and $a_0 \in \mathbb{R}$ be such that there exist a neighborhood of a_0, I, and a function $\rho : I \to \mathbb{R}$, $\rho \in C^2(I)$ such that $(a, \rho(a)) \in \Lambda$ $\forall a \in I$. If the uniparametric family

$$z = \varphi(x, y, a, \rho(a))$$

has a real envelope in a neighborhood of (x_0, y_0, a_0), then this is an integral surface of (1).

Proof. We obtain the envelope of the family $z = \varphi(x, y, a, \rho(a))$ by eliminating a from the system

$$\begin{cases} z = \varphi(x, y, a, \rho(a)), \\ 0 = \varphi_a(x, y, a, \rho(a)) + \varphi_b(x, y, a, \rho(a)) \cdot \rho'(a) \end{cases} \tag{4}$$

in a neighborhood of a_0, which we also denote I.

The second equation of (4) determines a function, defined in a neighborhood of (x_0, y_0),

$$a = a(x, y) \quad (\text{with} \quad a_0 = a(x_0, y_0)),$$

and placing this in the first equation we obtain

$$z = \varphi(x, y, a(x, y), \rho(a(x, y))).$$

This surface is an integral surface because

$$z_x = \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial a}\frac{\partial a}{\partial x} + \frac{\partial \varphi}{\partial b}\rho'\frac{\partial a}{\partial x} = \frac{\partial \varphi}{\partial x},$$

$$z_y = \frac{\partial \varphi}{\partial y} + \frac{\partial \varphi}{\partial a}\frac{\partial a}{\partial y} + \frac{\partial \varphi}{\partial b}\rho'\frac{\partial a}{\partial y} = \frac{\partial \varphi}{\partial y},$$

and so

$$F(x, y, z, z_x, z_y) = F\left(x, y, \varphi, \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}\right)\Bigg|_{\substack{a=a(x,y)\\ b=\rho(a(x,y))}} = 0. \qquad \square$$

Remark. A sufficient condition for the existence of a real envelope is

$$\varphi_{aa} + 2\varphi_{ab}\rho' + \varphi_{bb}\rho'^2 + \varphi_b\rho''\rfloor_{a=a_0} \neq 0.$$

Furthermore, we can sometimes obtain from the complete integral other integral surfaces that we will call singular integral surfaces.

THEOREM 2. *Let $z = \varphi(x, y, a, b)$, $\varphi \in C^2(G \times \Lambda)$ be a family of integral surfaces determined by the complete integral of (1). Let $P \equiv (x_0, y_0, a_0, b_0) \in G \times \Lambda$ and denote $u_0 = \varphi(x_0, y_0, a_0, b_0)$. Suppose that*
(i) $\varphi_{ax}(P)\varphi_{by}(P) \neq \varphi_{ay}(P)\varphi_{bx}(P)$,
(ii) $\varphi_{aa}(P)\varphi_{bb}(P) \neq \varphi^2_{ab}(P)$.

Then there exists a unique singular integral surface defined in a neighborhood of (x_0, y_0), envelope of the biparametric family, which satisfies

$$F_p(x_0, y_0, u_0, \varphi_x(x_0, y_0, a_0, b_0), \varphi_y(x_0, y_0, a_0, b_0)) = 0,$$
$$F_q(x_0, y_0, u_0, \varphi_x(x_0, y_0, a_0, b_0), \varphi_y(x_0, y_0, a_0, b_0)) = 0.$$

Proof. Let us consider the system

$$\begin{cases} u = \varphi(x, y, a, b), \\ \varphi_a(x, y, a, b) = 0, \\ \varphi_b(x, y, a, b) = 0. \end{cases}$$

Since by (ii)

$$\frac{\partial(\varphi_a, \varphi_b)}{\partial(a, b)}(P) \neq 0,$$

it follows that there exist two functions that we also denote a and b:

$$a, \quad b : G_0 \subset G \to \Lambda_0 \subset \Lambda$$

such that

$$\varphi_a(x, y, a(x, y), b(x, y)) = \varphi_b(x, y, a(x, y), b(x, y)) = 0 \quad \forall (x, y) \in G_0.$$

Substituting these values in the first equation, we get

$$u = \varphi(x, y, a(x, y), b(x, y)) \quad (x, y) \in G_0,$$

a surface which contains the envelope of the biparametric family. We will prove that this envelope is a singular integral surface.

In fact, let φ be the integral surface derived from the complete integral

$$F(x, y, \varphi(x, y, a, b), \varphi_x(x, y, a, b), \varphi_y(x, y, a, b)) = 0 \quad (x, y, a, b) \in G \times \Lambda.$$

Differentiating with respect to a and b,

$$\begin{cases} F_u\varphi_a + F_p\varphi_{xa} + F_q\varphi_{ya} = 0, \\ F_u\varphi_b + F_p\varphi_{xb} + F_q\varphi_{yb} = 0. \end{cases}$$

Taking $(x, y) \in G_0$, it follows that $\varphi_a = \varphi_b = 0$ and

$$\begin{cases} F_p\varphi_{xa} + F_q\varphi_{ya} = 0, \\ F_p\varphi_{xb} + F_q\varphi_{yb} = 0, \end{cases}$$

and by (i) we will have $F_p = F_q = 0$. $\square$

3. The Lagrange–Charpit method. We will look for a complete integral for (1) of the form

$$\Phi(x, y, z, a, b) = \Psi(x, y, z, a) - b.$$

For every fixed b, the equivalence

$$\Phi(x, y, z, a, b) = 0 \Longleftrightarrow \Psi(x, y, z, a) = b$$

represents a uniparametric family of surfaces whose normal vector at every point is $(\frac{\partial\Psi}{\partial x}, \frac{\partial\Psi}{\partial y}, \frac{\partial\Psi}{\partial z})$. But if we consider the explicit equations defined implicitly by the former family, $z = \varphi(x, y, a, b)$, we have

$$\left(\frac{\partial\Psi}{\partial x}, \frac{\partial\Psi}{\partial y}, \frac{\partial\Psi}{\partial z}\right) = -\frac{\partial\Psi}{\partial z}(p, q, -1).$$

This would mean that Pfaff's equation

$$p(x, y, z, a)dx + q(x, y, z, a)dy - dz = 0$$

is integrable. So we must seek expressions $p(x, y, z, a)$ and $q(x, y, z, a)$ that satisfy (1) and that permit us to build an integrable Pfaff's equation.

We do this with the following theorem.

THEOREM 3. *Let $(x_0, y_0, z_0) \in U$, $(p_0, q_0) \in V$, $a_0 \in \tilde{\Lambda} \subset \mathbb{R}$ open be such that $F(x_0, y_0, z_0, p_0, q_0) = 0$ and let $G : U \times V \times \tilde{\Lambda} \to \mathbb{R}$ be a C^2 function. Suppose that*

$$\exists P_0 \equiv (x_0, y_0, z_0, p_0, q_0, a_0) \in U \times V \times \tilde{\Lambda} \quad \text{such that} \quad \frac{\partial(F, G)}{\partial(p, q)}(P_0) \neq 0. \tag{5}$$

Then there exists an open neighborhood Δ of $(x_0, y_0, z_0, a_0) \in \mathbb{R}^4$ and two unique C^1 functions $p, q : \Delta \subset \mathbb{R}^4 \to \mathbb{R}$ defined in Δ which satisfy

(i)

$$\left.\begin{aligned} F(x, y, z, p(x, y, z, a), q(x, y, z, a)) &= 0 \\ G(x, y, z, p(x, y, z, a), q(x, y, z, a), a) &= 0 \end{aligned}\right\} \quad \forall (x, y, z, a) \in \Delta, \tag{6}$$

(ii) $\forall (x, y, z, a) \in \Delta$,

$$\begin{aligned} &\frac{\partial(F, G)}{\partial(p, q)}\left[q\frac{\partial p}{\partial z} - p\frac{\partial q}{\partial z} + \frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}\right] \\ &-F_pG_x - F_qG_y - (pF_p + qF_q)G_z + (F_x + pF_z)G_p + (F_y + qF_z)G_q = 0. \end{aligned} \tag{7}$$

Proof. By the implicit function theorem

$$\exists \varepsilon, \delta > 0, \quad \exists!\ p, q : B_1((x_0, y_0, z_0, a_0), \delta) \to B_2((p_0, q_0), \varepsilon)$$

of regularity C^1 such that

(i) $p(x_0, y_0, z_0, a_0) = p_0,\ q(x_0, y_0, z_0, a_0) = q_0,$

(ii)

$$\left.\begin{aligned} F(x,y,z,p(x,y,z,a),q(x,y,z,a)) &= 0 \\ G(x,y,z,p(x,y,z,a),q(x,y,z,a),a) &= 0 \end{aligned}\right\} \quad \forall (x,y,z,a) \in B_1 \equiv \Delta,$$

which is (6).

If we differentiate the former equations with respect to x,

$$\begin{cases} F_x + F_p \dfrac{\partial p}{\partial x} + F_q \dfrac{\partial q}{\partial x} = 0, \\ G_x + G_p \dfrac{\partial p}{\partial x} + G_q \dfrac{\partial q}{\partial x} = 0, \end{cases}$$

from which

$$F_x G_p - F_p G_x - \frac{\partial(F,G)}{\partial(p,q)} \frac{\partial q}{\partial x} = 0. \tag{8}$$

Differentiating with respect to y and eliminating $\frac{\partial q}{\partial y}$, we get

$$F_y G_q - F_q G_y + \frac{\partial(F,G)}{\partial(p,q)} \frac{\partial p}{\partial y} = 0. \tag{9}$$

Differentiating with respect to z and eliminating $\frac{\partial p}{\partial z}$ and $\frac{\partial q}{\partial z}$, we obtain, respectively,

$$F_z G_p - F_p G_z - \frac{\partial(F,G)}{\partial(p,q)} \frac{\partial q}{\partial z} = 0, \tag{10}$$

$$F_z G_q - F_q G_z + \frac{\partial(F,G)}{\partial(p,q)} \frac{\partial p}{\partial z} = 0. \tag{11}$$

Then, multiplying (11) by q and (10) by p and adding with (9) and (8), we obtain (7). $\square$

THEOREM 4. *Let $G : U \times V \times \tilde{\Lambda} \to \mathbb{R}$ be a C^2 solution of the linear p.d.e.*

$$F_p G_x + F_q G_y + (pF_p + qF_q) G_z - (F_x + pF_z) G_p - (F_y + qF_z) G_q = 0 \tag{12}$$

which verifies hypothesis (5).

Let p, q be the functions whose existence is proved in Theorem 3. *Then Pfaff's equation*

$$p(x,y,z,a)dx + q(x,y,z,a)dy = dz \tag{13}$$

is integrable and its general solution is a complete integral of (1).

Proof. It suffices to prove that the necessary and sufficient condition for the integrability of (13) (see [8]) is

$$(p, q, -1) \cdot \text{curl } (p, q, -1) = 0 \Longleftrightarrow q \frac{\partial p}{\partial z} - p \frac{\partial q}{\partial z} + \frac{\partial p}{\partial y} - \frac{\partial q}{\partial x} = 0$$

and that this condition follows from (7) and (12).

The definition of integrability implies that there exist $\mu : \Delta \to \mathbb{R}$, C^1 such that $\mu(\bar{x}) \neq 0\ \forall \bar{x} \in \Delta$ and a function $\Psi : \Delta \to \mathbb{R}$ such that

$$\begin{cases} \dfrac{\partial \Psi}{\partial x}(x,y,z,a) = \mu(x,y,z,a)p(x,y,z,a), \\ \dfrac{\partial \Psi}{\partial y}(x,y,z,a) = \mu(x,y,z,a)q(x,y,z,a), \\ \dfrac{\partial \Psi}{\partial z}(x,y,z,a) = -\mu(x,y,z,a). \end{cases}$$

As we know, the general solution of (13) is the expression

$$\Psi(x,y,z,a) = b.$$

And since $\frac{\partial \Psi}{\partial z}(x,y,z,a) \neq 0$, the implicit function theorem allows us to define a function $\varphi : B_1((x_0,y_0,a_0,b_0),\delta_1) \to B_2(z_0,\delta_2)$, $z = \varphi(x,y,a,b)$ such that

$$\Psi(x,y,\varphi(x,y,a,b),a) = b \quad \forall (x,y,a,b) \in B_1.$$

Then

$$\begin{aligned} &\frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial z}\frac{\partial \varphi}{\partial x} = 0 \\ &\Rightarrow \mu(x,y,\varphi(x,y,a,b),a)p(x,y,\varphi(x,y,a,b),a) - \mu(x,y,\varphi(x,y,a,b),a)\frac{\partial \varphi}{\partial x} = 0 \\ &\Rightarrow \frac{\partial \varphi}{\partial x}(x,y,a,b) = p(x,y,\varphi(x,y,a,b),a). \end{aligned}$$

Analogously,

$$\frac{\partial \varphi}{\partial y}(x,y,a,b) = q(x,y,\varphi(x,y,a,b),a).$$

And so from (6) it follows that

$$F\left(x,y,\varphi(x,y,a,b),\frac{\partial \varphi}{\partial x}(x,y,a,b),\frac{\partial \varphi}{\partial y}(x,y,a,b)\right) = 0 \quad \forall (x,y,a,b) \in B_1.$$

Condition (3) is

$$\operatorname{rank}\begin{pmatrix} \Psi_a & \Psi_{xa} & \Psi_{ya} & \Psi_{za} \\ -1 & 0 & 0 & 0 \end{pmatrix} = 2$$

and is clearly true. $\square$

Is well known that equation (12) furnishes C^1 first integrals of the characteristic system

$$\text{(14)} \qquad \frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dz}{pF_p + qF_q} = \frac{dp}{-(F_x + pF_z)} = \frac{dp}{-(F_y + qF_z)}.$$

The Lagrange–Charpit method consists of
(i) finding a first integral of (14) which satisfies (5),

$$G(x, y, z, p, q) = a,$$

(ii) obtaining p and q from the system

$$\begin{cases} F(x, y, z, p, q) = 0, \\ G(x, y, z, p, q, a) = 0, \end{cases}$$

(iii) solving Pfaff's equation

$$p(x, y, z, a)dx + q(x, y, z, a)dy - dz = 0,$$

(iv) the expression $\Psi(x, y, z, a) = b$ being a complete integral of (1).

Remark. Hypothesis (5) can be weakened. It suffices to require that the Jacobian of F and G with respect to two variables is not zero (see [4]).

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