DYNAMICAL ZETA FUNCTIONS AND KUMMER CONGRUENCES

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Abstract. We establish a connection between the coefficients of Artin-Mazur zeta-functions and Kummer congruences.

This allows to settle positively the question of the existence of a map $T\colon X \to X$ such that the number of fixed points of T^n are $|E_{2n}|$, where E_{2n} are the Euler numbers. Also we solve a problem of Gabcke related to the coefficients of Riemann-Siegel formula.

Introduction

In this paper we establish a connection between two important topics the Artin-Mazur zeta function and Kummer's congruences. Some connection between Kummer's congruences and periodic points are pointed in the paper by Everest, van der Poorten, Puri and Ward [4].

Inspired by the Hasse-Weil zeta function of an algebraic variety over a finite field, Artin and Mazur [2] defined the Artin-Mazur zeta function for an arbitrary map $T\colon X \to X$ of a topological space X:

$$Z(T;x) := \exp\left(\sum_{n=1}^{\infty} \frac{\operatorname{Fix} T^n}{n} x^n\right).$$

Where $\operatorname{Fix} T^n$ is the number of isolated fixed points of T^n.

Manning [11] proved the rationality of the Artin-Mazur zeta function for diffeomorphisms of a smooth compact manifold satisfying Smale axiom A.

Following [12], call a sequence $a = (a_n)_{n\geq 1}$ of non-negative integers *realizable* if there is a set X and a map $T\colon X \to X$ such that a_n is the number of fixed points of T^n.

We must notice that in [12] it is proved that if (a_n) is realizable, then there exists a compact space X and a homeomorphism $T\colon X \to X$, such that $a_n = \operatorname{Fix} T^n$.

Puri and Ward [13] proved that a sequence of non-negative integers $(a_n)_{n\geq 1}$ is realizable if and only if $\sum_{d|n} \mu(n/d) a_d$ is non negative and divisible by n for all $n \geq 1$. Here $\mu(n)$ denotes the well known Möbius function (see [1]), defined by $\mu(n) = (-1)^k$ if n is a product of k different prime numbers, and $\mu(n) = 0$ if n is not squarefree.

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We shall delete the positivity condition, so we shall say that the sequence of integers $(a_n)_{n=1}^{\infty}$ is *pre-realizable* if $\sum_{d|n}\mu(n/d)a_d$ is divisible by n for every natural number.

In 1851 Kummer [9] discovered what we call Kummer's congruences for Bernoulli numbers, (see the book by Nielsen [10]). Carlitz [3] extended these congruences to the generalized Bernoulli numbers of Leopoldt. Some restrictions of Carlitz's results has been removed by the work of Fresnel [5]. These congruences are important for the definition of the p-adic L-functions.

We establish a connection between these concepts that we can formulate as in the following theorem.

Theorem 0.1. *Let (a_n) be a sequence that satisfies Kummer congruences for every rational prime, then for every natural number b the sequence $(a_{b+n})_{n=1}^{\infty}$ is pre-realizable.*

This theorem allow us to solve a problem posed by Gabcke [6]. This is connected with the Riemann-Siegel formula. In the investigation of the zeta function of Riemann it is important to compute the values of this function $\zeta(1/2+it)$ at points on the critical line with t very high. Riemann found a very convenient formula for these computations, yet he does not publish anything about this formula. In 1932 C. L. Siegel was able to recover it from Riemann's nachlass. Now this formula is known as the Riemann-Siegel formula.

To obtain the terms of this formula play a role certain numbers λ_n that can be defined by a recurrence relation

$$\lambda_0 = 1,$$
$$(n+1)\lambda_{n+1} = \sum_{k=0}^{n} 2^{4k+1}|E_{2k+2}|\lambda_{n-k}.$$

Here E_{2n} denotes Euler numbers defined by

$$\frac{1}{\cosh x} = \sum_{n=0}^{\infty}\frac{E_n}{n!}x^n.$$

Hence $E_{2n+1}=0$ for $n\ge 0$ and

$$E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad E_6 = -61, \quad E_8 = 1\,385, \quad \dots$$

Gabcke [6] observed that the first six numbers λ_n are integers and conjectured that this is so for all of them. Gabcke also considers analogous sequences (ϱ_n) and (μ_n). Although he does not mention it, the same motivations for his conjecture also supports that these too are integers sequences. We prove all these conjectures. The proof of these assertions was the first motivation of this paper.

In [13] Puri and Ward ask if the sequence $(|E_{2n}|)_{n\ge 1}$ is realizable. As we will see the solution of Gabcke's problem is connected with this one. We shall prove that in fact it is realizable.

Notations: When p is a prime number and m an integer we shall put $p^{\alpha}\parallel m$ to indicate that p^{α} is the greatest power of p that divides m. We indicate this relation also by $\nu_p(m)=\alpha$. We shall put $n\perp m$ to say that n and m are relatively prime.

1. Dynamical Zeta Function

Theorem 1.1. *Let $(a_n)_{n=1}^\infty$ be a sequence of complex numbers and define the sequence $(b_n)_{n=1}^\infty$ by*

$$nb_n = \sum_{d|n} \mu(n/d) a_d. \tag{1.1}$$

Then we have the equality between formal power series

$$\prod_{n=1}^\infty (1-x^n)^{-b_n} = \exp\Big(\sum_{n=1}^\infty \frac{a_n}{n}\Big).$$

Proof. By the well known Möbius inversion formula the relation (1.1) is equivalent to

$$a_n = \sum_{d|n} db_d, \tag{1.2}$$

therefore we have the following equalities between formal power series

$$\begin{aligned}\log \prod_{n=1}^\infty (1-x^n)^{-b_n} &= -\sum_{n=1}^\infty b_n \log(1-x^n) = \sum_{n=1}^\infty \sum_{k=1}^\infty b_n \frac{x^{nk}}{k} \\ &= \sum_{m=1}^\infty \frac{x^m}{m}\Big(\sum_{n|m} nb_n\Big) = \sum_{m=1}^\infty \frac{x^m}{m} a_m.\end{aligned}$$

And this is equivalent to the equality we want to prove. □

Theorem 1.2. *Let $(a_n)_{n=1}^\infty$ be a sequence of complex numbers and define the sequence $(A_n)_{n=0}^\infty$ by the recurrence relation*

$$\begin{aligned} A_0 &= 1, \\ (n+1)A_{n+1} &= \sum_{k=0}^n A_{n-k} a_{k+1}, \qquad n \geq 0. \end{aligned}$$

Then we have the equality between formal power series

$$\sum_{n=0}^\infty A_n x^n = \exp\Big(\sum_{n=1}^\infty \frac{a_n}{n} x^n\Big).$$

Proof. First we have the equality between formal power series

$$\sum_{n=1}^\infty nA_n x^{n-1} = \Big(\sum_{n=0}^\infty A_n x^n\Big)\Big(\sum_{n=1}^\infty a_n x^{n-1}\Big)$$

because by the hypothesis the coefficient of x^n is equal in both members.

Since $A_0 = 1$ integrating formally give us

$$\log\Big(\sum_{n=0}^\infty A_n x^n\Big) = \sum_{n=1}^\infty \frac{a_n}{n} x^n.$$

That is equivalent to the relation we wanted to prove. □

The following theorem gives various equivalent conditions for a sequence of integers $(a_n)_{n=1}^\infty$ to be pre-realizable.

Theorem 1.3. *Given a sequence $(a_n)_{n\geq 1}$ of integers, the following conditions are equivalent:*

(a) *The numbers $(b_n)_{n\geq 1}$ defined by*

$$nb_n = \sum_{d|n} \mu(n/d) a_d$$

are integers for every $n \in \mathbf{N}$.

(b) *The numbers $(A_n)_{n\geq 0}$ defined by*

$$A_0 = 1,$$

$$(n+1)A_{n+1} = \sum_{k=0}^{n} A_{n-k} a_{k+1}, \qquad n \geq 0 \tag{1.3}$$

are integers for every $n \geq 0$.

(c) *For every prime number p and natural numbers n, α with $p \perp n$ we have*

$$a_{np^\alpha} \equiv a_{np^{\alpha-1}} \pmod{p^\alpha}.$$

Proof. First we prove the equivalence of (a) and (b).

(a)$\Longrightarrow$(b). Assume (a). By the definition of the (b_n) and Theorem 1.1 we have

$$\prod_{n=1}^{\infty} (1-x^n)^{-b_n} = \exp\Bigl(\sum_{n=1}^{\infty} \frac{a_n}{n} x^n\Bigr).$$

and by the condition (a) of the theorem the b_n are integers. Let $(A_n)_{n=0}^{\infty}$ the numbers defined by 1.3, we have to show that they are integers. By Theorem 1.2 these numbers satisfies the relation

$$\sum_{n=0}^{\infty} A_n x^n = \exp\Bigl(\sum_{n=1}^{\infty} \frac{a_n}{n} x^n\Bigr).$$

Thus we have

$$\sum_{n=0}^{\infty} A_n x^n = \prod_{n=1}^{\infty} (1-x^n)^{-b_n}.$$

Expanding this product, since the b_n are integers, we get that the A_n are also integers. Hence we have that (a) implies (b).

(b)$\Longrightarrow$(a). Now, by hypothesis the numbers $(A_n)_{n\geq 0}$ are integers. We can determine inductively a unique sequence of integers c_n such that

$$\sum_{n=0}^{\infty} A_n x^n = \prod_{n=1}^{\infty} (1-x^n)^{-c_n}.$$

In the first step observe that the coefficients of x in both members must be the same, hence

$$A_1 = c_1.$$

Then observe that

$$(1-x)^{c_1} \Bigl(\sum_{n=0}^{\infty} A_n x_n\Bigr) = 1 + \sum_{n=2}^{\infty} A_n^{(2)} x^n,$$

where the numbers $A_n^{(2)}$ are integers.

Assume by induction that we have determined integers c_j, for $j = 1, 2, \ldots n-1$ such that
$$\prod_{j=1}^{n-1}(1-x^j)^{c_j}\Big(\sum_{n=0}^{\infty} A_n x^n\Big) = 1 + \sum_{k=n}^{\infty} A_k^{(n)} x^k.$$
Then the $A_k^{(n)}$ are integers and we can define $c_n = A_n^{(n)}$, that satisfies the induction hypothesis. Now we have
$$\prod_{j=1}^{\infty}(1-x^j)^{c_j}\Big(\sum_{n=0}^{\infty} A_n x^n\Big) = 1.$$
By the hypothesis and Theorem 1.2
$$\sum_{n=0}^{\infty} A_n x^n = \exp\left(\sum_{n=1}^{\infty}\frac{a_n}{n}x^n\right).$$
Therefore
$$\prod_{n=1}^{\infty}(1-x^n)^{-c_n} = \sum_{n=0}^{\infty} A_n x^n = \exp\left(\sum_{n=1}^{\infty}\frac{a_n}{n}x^n\right).$$
Now take logarithms in both members to obtain
$$\sum_{n=1}^{\infty} c_n \sum_{k=1}^{\infty}\frac{x^{kn}}{k} = \sum_{n=1}^{\infty}\frac{a_n}{n}x^n.$$
Reasoning as in the proof of Theorem 1.1 we get
$$a_m = \sum_{n|m} n c_n.$$
Therefore by the Möbius inversion formula $c_n = b_n$ the numbers defined on condition (a), and by construction these numbers c_n are integers. Thus we have proved (a).

(a)$\Longrightarrow$(c). We know that condition (a) is equivalent to the existence of integers b_n that satisfy the equation (1.2).

Assume that p is a prime number and n and α natural numbers such that $p \perp n$. Then
$$a_{np^\alpha} = \sum_{d|np^\alpha} d b_d = \sum_{k=0}^{\alpha}\sum_{d|n} dp^k b_{dp^k}.$$
Analogously
$$a_{np^{\alpha-1}} = \sum_{k=0}^{\alpha-1}\sum_{d|n} dp^k b_{dp^k}.$$
Therefore
$$a_{np^\alpha} - a_{np^{\alpha-1}} = \sum_{d|n} dp^\alpha b_{dp^\alpha} \equiv 0 \pmod{p^\alpha}.$$

(c)$\Longrightarrow$(a). Let n be an integer. We have to show that
$$\sum_{d|n}\mu(n/d)a_d$$
is divisible by n. Let $p^\alpha \parallel n$, with $\alpha \geq 1$, then $n = p^\alpha m$ with $p \perp m$.

Since $\mu(k)\neq 0$ only when k is squarefree, we get

$$\sum_{d|n}\mu(n/d)a_d=\sum_{d|m}\mu(m/d)a_{dp^\alpha}-\sum_{d|m}\mu(m/d)a_{dp^{\alpha-1}}$$
$$=\sum_{d|m}\mu(m/d)\big(a_{dp^\alpha}-a_{dp^{\alpha-1}}\big)\equiv 0\pmod{p^\alpha}.$$

The sum is divisible for every primary divisor of n, and therefore divisible by n. □

2. Kummer congruences

In 1851 Kummer [9] proved the following theorem:

Theorem 2.1 (Kummer)**.** *Let p be a prime number. Assume that*

$$\sum_{n=0}^\infty a_n\frac{x^n}{n!}=\sum_{k=0}^\infty c_k(e^{bx}-e^{ax})^k,$$

where a, b and the c_k are integral $\pmod p$. Then the a_n are integers $\pmod p$ and for $e\geq 1$, $n\geq 1$, $m\geq 0$, and $p^{e-1}(p-1)\mid w$ we have

$$\sum_{s=0}^n(-1)^s\binom{n}{s}a_{m+sw}\equiv 0\pmod{(p^m,p^{ne})}. \tag{2.1}$$

The congruences (2.1) are usually called Kummer congruences. We shall say that the sequence (a_n) satisfies Kummer congruences if we have (2.1) for every prime number p. By Kummer theorem these sequences exist, but we are interested in some particular sequences.

Theorem 2.2. *The sequence $(E_{2n})_{n=1}^\infty$ satisfies Kummer congruences.*

Proof. Since

$$\frac{1}{\cosh x}=\frac{2}{e^x+e^{-x}}=\frac{2}{2+(e^{x/2}-e^{-x/2})^2},$$

Kummer theorem proves that $(E_n)_{n=1}^\infty$ satisfies Kummer congruence (2.1) for every odd prime number p. Therefore the sequence $(E_{2n})_{n=1}^\infty$ satisfies these congruences for every odd prime number p. This reasoning can be found in Kummer [9].

The above procedure does not give the case $p=2$, but Fresnel [5] has extended Kummer congruences for Euler number, as we see in the following lines.

Let χ the function from $\mathbf{Z}$ to $\{-1,0,1\}$, such that $\chi(n)=0$ if n is even, $\chi(n)=1$ if $n\equiv 1\pmod 4$ and $\chi(n)=-1$ if $n\equiv 3\pmod 4$, then the generalized Bernoulli numbers associated to this character, —see [5] for details— are related to Euler numbers as

$$\frac{B^n(\chi)}{n}=-\frac{E_{n-1}}{2}.$$

In [5] p. 319 we found that, when $2^e\parallel w$, with $e\geq 1$

$$\sum_{s=0}^n(-1)^s\binom{n}{s}\frac{B_{m+sw}(\chi)}{m+sw}\equiv 0\pmod{(2^{n(e+2)},2^{m-1})}.$$

With a change of notation this is equivalent to

$$\sum_{s=0}^n(-1)^s\binom{n}{s}\frac{E_{2m+sw}}{2}\equiv 0\pmod{(2^{n(e+2)},2^{2m})}.$$

Obviously this implies that for $2^{e-1} \mid w$ we have

$$\sum_{s=0}^{n}(-1)^s\binom{n}{s}E_{2(m+sw)} \equiv 0 \pmod{(2^{ne}, 2^m)}.$$

□

The above theorem is a model of many more interesting examples. In Carlitz [3], it is proved that if χ is a primitive character mod$f$, and f is divisible by at least two distinct rational primes, then $B^n(\chi)/n$ is an algebraic integer and

$$\sum_{s=0}^{n}(-1)^s\binom{n}{s}\frac{B^{n+1+sw}(\chi)}{n+1+sw} \equiv 0 \pmod{(p^n, p^{en})},$$

if $p^{e-1}(p-1) \mid w$.

Thus the sequence $(a_n)_{n=1}^{\infty}$, with $a_n = B^{n+1}(\chi)/(n+1)$ satisfies Kummer congruences, if the character χ is real. When χ is complex the sequence defined by $a_n = \mathrm{Tr}(B^{n+1}(\chi)/(n+1))$ satisfies Kummer congruences.

The sequences that satisfies Kummer congruences are pre-realizable, as we will see in the following theorem.

Theorem 2.3. *Let $(a_n)_{n=1}^{\infty}$ a sequence that satisfies Kummer congruences. Then*

$$a_{b+np^{\alpha}} \equiv a_{b+np^{\alpha-1}} \pmod{p^{\alpha}},$$

for every natural numbers b, n, α and prime number p such that $p \perp n$.

That is to say that if (a_n) satisfies Kummer congruences then for every natural number b, the sequence $(a_{b+n})_{n=1}^{\infty}$ is pre-realizable.

Proof. By (2.1), with $n = 1$ we have

$$a_{m+p^{e-1}(p-1)} \equiv a_m \pmod{(p^m, p^e)}.$$

Therefore, for every natural number k, and assuming $m \geq e$

$$a_{m+kp^{e-1}(p-1)} \equiv a_m \pmod{p^e}.$$

Now take $m = b + np^{\alpha-1}$, $k = n$ and $e = \alpha$. If $b + np^{\alpha-1} \geq \alpha$, we get

$$a_{b+np^{\alpha-1}+np^{\alpha-1}(p-1)} \equiv a_{b+np^{\alpha-1}} \pmod{p^{\alpha}}.$$

Since $p^{\alpha-1} \geq \alpha$ for p prime and $\alpha \geq 1$, the condition is satisfied and we get

$$a_{b+np^{\alpha}} \equiv a_{b+np^{\alpha-1}} \pmod{p^{\alpha}}.$$

□

3. Euler numbers as numbers of fixed points

We are now in position to solve the problem posed by Puri and Ward in [13], they ask if the sequence $(|E_{2n}|)_{n=1}^{\infty}$ is realizable. We shall show that this is true.

Theorem 3.1. *There exists a map $T\colon X \to X$, such that*

$$|E_{2n}| = \mathrm{Fix}\, T^n.$$

Proof. First we show that $|E_{2n}|$ is a pre-realizable sequence. By Theorem 2.2 the sequence $(E_{2n})_{n=1}^{\infty}$ satisfies Kummer congruences. Thus by Theorem 2.3, for $p \perp m$,

$$E_{2mp^{\alpha}} \equiv E_{2mp^{\alpha-1}} \pmod{p^{\alpha}}.$$

Therefore,

$$|E_{2mp^{\alpha}}| \equiv |E_{2mp^{\alpha-1}}| \pmod{p^{\alpha}}.$$

By Theorem 1.3 it follows that the numbers b_n, defined by

$$nb_n = \sum_{d|n} \mu(n/d)|E_{2d}|,$$

are integers.

Now we must show that the numbers b_n are non negative. To this end we observe that

$$nb_n \geq |E_{2n}| - \sum_{d=1}^{n/2} |E_{2d}|.$$

Now we apply the well known formula

$$1 \leq \Bigl(\frac{\pi}{2}\Bigr)^{2d+1} \frac{|E_{2d}|}{(2d)!} = 2\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2d+1}} \leq 2.$$

Thus

$$\begin{aligned} nb_n &\geq (2n)!\Bigl(\frac{2}{\pi}\Bigr)^{2n+1} - 2\sum_{d=1}^{n/2}(2d)!\Bigl(\frac{2}{\pi}\Bigr)^{2d+1} \\ &\geq (2n)!\Bigl\{\Bigl(\frac{2}{\pi}\Bigr)^{2n+1} - 2\frac{(n)!}{(2n)!}\sum_{d=1}^{\infty}\Bigl(\frac{2}{\pi}\Bigr)^{2d+1}\Bigr\}. \end{aligned}$$

We can compute the last sum and we get $0.433\ldots$, therefore

$$nb_n \geq (2n)!\Bigl\{\Bigl(\frac{2}{\pi}\Bigr)^{2n+1} - \frac{(n)!}{(2n)!}\Bigr\}.$$

This is positive for $n \geq 2$, and we have $b_1 = 1 \geq 0$. $\square$

The first values of these three sequences in this case are the following:

a_n		1	5	61	1385	50521	2702765	199360981	$\ldots$
b_n		1	2	20	345	10104	450450	28480140	$\ldots$
A_n	1	3	23	371	10515	461869	28969177	2454072147	$\ldots$

4. Solution of Gabcke problem

By Theorem 1.3 the assertion of Gabcke, —that the numbers λ_n are integers—, is equivalent to say that the sequence $a_k = 2^{4k-3}|E_{2k}|$ is pre-realizable. We shall show that in fact it is realizable.

If (a_n) and (a'_n) are realizable, then the sequence $(a_n a'_n)$ is also realizable. In fact given $T\colon X \to X$ and $T'\colon Y \to Y$ such that $a_n = \operatorname{Fix} T^n$ and $a'_n = \operatorname{Fix} T'^n$, then it is easy to see that $T \times T'\colon X \times Y \to X \times Y$ satisfies $a_n a'_n = \operatorname{Fix}(T \times T')$.

Therefore by Theorem 3.1 what we need is to prove that the sequence 2^{4n-3} is realizable. This follows from the following theorem.

Theorem 4.1. *Let a and $b \in \mathbf{N}$ such that $b \mid a$ and for every prime number $p \mid a$ $p \mid (a/b)$. Then the sequence a^n/b is realizable.*

Proof. By the result of Puri and Ward we must show that the sequence a^n/b is pre-realizable and that the corresponding b_n are non-negative integers.

First let $p \perp a$ be a prime number, and let $n \perp p$ and α be natural numbers. We must show that

$$a^{np^\alpha}/b \equiv a^{np^{\alpha-1}}/b \pmod{p^\alpha}.$$

Since $b \perp p$, this is equivalent to

$$a^{np^\alpha} \equiv a^{np^{\alpha-1}} \pmod{p^\alpha}.$$

Now for $\alpha = 1$ this is Fermat's little theorem, and for a general α it follows, by induction, from the fact that for $\alpha \geq 1$ if $a \equiv b \bmod p^\alpha$, then $a^p \equiv b^p \bmod p^{\alpha+1}$.

Now if $p \mid a$, assume that $p^r \parallel a$ and $p^s \parallel b$. By hypothesis we have $r \geq s+1$. We have to show that

$$a^{np^\alpha}/b \equiv a^{np^{\alpha-1}}/b \pmod{p^\alpha},$$

where $p \perp n$ and $\alpha \geq 1$. But the two numbers are divisible by $p^{rnp^{\alpha-1}-s}$. All we have to show is that $rnp^{\alpha-1} \geq s+\alpha$. We can assume that $n=1$. For $\alpha=1$ this is $r \geq s+1$ that is true by hypothesis. For other values of α, $\alpha \geq 2$ and we have

$$rp^{\alpha-1} = r(p^{\alpha-1}-1) + r \geq (\alpha-1) + (s+1).$$

Now we define the numbers b_n by

$$nb_n = \sum_{d|n} \mu(n/d) a^d/b.$$

By the previous reasoning we know the b_n are integers. If $a=1$, it is easy to see that $b_1 = 1$ and $b_n = 0$ for $n>1$. In other case $a \geq 2$ and we have

$$nb_n \geq \frac{1}{b}\Big(a^n - \sum_{d=1}^{n/2} a^d\Big).$$

This is easyly seen to be non-negative. □

Corollary 4.2. *The sequence* $(a_n)_{n=1}^\infty$, *where* $a_n = 2^{4n-3}$ *is realizable.*

The three sequences associated to this realizable sequence are

a_n		2	32	512	8192	131072	2097152	33554432	536870912	...
b_n		2	15	170	2040	26214	349435	4793490	67107840	...
A_n	1	2	18	204	2550	33660	460020	6440280	91773990	...

Now we are in position to prove Gabcke's conjecture.

Theorem 4.3. *Let* λ_n *the numbers defined by*

$$\lambda_0 = 1,$$

$$(n+1)\lambda_n = \sum_{k=0}^{n} 2^{4k+1}|E_{2k+2}|\lambda_{n-k}, \qquad (n \geq 0), \tag{4.1}$$

ϱ_n *those defined by*

$$\varrho_0 = -1,$$

$$(n+1)\varrho_n = -\sum_{k=0}^{n} 2^{4k+1}|E_{2k+2}|\varrho_{n-k}, \qquad (n \geq 0), \tag{4.2}$$

and finally let $\mu_n = (\lambda_n + \varrho_n)/2$. *All those numbers are integers.*

Proof. By Theorem 3.1, and Corollary 4.2 the sequences $(|E_{2n}|)_{n=1}^{\infty}$ and $(2^{4n-3})_{n=1}^{\infty}$ are realizable. Since the product of two realizable sequences is realizable, the sequence $(2^{4n-3}|E_{2n}|)_{n=1}^{\infty}$ is realizable. Therefore it satisfies condition (a) of Theorem 1.3. So it satisfies condition (b), but this is precisely that the numbers λ_n are integers.

Now condition (c) of the same Theorem gives us that with $a_n = 2^{4n-3}|E_{2n}|$ we have for every prime number p and natural numbers $n \perp p$ and α that

$$a_{np^\alpha} \equiv a_{np^{\alpha-1}} \pmod{p^\alpha}.$$

Thus the same congruences are satisfied by the numbers $a'_n = -a_n$. Once again Theorem 3.1 says that the numbers a'_n satisfies condition (b). This is the same as saying that the numbers A'_n defined by

$$\begin{aligned} A'_0 &= 1 \\ (n+1)A'_n &= -\sum_{k=0}^{n} 2^{4k+1}|E_{2k+2}|A'_{n-k} \qquad (n \geq 0) \end{aligned} \tag{4.3}$$

are integers. But it is easyly seen that $\varrho_n = -A'_n$.

The affirmation about the numbers μ_n follows from the fact that $\lambda_n \equiv \varrho_n \pmod 2$. That we prove in Theorem 4.5. □

The following theorem is well known. I give a proof for completeness.

Theorem 4.4. *Let $s(n)$ be the sum of the digits of the binary representation of n, then*

$$s(n) = n - \sum_{j=1}^{\infty} \Big\lfloor \frac{n}{2^j} \Big\rfloor.$$

Proof. Let the binary representation of n be of type $\cdots 0\overbrace{11\cdots 1}^{k \text{ times}}$, with $k \geq 0$, then $n+1 = \cdots 1\overbrace{00\cdots 0}^{k \text{ times}}$. Therefore

$$s(n) - k = s(n+1) - 1.$$

Also $k = \nu_2(n+1)$ the exponent of 2 in the prime factorization of $n+1$.

Thus we have proved that for every integer $n \geq 0$

$$s(n+1) + \nu_2(n+1) = s(n) + 1. \tag{4.4}$$

We add this equalities for $n = 0, 1, \ldots, n-1$ to get

$$s(n) + \sum_{k=1}^{n} \nu_2(k) = n.$$

It is easily checked that

$$\sum_{k=1}^{n} \nu_2(k) = \sum_{j=1}^{\infty} \Big\lfloor \frac{n}{2^j} \Big\rfloor.$$

□

Theorem 4.5. *The numbers λ_n and ϱ_n defined by Equations (4.1) and (4.2) satisfy*

$$\nu_2(\lambda_n) = \nu_2(\varrho_n) = s(n).$$

Proof. First consider the sequence λ_n. Clearly the theorem is true for the first λ_n which are

$$\lambda_0 = 1, \quad \lambda_1 = 2, \quad \lambda_3 = 82, \quad \lambda_4 = 10572.$$

Since Euler numbers E_{2k} are odd, from the definition of λ_n it follows that

$$\nu_2(n+1) + \nu_2(\lambda_{n+1}) = \nu_2\Big(\sum_{k=0}^{n} 2^{4k+1}|E_{2k+2}|\lambda_{n-k}\Big). \tag{4.5}$$

By induction the terms of this sum are exactly divided by the powers of 2 of exponents

$$1 + s(n), \quad 5 + s(n-1), \quad 9 + s(n-2), \quad \dots \quad (4n+1) + s(0)$$

This is a strictly increasing sequence, since

$$s(n) - s(n-1) = 1 - \nu_2(n) < 4.$$

Hence from (4.5) we get

$$\nu_2(n+1) + \nu_2(\lambda_{n+1}) = 1 + s(n).$$

By (4.4)

$$\nu_2(\lambda_{n+1}) = s(n) - \nu_2(n+1) + 1 = s(n+1).$$

The same proof applies to the sequence (ϱ_n). □

5. Examples

We give here some examples of numbers satisfying our Theorem 1.3. First consider the case of the numbers of Gabcke $A_n = \lambda_n$. The first terms of the associated sequences are given by the following table.

a_n		2	160	31232	11345920	947622146676	957663025230936	...
b_n		2	79	10410	2836440	1324377702	944684832315	...
λ_n	1	2	82	10572	2860662	1330910844	947622146676	...

We can give arbitrarily a sequence of integers (b_n) and obtain sequences (a_n) and (A_n) that automatically satisfy our theorems. We give two simple examples.

With $b_n = 1$ for every n, we get $a_n = \sigma(n)$.

a_n		1	3	4	7	6	12	8	15	13	18	12	28	...
b_n		1	1	1	1	1	1	1	1	1	1	1	1	...
A_n	1	1	2	3	5	7	11	15	22	30	42	56	77	...

With $b_n = -24$, the numbers A_n are given by Ramanujan's τ function $A_n = \tau(n+1)$.

a_n		-24	-72	-96	-168	-144	-288	-192	-360	...
b_n		-24	-24	-24	-24	-24	-24	-24	-24	...
A_n	1	-24	252	-1472	4830	-6048	-16744	84480	-113643	...

Finally let (T_n) be the tangent numbers with the notation of [8]. We have $T_{2n} = 0$

$$T_{2n+1} = (-1)^n \frac{4^{n+1}(4^{n+1}-1)B_{2n+2}}{2n+2}.$$

It can be proved that $a_n = (-1)^n T_{2n+1}$ satisfies Kummer congruences. It follows that the sequence (T_{2n+1}) is realizable, in this case the three sequences are

a_n		2	16	272	7936	353792	22368256	1903757312	...
b_n		2	7	90	1980	70758	3727995	271965330	...
A_n	1	2	10	108	2214	75708	3895236	280356120	...

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