

Results on Existence of Solution for an Optimal Design Problem

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1. INTRODUCTION

In this paper we study a control problem for elliptic nonlinear monotone problems with Dirichlet boundary conditions where the control variables are the coefficients of the equation and the open set where the partial differential problem is studied.

More exactly, we consider a bounded open set $\Omega \subset \mathbb{R}^N$ and a monotone operator A from $H^1(\Omega)$ to $H^{-1}(\Omega)$, mapping $y \in H^1(\Omega)$ in $Ay = -\operatorname{div} a(x, \nabla y) \in H^{-1}(\Omega)$, where $a : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function which defines a monotone Leray-Lions operator of order 2.

Our problem is to find an open set $\tilde{\Omega} \subset \Omega$ and A on the conditions above, such that for $f \in H^{-1}(\Omega)$, the solution y of

$$\begin{cases} Ay = f & \text{in } \tilde{\Omega} \\ y \in H_0^1(\tilde{\Omega}) \end{cases} \quad (1.1)$$

minimize a functional $J : H_0^1(\Omega) \rightarrow \mathbb{R}$ (the solution of (1.1) will be considered extended by zero outside $\tilde{\Omega}$ and then, defined as an element of $H_0^1(\Omega)$).

When $\tilde{\Omega}$ is fixed (see [18], [19], [20]) or A is fixed (see [3], [4]), the problem has been studied in several papers, usually for linear problems. It is well known that these problems have no solution in general. In the present paper, we show the existence of solution when the controls are searched in a large set.

Our results can be generalized to systems of M equations and operators of order $p \in (1, +\infty)$ (see [5]). Here, by simplicity, we study the scalar case with $p = 2$.

From the point of view of the applications, the results exposed in the present paper are related with the selection of optimal shape material (take into account that the coefficients of the equation depend on the choice of the materials).

2. NOTATION AND PRELIMINARIES

Let Ω be a bounded open subset of \mathbb{R}^N . For a measure μ , we denote by $L^2_\mu(\Omega)$, the space of the functions which are μ -measurable and have its power two μ -integrable. If μ is the Lebesgue measure, we write $L^p(\Omega, \mathbb{R}^M)$.

We denote by $H^1_0(\Omega)$ the closure of the C^∞ functions with compact support for the norm $\|u\|_{H^1_0(\Omega)} = \|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)^N}$. The dual space of $H^1_0(\Omega)$, it is denoted by $H^{-1}(\Omega)$.

For every subset $B \subset \Omega$, and $p \in (1, +\infty)$, we denote by $C(B, \Omega)$ the capacity of B (in Ω), which is defined as the infimum of

$$\int_{\Omega} |\nabla y|^2 dx$$

over the set of the functions $y \in H^1_0(\Omega)$ such that $y \geq 1$ a.e. in a neighbourhood of B .

We say that a property $\mathcal{P}(x)$ holds C -quasi everywhere (abbreviated as q.e.) in a set E , if there exists $N \subset E$ with $C(N, \Omega) = 0$ such that $\mathcal{P}(x)$ holds for all $x \in E \setminus N$.

A function $y : \Omega \rightarrow \mathbb{R}$ is said to be quasi continuous, if for every $\varepsilon > 0$ there exists $N \subset \Omega$, with $C(N, \Omega) < \varepsilon$, such that the restriction of y to $\Omega \setminus N$ is continuous. It is well know that every $y \in H^1_0(\Omega)$ has a quasi continuous representative (see [15], [16], [24]). We always identify y with its quasi continuous representative.

A subset $A \subset \Omega$ is said to be quasi open in Ω , if for every $\varepsilon > 0$ there exists an open subset $U \subset \Omega$, with $C(U, \Omega) < \varepsilon$, such that $A \cup N$ is open.

We denote by $\mathcal{M}_0^2(\Omega)$ the class of all Borel measures which vanish on the sets of capacity zero and satisfy

$$\mu(B) = \inf\{\mu(A) : A \text{ quasi open, } B \subseteq A \subseteq \Omega\}$$

for every Borel set $B \subseteq \Omega$.

DEFINITION 2.1. For $\alpha, \gamma > 0$, we denote by $\mathcal{A}(\alpha, \gamma)$ the set of Carathéodory functions $a : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ such that

- (i) $a(x, 0) = 0$ for a.e. $x \in \Omega$;
- (ii) $(a(x, \xi_1) - a(x, \xi_2))(\xi_1 - \xi_2) \geq \max\{\alpha|\xi_1 - \xi_2|^2, \gamma|a(x, \xi_1) - a(x, \xi_2)|^2\}$ for all $\xi_1, \xi_2 \in \mathbb{R}^N$, a.e. $x \in \Omega$.

Remark 2.2. If a belongs to $\mathcal{A}(\alpha, \gamma)$, then a satisfies

- (iii) $|a(x, \xi_1) - a(x, \xi_2)| \leq \frac{1}{\gamma}|\xi_1 - \xi_2|$ for all $\xi_1, \xi_2 \in \mathbb{R}$, a.e. $x \in \Omega$.

Reciprocally, if a function a satisfies

$$(a(x, \xi_1) - a(x, \xi_2))(\xi_1 - \xi_2) \geq \alpha|\xi_1 - \xi_2|^2$$

for all $\xi_1, \xi_2 \in \mathbb{R}$, a.e. $x \in \Omega$, and there exists $\beta > 0$ such that

$$|a(x, \xi_1) - a(x, \xi_2)| \leq \beta|\xi_1 - \xi_2|$$

for all $\xi_1, \xi_2 \in \mathbb{R}$, a.e. $x \in \Omega$, then a satisfies (ii) with $\gamma = \frac{\alpha}{\beta^2}$.

DEFINITION 2.3. We denote by $\mathcal{U}(\alpha, \gamma)$ the set of pairs (μ, F) such that $\mu \in \mathcal{M}_0^2(\Omega)$ and $F : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies

- (a) $F(\cdot, s)$ is μ -measurable for very $s \in \mathbb{R}$;
- (b) $F(x, 0) = 0$, μ -a.e. $x \in \Omega$;
- (c) $(F(x, s_1) - F(x, s_2))(s_1 - s_2) \geq \max\{\alpha|s_1 - s_2|^2, \gamma|F(x, s_1) - F(x, s_2)|^2\}$ for all $s_1, s_2 \in \mathbb{R}$, μ -a.e. $x \in \Omega$.

Remark 2.4. Hypothesis (c) is equivalent to:

$$\frac{1}{\gamma}(s_1 - s_2) \geq F(x, s_1) - F(x, s_2) \geq \alpha(s_1 - s_2)$$

for all $s_1, s_2 \in \mathbb{R}$, $s_1 \geq s_2$, μ -a.e. $x \in \Omega$.

We consider a functional $J : H_0^1(\Omega) \rightarrow \mathbb{R}$ which is sequentially weakly lower semicontinuous, i.e.:

$$y_n \rightharpoonup y \quad \Rightarrow \quad \liminf_{n \rightarrow \infty} J(y_n) \geq J(y). \tag{2.2}$$

3. EXISTENCE OF SOLUTION FOR THE OPTIMAL DESIGN PROBLEM

For $f \in H^{-1}(\Omega)$, $\tilde{\Omega} \subset \Omega$ and $a \in \mathcal{A}$, we consider the partial differential problem

$$\begin{cases} -\operatorname{div} a(x, \nabla y) = f & \text{in } \tilde{\Omega} \\ y \in H_0^1(\tilde{\Omega}). \end{cases} \quad (3.3)$$

Our purpose is to find $\tilde{\Omega}$ and $a \in \mathcal{A}$ which solve the minimum problem

$$\begin{cases} \min J(y) \\ a \in \mathcal{A}, \tilde{\Omega} \subset \Omega. \end{cases} \quad (3.4)$$

In order to show the existence of solution for (3.4), we can try to use the direct method of calculus of variations. For that, we consider $\Omega_n \subset \Omega$ opens, and $a_n \in \mathcal{A}$ such that the sequence y_n of solutions of

$$\begin{cases} -\operatorname{div} a_n(x, \nabla y_n) = f & \text{in } \Omega_n \\ y_n \in H_0^1(\Omega_n) \end{cases} \quad (3.5)$$

is minimizing, i.e.:

$$\liminf_{n \rightarrow \infty} J(y_n) = I$$

where

$$I = \inf\{J(y) : a \in \mathcal{A}, \tilde{\Omega} \subset \Omega, y \text{ satisfies (3.3)}\}.$$

Taking y_n as test function in (3.5), we deduce

$$\int_{\Omega} a_n(x, \nabla y_n) \nabla y_n \, dx = \int_{\Omega} f y_n \, dx,$$

which by (ii) implies

$$\|y_n\|_{H_0^1(\Omega)} \leq \frac{\|f\|_{H^{-1}(\Omega)}}{\sqrt{\alpha}},$$

where we have identified y_n with its extension by zero to $\Omega \setminus \Omega_n$. So, there exists a subsequence (still denoted by y_n) which converges weakly to a function y in $H_0^1(\Omega)$. By the lower semicontinuity (2.2) of J , we have $J(y) \leq I$. If there exists $\tilde{\Omega} \subset \Omega$ and $a \in \mathcal{A}$ such that y satisfies (3.3), then $J(y) = I$ and the problem is solved.

Therefore, we need to find the equation satisfied by the function y and to know if it is of the same type that (3.3). Thus, we need to study the homogenization problem

$$\begin{cases} -\operatorname{div} a_n(x, \nabla y_n) = f & \text{in } \mathcal{D}'(\Omega_n) \\ y_n \in H_0^1(\Omega_n), \end{cases} \quad (3.6)$$

where $a_n \in \mathcal{A}$ and Ω_n is a sequence of arbitrary open sets contained in a given bounded open set $\Omega \subset \mathbb{R}^N$. This is a question which is well known when Ω_n or a_n is fixed.

When Ω_n is fixed it has been proved (see for example [21], for the linear problem and [22], [23] for the nonlinear one) that there exists a function $a \in \mathcal{A}$ such that (for a sequence) the solutions y_n of (3.6) with $\Omega_n = \tilde{\Omega}$ fixed, converge weakly in $H_0^1(\Omega)$ to the solution y of

$$\begin{cases} -\operatorname{div} a(x, \nabla y) = f & \text{in } \tilde{\Omega} \\ y \in H_0^1(\tilde{\Omega}), \end{cases}$$

where a does not depend of f . In particular, this implies that the initial problem (3.4), has a solution if we assume that $\tilde{\Omega}$ is not variable (control coefficients problem).

However, when a_n is fixed, it is not true in general that there exists a subsequence of Ω_n , still denoted by Ω_n , and open set $\tilde{\Omega} \subset \Omega$ such that the solutions of (3.6) with $a_n = a$ fixed, converge weakly in $H_0^1(\Omega)$ to the solution y of

$$\begin{cases} -\operatorname{div} a(x, \nabla y) = f & \text{in } \mathcal{D}'(\tilde{\Omega}) \\ y \in H_0^1(\tilde{\Omega}). \end{cases}$$

For example, if $N = 3$, and $\Omega_n = \Omega \setminus \bigcup_{k \in \mathbb{Z}^N} B(\frac{k}{n}, \frac{1}{n^3})$, it has been proved in [8], that the sequence of solutions y_n of (3.6) with $a(x, \xi) = \xi$, for all $\xi \in \mathbb{R}$, a.e. $x \in \Omega$, (laplacian operator) converges weakly in $H_0^1(\Omega)$ to the unique solution y of

$$\begin{cases} -\Delta y + \frac{4\pi}{3}y = f & \text{in } \Omega \\ y \in H_0^1(\Omega). \end{cases} \tag{3.7}$$

As a consequence of this result, let us now prove

THEOREM 3.1. *The problem (3.4) has no solution in general.*

Proof. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. We respectively denote by y_0 the solution of (3.7) with $f = 1$ and by \bar{y} the solution of

$$\begin{cases} -\Delta \bar{y} = 1 & \text{in } \Omega \\ \bar{y} \in H_0^1(\Omega). \end{cases} \tag{3.8}$$

We consider $J : H_0^1(\Omega) \rightarrow \mathbb{R}$ as $J(y) = \int_{\Omega} |y - y_0|^2 dy$, for all $y \in H_0^1(\Omega)$ and $\alpha = 1 - \varepsilon$, $\gamma = \frac{1}{1+\varepsilon}$, with ε small enough such that

$$\left(\frac{\varepsilon^2 + 4\varepsilon}{1 - \varepsilon}\right)^2 \int_{\Omega} |\nabla \bar{y}|^2 dy < \int_{\Omega} |\nabla(\bar{y} - y_0)|^2 dy. \tag{3.9}$$

Remark that $y_n \rightharpoonup y$ in $H_0^1(\Omega)$ implies $J(y_n) \rightarrow J(y)$ in \mathbb{R} .

It is clear for the result of Cianorescu-Murat mentioned above, that in this case $I = 0$. So, if there exists $(a, \tilde{\Omega})$ solution of (3.4), then

$$\begin{cases} -\operatorname{div} a(x, \nabla y_0) = 1 & \text{in } \tilde{\Omega} \\ y_0 \in H_0^1(\tilde{\Omega}). \end{cases}$$

Now, $y_0 \in H_0^1(\tilde{\Omega})$, implies that $y_0 = 0$ q.e. in $\Omega \setminus \tilde{\Omega}$, but the strong maximum principle implies that $y_0 > 0$ in Ω . So, $\Omega \setminus \tilde{\Omega}$ has capacity zero, but then $H_0^1(\Omega)$ is equal to $H_0^1(\tilde{\Omega})$, and y_0 is also the solution of the problem

$$\begin{cases} -\operatorname{div} a(x, \nabla y_0) = 1 & \text{in } \Omega \\ y_0 \in H_0^1(\Omega). \end{cases} \tag{3.10}$$

On the other hand, by (ii), for every $\xi \in \mathbb{R}^N$ and a.e. $x \in \Omega$, we have

$$\begin{aligned} |\xi - a(x, \xi)| &= |\xi|^2 + |a(x, \xi)|^2 - 2a(x, \xi)\xi \\ &\leq |\xi|^2 + (1 + \varepsilon)^2|\xi|^2 - 2(1 - \varepsilon)|\xi|^2 = (4\varepsilon + \varepsilon^2)|\xi|^2. \end{aligned} \tag{3.11}$$

Taking $y_0 - \bar{y}$ as test function in the difference of (3.8) and (3.10), we deduce

$$\int_{\Omega} [a(x, \nabla y_0) - \nabla \bar{y}] \nabla (y_0 - \bar{y}) \, dy = 0,$$

and then, using (ii) and (iii), we obtain

$$\begin{aligned} (1 - \varepsilon) \int_{\Omega} |\nabla (y_0 - \bar{y})|^2 \, dy &\leq \int_{\Omega} [a(x, \nabla y_0) - a(x, \nabla \bar{y})] \nabla (y_0 - \bar{y}) \, dy \\ &\leq \int_{\Omega} [\nabla \bar{y} - a(x, \nabla \bar{y})] \nabla (y_0 - \bar{y}) \, dy \\ &\leq (4\varepsilon + \varepsilon^2) \left(\int_{\Omega} |\nabla \bar{y}|^2 \, dy \right)^{\frac{1}{2}} \left(\int_{\Omega} |\nabla (y_0 - \bar{y})|^2 \, dy \right)^{\frac{1}{2}}. \end{aligned} \tag{3.12}$$

From (3.9) and (3.12) we deduce the absurd. ■

Our interest in the following is to show that the control problem has a solution if we search for the control variables in a more large set. For this purpose, following G. Dal Maso and U. Mosco (see [11]), we remark that defining for $\tilde{\Omega} \subset \Omega$ open, the measure $\mu \in \mathcal{M}_0^2(\Omega)$ as

$$\mu(B) = \begin{cases} 0 & \text{if } \operatorname{cap}((\Omega \setminus \tilde{\Omega}) \cap B) = 0 \\ +\infty & \text{if } \operatorname{cap}((\Omega \setminus \tilde{\Omega}) \cap B) > 0, \end{cases}$$

and taking F , such that the pair (μ, F) belongs to $\mathcal{U}(\alpha, \gamma)$ (it always exists) the problem (3.3) is equivalent to the variational problem

$$\left\{ \begin{array}{l} y \in H_0^1(\Omega) \cap L_\mu^2(\Omega) \\ \int_\Omega a(x, \nabla y) \nabla v \, dx + \int_\Omega F(x, y) v \, d\mu = \langle f, v \rangle \\ \forall v \in H_0^1(\Omega) \cap L_\mu^2(\Omega) \end{array} \right. \quad (3.13)$$

Then, at the place of the original control problem, we can consider the following one

$$\min\{J(y) : a \in \mathcal{A}(\alpha, \gamma), (F, \mu) \in \mathcal{U}(\alpha, \gamma)\}, \quad (3.14)$$

where for $\mathcal{A}(\alpha, \gamma)$, $(F, \mu) \in \mathcal{U}(\alpha, \gamma)$, y is the unique solution of (3.13). The advantage of the new formulation is clear from the following theorem.

THEOREM 3.2. *For every sequences a_n in $\mathcal{A}(\alpha, \gamma)$ and (F_n, μ_n) in $\mathcal{U}(\alpha, \gamma)$, there exists a subsequence, still denoted by n , such that for every $f \in H^{-1}(\Omega)$, the solution y_n of*

$$\left\{ \begin{array}{l} y_n \in H_0^1(\Omega) \cap L_{\mu_n}^2(\Omega) \\ \int_\Omega a_n(x, \nabla y_n) \nabla v \, dx + \int_\Omega F_n(x, y_n) v \, d\mu_n = \langle f, v \rangle \\ \forall v \in H_0^1(\Omega) \cap L_{\mu_n}^2(\Omega) \end{array} \right. \quad (3.15)$$

converges weakly in $H_0^1(\Omega)$ to the solution y of (3.13).

Theorem 3.2 has been proved by the authors in [5], in fact it is true for operators of order $p \in (1, +\infty)$ and for systems. In particular, it gives the form of the limit problem of (3.6) for arbitrary Ω_n and a_n . When μ_n is zero for every n , the result can be found in [22] and [23]. For the case a_n constant, the theorem has been shown in [7], although it is not proved that the pair (F, μ) which appears in the limit problem is in $\mathcal{U}(\alpha, \gamma)$ (see also [6], [8], [9], [10], [11], [12], ...). When a_n and F_n are linear, the result appears in [14].

For the double homogenization problem, with monotone operators, a previous result has been proved in [17], but in this work μ_n are not general, they correspond to a sequence of open sets Ω_n , such that the measure μ in the limit is the Lebesgue measure.

Using Theorem 3.2, we can now apply the direct method of the calculus of variations as above to prove

THEOREM 3.3. *The problem (3.14) admits at least a solution $a \in \mathcal{A}(\alpha, \gamma)$, $(F, \mu) \in \mathcal{U}(\alpha, \gamma)$.*

The question which remains is to know if the problem (3.15) is a relaxation of (3.4), i.e., if for every $a \in \mathcal{A}(\alpha, \gamma)$ and $(F, \mu) \in \mathcal{U}(\alpha, \gamma)$, there exists $a_n \in \mathcal{A}$ and $\Omega_n \subset \Omega$ open, such that the solutions y_n of (3.6) converge weakly in $H_0^1(\Omega)$ to the solution y of (3.13). This is true if we ask for the elements of \mathcal{A} and \mathcal{U} to be linear in its second variable (see [14]).

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