Gevrey solutions of the irregular hypergeometric system associated with an affine monomial curve

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Abstract

We describe the Gevrey series solutions at singular points of the irregular hypergeometric system (GKZ system) associated with an affine monomial curve. We also describe the irregularity complex of such a system with respect to its singular support.

Introduction

We explicitly describe the Gevrey solutions of the hypergeometric system associated with an affine monomial curve in $\mathbb{C}^n$ by using Γ–series introduced by I.M. Gel'fand, M.I. Graev, M.M. Kapranov and A.V. Zelevinsky [8], [10] and also used by M. Saito, B. Sturmfels and N. Takayama [30] in a very useful and slightly different form. We use these Γ–series to study the *Gevrey filtration of the irregularity complex* –as defined by Z. Mebkhout in [25, Sec. 6]– of the corresponding analytic hypergeometric $\mathcal{D}$–module with respect to its singular support. Here $\mathcal{D}$ is the sheaf of linear differential operators with holomorphic coefficients on $\mathbb{C}^n$.

A general hypergeometric system $M_A(\beta)$ is associated with a pair (A, β) where $A = (a_{ij})$ is an integer $d \times n$ matrix of full rank d and β is a parameter vector in $\mathbb{C}^d$ ([8], [9], [10]). The system $M_A(\beta)$ is defined by the following system of linear partial differential equations in the unknown φ:

$$\prod_{i=1}^{n} \left(\frac{\partial}{\partial x_i}\right)^{u_i} (\varphi) - \prod_{i=1}^{n} \left(\frac{\partial}{\partial x_i}\right)^{v_i} (\varphi) = 0 \ \text{ for } (u, v \in \mathbb{N}^n, \ Au = Av)$$

$$\sum_{j=1}^{n} a_{ij} x_j \frac{\partial \varphi}{\partial x_j} - \beta_i \varphi = 0 \ \text{ for } i = 1, \ldots, d.$$

If $d = 1$ (i.e. if $A = (a_1 \, a_2 \, \cdots \, a_n)$) we say that the hypergeometric system $M_A(\beta)$ is associated with the affine monomial curve defined by A in $\mathbb{C}^n$. If $d = 2$ and the vector $(1, \cdots, 1)$ is in the $\mathbb{Q}$-row span of A then we say that $M_A(\beta)$ is associated with the projective monomial curve defined by A in $\mathbb{P}_{n-1}(\mathbb{C})$.

Rational solutions of the hypergeometric system associated with a projective monomial curve have been studied in [6]. The holomorphic solutions of general hypergeometric systems at a generic point

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in $\mathbb{C}^n$ have been widely studied (see e.g. [9], [10], [1], [30], [27]). The rank of a general $M_A(\beta)$ is by definition the number of linearly independent holomorphic solutions of $M_A(\beta)$ at a generic point in $\mathbb{C}^n$. This rank equals, for generic $\beta \in \mathbb{C}^d$ and assuming $\mathbb{Z}A = \mathbb{Z}^d$, the normalized volume $vol(\Delta)$ of the convex hull Δ of the columns of A and the origin, considered as points in $\mathbb{R}^d$ ([10] and [1]).

Several recent papers are devoted to the study of the exceptional parameters β, i.e. for which the rank jumps (see [23] and the references therein, see also [4]).

To setup the problem of computing the Gevrey solutions of a left $\mathcal{D}$–module, we will first recall the situation in the one dimensional case.

Let $P = a_m \frac{d^m}{dx^m} + \cdots + a_1 \frac{d}{dx} + a_0$ be an ordinary linear differential operator, of order m, with $a_i = a_i(x)$ a holomorphic function at the origin in $\mathbb{C}$. The *slopes* of P are the slopes of the Newton polygon $N(P)$ of P defined as the convex hull of the union of the quadrants $(i, i - v(a_i)) + (\mathbb{Z}_{\leq 0})^2$ for $i = 0, \ldots, m$, where $v(a_i)$ is the multiplicity of the zero of $a_i(x)$ at $x = 0$. By Fuchs' Theorem, P is regular at $x = 0$ if and only if $N(P)$ is the quadrant $(m, m - v(a_m)) + (\mathbb{Z}_{\leq 0})^2$.

Malgrange-Komatsu's comparison Theorem states that P is regular at $x = 0$ if and only if the solution space $\mathrm{Sol}(P; \widehat{\mathcal{O}}/\mathcal{O})$ is zero ([16], [22]). Here P acts naturally on the quotient $\widehat{\mathcal{O}}/\mathcal{O}$ where $\widehat{\mathcal{O}} = \mathbb{C}[[x]]$ (resp. $\mathcal{O} = \mathbb{C}\{x\}$) is the ring of formal (resp. convergent) power series in the variable x. This solution space measures the irregularity of P at $x = 0$ and will be denoted $\mathrm{Irr}_0(P) := \mathrm{Sol}(P; \widehat{\mathcal{O}}/\mathcal{O})$.

Moreover, the vector space $\mathrm{Irr}_0(P)$ is filtered by the so-called Gevrey solutions of the equation $P(u) = 0$. Let us denote by $\widehat{\mathcal{O}}_s \subset \widehat{\mathcal{O}}$ the subring of Gevrey series of order less than or equal to s (where $s \geq 1$ is a real number). A formal power series $f = \sum_i f_i x^i \in \widehat{\mathcal{O}}$ is in $\widehat{\mathcal{O}}_s$ if and only if the series $\rho_s(f) := \sum_i (i!)^{1-s} f_i x^i$ is convergent at $x = 0$. The irregularity $\mathrm{Irr}_0(P)$ is filtered by the solution subspaces $\mathrm{Irr}_0^{(s)}(P) := \mathrm{Sol}(P; \widehat{\mathcal{O}}_s/\mathcal{O})$. By the comparison Theorem of J.P. Ramis [28] the jumps in this filtration are in bijective correspondence with the slopes of $N(P)$ and the dimension of each $\mathrm{Irr}_0^{(s)}(P)$ can also be read on $N(P)$.

In higher dimension, one has analogous results but the situation is much more involved. Let X be a complex manifold of dimension $n \geq 1$ and let us denote by $\mathcal{O}_X$ (resp. $\mathcal{D}_X$) the sheaf of rings of holomorphic functions on X (resp. of linear differential operators with holomorphic coefficients). The role of the previous equation $P(u) = 0$ is played here by a maximally overdetermined system of linear partial differential equations on X, or intrinsically by a holonomic $\mathcal{D}_X$–module $\mathcal{M}$. Z. Mebkhout introduced in [25] the irregularity of $\mathcal{M}$ along a smooth hypersurface $Y \subset X$ as the complex of sheaves of vector spaces $\mathrm{Irr}_Y(\mathcal{M}) := \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{Q}_Y)$ where $\mathcal{Q}_Y$ is the quotient $\mathcal{Q}_Y = \frac{\mathcal{O}_{\widehat{X|Y}}}{\mathcal{O}_{X|Y}}$ and $\mathcal{O}_{\widehat{X|Y}}$ is the formal completion of $\mathcal{O}_X$ with respect to Y. This irregularity is nothing but the solution complex of $\mathcal{M}$ with values in $\mathcal{Q}_Y$. Z. Mebkhout also introduced the irregularity of order s of $\mathcal{M}$ along Y as the solution complex of $\mathcal{M}$ with values in $\mathcal{Q}_Y(s)$, i.e. $\mathrm{Irr}_Y^{(s)}(\mathcal{M}) := \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{Q}_Y(s))$ where $\mathcal{Q}_Y(s)$ is the quotient of the Gevrey series $\mathcal{O}_{\widehat{X|Y}}(s)$ of order less than or equal to s along Y by the sheaf $\mathcal{O}_{X|Y}$. Z. Mebkhout proved [25] that these irregularity complexes belong to $\mathrm{Per}(\mathbb{C}_Y)$, the abelian category of perverse sheaves on Y and that $\mathrm{Irr}_Y^{(s)}(\mathcal{M})$ is an increasing filtration of the irregularity $\mathrm{Irr}_Y(\mathcal{M})$. A real number $s > 1$ is said to be an *analytic slope* of $\mathcal{M}$ along Y at a point $p \in Y$ if the inclusion $\mathrm{Irr}_Y^{(s')}(\mathcal{M}) \subset \mathrm{Irr}_Y^{(s)}(\mathcal{M})$ is strict in a neighborhood of $p \in Y$ for $1 < s' < s$.

On the other hand, Y. Laurent defined the *algebraic slopes* of $\mathcal{M}$ by using interpolations of the order filtration and the Malgrange-Kashiwara filtration along Y [18]. Y. Laurent also proved that these algebraic slopes form a finite set of rational numbers.

The comparison Theorem of Y. Laurent and Z. Mebkhout [19] proves that the analytic and the algebraic slopes coincide. Although the slopes of a $\mathcal{D}_X$–module $\mathcal{M}$ can be computed by an algorithm when $\mathcal{M}$ is algebraic [2], computations of the irregularity $\mathrm{Irr}_Y^{(s)}(\mathcal{M})$ are very rare. More precisely, under some conditions on the matrix A, the slopes of a hypergeometric system $M_A(\beta)$ can be combinatorially described from the matrix A ([32], see also [5], [11]).

In this paper we explicitly describe the irregularity $\mathrm{Irr}_Y^{(s)}(\mathcal{M})$ for any s and for $\mathcal{M} = \mathcal{M}_A(\beta)$ the hypergeometric system associated with an affine monomial curve when the associated semigroup is positive. Here Y is the singular support of the system.

The paper has the following structure. In Sections 1 and 2 we recall the definition of Gevrey series and some essential results, proved by Z. Mebkhout [25], about the *irregularity complex* $\mathrm{Irr}_Y(\mathcal{M})$ of a holonomic $\mathcal{D}$-module $\mathcal{M}$ with respect to a hypersurface Y in a complex manifold. We also recall the comparison theorem of Y. Laurent and Z. Mebkhout for the *algebraic* and *analytic slopes* of a holonomic $\mathcal{D}$–module [19].

Section 3 deals with the basic properties of the Γ–series associated with the pair (A, β) following [8], [10, Sec. 1] and [30, Sec. 3.4].

Sections 4 and 5 are devoted to the computation of the cohomology of the irregularity complex $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ of the hypergeometric system $\mathcal{M}_A(\beta)$ associated with a smooth monomial curve and with a general monomial curve respectively, at any point of the singular support Y of the system. In both cases the associated semigroup is assumed to be positive.

Proofs are fulfilled by reducing the number of variables, by using the restriction functor in $\mathcal{D}$–module theory, and then applying the results from the 2 dimensional case [7].

This paper can be considered as a natural continuation of [7], [5] and [11] and its results should be related to the ones of [33]. We have often used some essential results of the book [30] about solutions of hypergeometric systems. Some of our results are related to [26] and also to [21] and [12].

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1 Gevrey series.

Let X be a complex manifold of dimension $n \geq 1$, $\mathcal{O}_X$ (or simply $\mathcal{O}$) the sheaf of holomorphic functions on X and $\mathcal{D}_X$ (or simply $\mathcal{D}$) the sheaf of linear differential operators with coefficients in $\mathcal{O}_X$. The sheaf $\mathcal{O}_X$ has a natural structure of left $\mathcal{D}_X$–module. Let Z be a hypersurface in X with defining ideal $\mathcal{I}_Z$. We denote by $\mathcal{O}_{X|Z}$ the restriction to Z of the sheaf $\mathcal{O}_X$ (and we will also denote by $\mathcal{O}_{X|Z}$ its extension by 0 on X). Recall that the formal completion of $\mathcal{O}_X$ along Z is defined as

$$\mathcal{O}_{\widehat{X|Z}} := \varprojlim_k \mathcal{O}_X/\mathcal{I}_Z^k.$$

By definition $\mathcal{O}_{\widehat{X|Z}}$ is a sheaf on X supported on Z and has a natural structure of left $\mathcal{D}_X$–module. We will also denote by $\mathcal{O}_{\widehat{X|Z}}$ the corresponding sheaf on Z. We denote by $\mathcal{Q}_Z$ the quotient sheaf defined by the following exact sequence

$$0 \to \mathcal{O}_{X|Z} \longrightarrow \mathcal{O}_{\widehat{X|Z}} \longrightarrow \mathcal{Q}_Z \to 0.$$

The sheaf $\mathcal{Q}_Z$ has then a natural structure of left $\mathcal{D}_X$–module.

Remark 1.1. *If $X = \mathbb{C}$ and $Z = \{0\}$ then $\mathcal{O}_{\widehat{X|Z},0}$ is nothing but $\mathbb{C}[[x]]$ the ring of formal power series in one variable x, while $\mathcal{O}_{\widehat{X|Z},p} = 0$ for any nonzero $p \in X$. In this case $\mathcal{Q}_{Z,0} = \frac{\mathbb{C}[[x]]}{\mathbb{C}\{x\}}$ and $\mathcal{Q}_{Z,p} = 0$ for $p \neq 0$.*

Definition 1.2. *Assume $Y \subset X$ is a smooth hypersurface and that around a point $p \in X$ the hypersurface Y is locally defined by $x_n = 0$ for some system of local coordinates $(x_1, \ldots, x_n)$. Let us consider a real number $s \geq 1$. A germ $f = \sum_{i \geq 0} f_i(x_1, \ldots, x_{n-1})x_n^i \in \mathcal{O}_{\widehat{X|Y},p}$ is said to be a Gevrey series of order s (along Y at the point p) if the power series*

$$\rho_s(f) := \sum_{i \geq 0} \frac{1}{i!^{s-1}} f_i(x_1, \ldots, x_{n-1})x_n^i$$

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is convergent at p.

The sheaf $\mathcal{O}_{\widehat{X|Y}}$ admits a natural filtration by the sub-sheaves $\mathcal{O}_{\widehat{X|Y}}(s)$ of Gevrey series of order s, $1 \leq s \leq \infty$ where by definition $\mathcal{O}_{\widehat{X|Y}}(\infty) = \mathcal{O}_{\widehat{X|Y}}$. So we have $\mathcal{O}_{\widehat{X|Y}}(1) = \mathcal{O}_{X|Y}$. We can also consider the induced filtration on $\mathcal{Q}_Y$, i.e. the filtration by the sub-sheaves $\mathcal{Q}_Y(s)$ defined by the exact sequence:

$$0 \to \mathcal{O}_{X|Y} \longrightarrow \mathcal{O}_{\widehat{X|Y}}(s) \longrightarrow \mathcal{Q}_Y(s) \to 0 \tag{1}$$

Definition 1.3. *Let Y be a smooth hypersurface in $X = \mathbb{C}^n$ and let p be a point in Y. The Gevrey index of a formal power series $f \in \mathcal{O}_{\widehat{X|Y},p}$ with respect to Y is the smallest $1 \leq s \leq \infty$ such that $f \in \mathcal{O}_{\widehat{X|Y}}(s)_p$.*

2 Irregularity complex and slopes

Let X be a complex manifold. We recall here the definition of the irregularity (also called the irregularity complex) of a left coherent $\mathcal{D}_X$–module given by Z. Mebkhout [25, (2.1.2) and page 98].

Recall that if $\mathcal{M}$ is a coherent left $\mathcal{D}_X$–module and $\mathcal{F}$ is any $\mathcal{D}_X$–module, the *solution complex* of $\mathcal{M}$ with values in $\mathcal{F}$ is by definition the complex

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{F})$$

which is an object of $D^b(\mathbb{C}_X)$ the derived category of bounded complexes of sheaves of $\mathbb{C}$–vector spaces on X. The cohomology sheaves of the solution complex are then $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{F})$ (or simply $\mathcal{E}xt^i(\mathcal{M}, \mathcal{F})$) for $i \in \mathbb{N}$.

Definition 2.1. [25, (2.1.2) and page 98] *Let Z be a hypersurface in X. The irregularity of $\mathcal{M}$ along Z (denoted by $\mathrm{Irr}_Z(\mathcal{M})$) is the solution complex of $\mathcal{M}$ with values in $\mathcal{Q}_Z$, i.e.*

$$\mathrm{Irr}_Z(\mathcal{M}) := \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{Q}_Z)$$

If Y is a smooth hypersurface in X we also have the following definition (see [25, Déf. 6.3.7])

Definition 2.2. *For each $1 \leq s \leq \infty$, the irregularity of order s of $\mathcal{M}$ with respect to Y is the complex $\mathrm{Irr}_Y^{(s)}(\mathcal{M}) := \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{Q}_Y(s))$.*

Remark 2.3. *Since $\mathcal{O}_{\widehat{X|Y}}(\infty) = \mathcal{O}_{\widehat{X|Y}}$ we have $\mathrm{Irr}_Y^{(\infty)}(\mathcal{M}) = \mathrm{Irr}_Y(\mathcal{M})$. The support of the irregularity of $\mathcal{M}$ along Z (resp. $\mathrm{Irr}_Y^{(s)}(\mathcal{M})$) is contained in Z (resp. in Y).*

If $X = \mathbb{C}$, $Z = \{0\}$ and $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X P$ is the $\mathcal{D}_X$–module defined by some nonzero linear differential operator $P(x, \frac{d}{dx})$ with holomorphic coefficients, then $\mathrm{Irr}_Z(\mathcal{M})$ is represented by the complex

$$0 \longrightarrow \frac{\mathbb{C}[[x]]}{\mathbb{C}\{x\}} \xrightarrow{P} \frac{\mathbb{C}[[x]]}{\mathbb{C}\{x\}} \longrightarrow 0$$

where P acts naturally on the quotient $\frac{\mathbb{C}[[x]]}{\mathbb{C}\{x\}}$.

Z. Mebkhout have proved [25, Th. 6.3.3] that for any holonomic $\mathcal{D}_X$–module $\mathcal{M}$ the complex $\mathrm{Irr}_Y^{(s)}(\mathcal{M})$ is a perverse sheaf on the smooth hypersurface $Y \subset X$ for any $1 \leq s \leq \infty$.

A complex $\mathcal{F}^\bullet \in D^b(\mathbb{C}_X)$ of sheaves of vector spaces is said to be *constructible* if there exists a stratification (X_λ) of X such that the cohomology sheaves $\mathcal{H}^i(\mathcal{F}^\bullet)$ are local systems on each X_λ. A constructible complex $\mathcal{F}^\bullet$ satisfies the *support condition* on X if the following conditions hold:

1. $\mathcal{H}^i(\mathcal{F}^\bullet) = 0$ for $i < 0$ and $i > n = dim(X)$.

2. The dimension of the support of $\mathcal{H}^i(\mathcal{F}^\bullet)$ is less than or equal to $n - i$ for $0 \le i \le n$.

A constructible complex $\mathcal{F}^\bullet$ is said to be *perverse* on X (or even a *perverse sheaf* on X) if both $\mathcal{F}^\bullet$ and its dual $\mathbb{R}\mathcal{H}om_{\mathbb{C}_X}(\mathcal{F}^\bullet, \mathbb{C}_X)$ satisfy the support condition.

The category $\mathrm{Per}(\mathbb{C}_X)$ of perverse sheaves on X is an abelian category (see [3]).

Remark 2.4. *From [25, Cor. 6.3.5] each* $\mathrm{Irr}_Y^{(s)}(-)$ *for* $1 \le s \le \infty$, *is an exact functor from the category of holonomic* $\mathcal{D}_X$-*modules to the category of perverse sheaves on* Y.

Moreover, the sheaves $\mathrm{Irr}_Y^{(s)}(\mathcal{M})$, $1 \le s \le \infty$, *form an increasing filtration of* $\mathrm{Irr}_Y^{(\infty)}(\mathcal{M}) = \mathrm{Irr}_Y(\mathcal{M})$. *This filtration is called the Gevrey filtration of* $\mathrm{Irr}_Y(\mathcal{M})$ *(see [25, Sec. 6]).*

Let us denote by

$$\mathrm{Gr}_s(\mathrm{Irr}_Y(\mathcal{M})) := \frac{\mathrm{Irr}_Y^{(s)}(\mathcal{M})}{\mathrm{Irr}_Y^{(<s)}(\mathcal{M})}$$

for $1 \le s \le \infty$ the graded object associated with the Gevrey filtration of the irregularity $\mathrm{Irr}_Y(\mathcal{M})$ (see [19, Sec. 2.4]).

We say, with [19, Sec. 2.4], that $1 \le s < \infty$ is an *analytic slope* of $\mathcal{M}$ along Y at a point $p \in Y$ if p belongs to the closure of the support of $\mathrm{Gr}_s(\mathrm{Irr}_Y(\mathcal{M}))$. Y. Laurent ([17], [18]) defines, in an algebraic way, the *algebraic slopes* of any coherent $\mathcal{D}_X$–module $\mathcal{M}$ along Y. These algebraic slopes can be algorithmically computed if the module $\mathcal{M}$ is defined by differential operators with polynomial coefficients [2].

In [19, Th. 2.5.3] Y. Laurent and Z. Mebkhout prove that for any holonomic $\mathcal{D}_X$–module the analytic and the algebraic slopes, with respect to any smooth hypersurface, coincide and that they are rational numbers. In [5] and [11] are described the slopes at the origin (with respect to any hyperplane in $\mathbb{C}^n$) of the hypergeometric system associated to any affine monomial curve. In [32] M. Schulze and U. Walther describe the slopes of any hypergeometric system with respect to any coordinate variety in $\mathbb{C}^n$ under some assumption on the semigroup associated with the system. By technical reasons the definition of slope given in [5] and [11] is slightly different to the one of Y. Laurent: a real number $-\infty \le r \le 0$ is called a slope in [5] and [11] if $\frac{r-1}{r}$ is an algebraic slope in the sense of Y. Laurent [18].

3 Hypergeometric systems and Γ–series

Hypergeometric systems are defined on $X = \mathbb{C}^n$. We denote by $A_n(\mathbb{C})$ or simply A_n the complex Weyl algebra of order n, i.e. the ring of linear differential operators with coefficients in the polynomial ring $\mathbb{C}[x] := \mathbb{C}[x_1, \ldots, x_n]$. The partial derivative $\frac{\partial}{\partial x_i}$ will be denoted by ∂_i.

Let $A = (a_{ij})$ be an integer $d \times n$ matrix with rank d and $\beta \in \mathbb{C}^d$. Let us denote by $E_i(\beta)$ for $i = 1, \ldots, d$, the operator $E_i(\beta) := \sum_{j=1}^n a_{ij} x_j \partial_j - \beta_i$. The toric ideal $I_A \subset \mathbb{C}[\partial] := \mathbb{C}[\partial_1, \ldots, \partial_n]$ associated with A is generated by the binomials $\square_u := \partial^{u_+} - \partial^{u_-}$ for $u \in \mathbb{Z}^n$ such that $Au = 0$ where $u = u_+ - u_-$ and u_+, u_- are both in $\mathbb{N}^n$ and have disjoint support.

The left ideal $A_n I_A + \sum_i A_n E_i(\beta) \subset A_n$ is denoted by $H_A(\beta)$ and it will be called the *hypergeometric ideal* associated with (A, β). The (global) hypergeometric module associated with (A, β) is by definition (see [8], [10]) the quotient $M_A(\beta) := A_n/H_A(\beta)$.

When $X = \mathbb{C}^n$ is considered as complex manifold, to the pair (A, β) we can also associate the corresponding analytic hypergeometric $\mathcal{D}_X$–module, denoted by $\mathcal{M}_A(\beta)$, which is the quotient of $\mathcal{D}_X$ modulo the sheaf of left ideals in $\mathcal{D}_X$ generated by $H_A(\beta)$.

In what follows we will use Γ–series following [8] and [10, Sec. 1] and in the way these objects are handled in [30, Sec. 3.4].

Let the pair (A, β) be as before. Assume $v \in \mathbb{C}^n$. We will consider the Γ–series

$$\varphi_v := x^v \sum_{u \in L_A} \frac{1}{\Gamma(v + u + \mathbf{1})} x^u \in x^v \mathbb{C}[[x_1^{\pm 1}, \ldots, x_n^{\pm n}]]$$

where $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{N}^n$, $L_A = \ker_{\mathbb{Z}}(A)$ and for $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{C}^n$ one has by definition $\Gamma(\gamma) = \prod_{i=1}^n \Gamma(\gamma_i)$ (where Γ is the Euler gamma function). Notice that the set $x^v \mathbb{C}[[x_1^{\pm 1}, \ldots, x_n^{\pm n}]]$ has a natural structure of left $A_n(\mathbb{C})$-module although it is not a $\mathcal{D}_{X,0}$–module. Nevertheless, if $Av = \beta$ then the expression φ_v formally satisfies the operators defining $\mathcal{M}_A(\beta)$. Let us notice that if $u \in L_A$ then $\varphi_v = \varphi_{v+u}$.

If $v \in (\mathbb{C} \setminus \mathbb{Z}_{<0})^n$ then the coefficient $\frac{1}{\Gamma(v+u+\mathbf{1})}$ is non-zero for all $u \in L_A$ such that $u_i + v_i \geq 0$ for all i with $v_i \in \mathbb{N}$. We also have the following equality

$$\frac{\Gamma(v + \mathbf{1})}{\Gamma(v + u + \mathbf{1})} = \frac{(v)_{u_-}}{(v + u)_{u_+}} \tag{2}$$

where for any $z \in \mathbb{C}^n$ and any $\alpha \in \mathbb{N}^n$ we have the convention

$$(z)_\alpha = \prod_{i, \alpha_i > 0} \prod_{j=0}^{\alpha_i - 1} (z_i - j).$$

Following [30, p. 132-133] the *negative support* of v (denoted by $\mathrm{nsupp}(v)$) is the set of indices i such that $v_i \in \mathbb{Z}_{<0}$. We say that v has *minimal negative support* if there is no $u \in L_A$ such that $\mathrm{nsupp}(v + u)$ is a proper subset of $\mathrm{nsupp}(v)$.

Assume $v \notin (\mathbb{C} \setminus \mathbb{Z}_{<0})^n$ has minimal negative support. The negative support of v is then a non-empty set and $\Gamma(v + \mathbf{1}) = \infty$. Moreover for each $u \in L_A$ at least one coordinate of $v + u$ must be strictly negative (otherwise $\mathrm{nsupp}(v + u) = \emptyset \subsetneq \mathrm{nsupp}(v)$). So $\Gamma(v + u + \mathbf{1}) = \infty$ for all $u \in L_A$ and $\varphi_v = 0$.

If $v \notin (\mathbb{C} \setminus \mathbb{Z}_{<0})^n$ does not have minimal negative support then there exists $u \in L_A$ such that $v + u$ has minimal negative support. If $\mathrm{nsupp}(v + u) = \emptyset$ then $\varphi_v = \varphi_{v+u} \neq 0$ while if $\mathrm{nsupp}(v + u) \neq \emptyset$ then $\varphi_v = \varphi_{v+u} = 0$.

Following *loc. cit.*, for any $v \in \mathbb{C}^n$ we will consider the series

$$\phi_v := x^v \sum_{u \in N_v} \frac{(v)_{u_-}}{(v + u)_{u_+}} x^u$$

where $N_v = \{u \in L_A \mid \mathrm{nsupp}(v + u) = \mathrm{nsupp}(v)\}$.

If $Av = \beta$ then ϕ_v is a solution of the hypergeometric ideal $H_A(\beta)$ (i.e. ϕ_v is formally annihilated by $H_A(\beta)$) if and only if v has minimal negative support [30, Prop. 3.4.13].

For $v \in (\mathbb{C} \setminus \mathbb{Z}_{<0})^n$ we have

$$\frac{\Gamma(v + \mathbf{1})}{\Gamma(v + u + \mathbf{1})} = \frac{(v)_{u_-}}{(v + u)_{u_+}}$$

and $\Gamma(v + \mathbf{1}) \varphi_v = \phi_v$.

If $v \notin (\mathbb{C} \setminus \mathbb{Z}_{<0})^n$ then the coefficient of x^v in ϕ_v is non-zero (in fact this coefficient is 1) while it is zero in φ_v.

In order to simplify notations we will adopt in the sequel the following convention: for $v \in \mathbb{C}^n$ and $u \in L_A$ we will denote

$$\Gamma[v; u] := \frac{(v)_{u_-}}{(v + u)_{u_+}}$$

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if $u \in N_v$ and $\Gamma[v; u] := 0$ otherwise. With this convention we have

$$\phi_v = x^v \sum_{u \in L_A} \Gamma[v; u] x^u.$$

In what follows we will describe the irregularity, along the singular locus, of the hypergeometric system associated with an affine monomial curve in $X = \mathbb{C}^n$ and with a parameter $\beta \in \mathbb{C}$.

4 The case of a smooth monomial curve

Let $A = (1 \, a_2 \, \cdots \, a_n)$ be an integer row matrix with $1 < a_2 < \cdots < a_n$ and $\beta \in \mathbb{C}$. Let us denote by $\mathcal{M}_A(\beta)$ the corresponding analytic hypergeometric system on $X = \mathbb{C}^n$. We will simply denote $\mathcal{D}$ the sheaf $\mathcal{D}_X$ of linear differential operators with holomorphic coefficients.

Although it can be deduced from general results (see [8] and [1, Th. 3.9]), a direct computation shows in this case that the characteristic variety of $\mathcal{M}_A(\beta)$ equals $T_X^* X \cup T_Y^* X$ where $Y = (x_n = 0)$. The module $\mathcal{M}_A(\beta)$ is then holonomic and its singular support is Y. Let us denote by $Z \subset \mathbb{C}^n$ the hyperplane $x_{n-1} = 0$.

One of the main results in this Section is

Theorem 4.1. *Let $A = (1 \, a_2 \, \cdots \, a_n)$ be an integer row matrix with $1 < a_2 < \cdots < a_n$ and $\beta \in \mathbb{C}$. Then the cohomology sheaves of $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ satisfy:*

i) $\mathcal{E}xt_{\mathcal{D}}^0(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s)) = 0$ for $1 \le s < a_n/a_{n-1}$.

ii) $\mathcal{E}xt_{\mathcal{D}}^0(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_{|Y \cap Z} = 0$ for $1 \le s \le \infty$.

iii) $\dim_{\mathbb{C}} (\mathcal{E}xt_{\mathcal{D}}^0(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p) = a_{n-1}$ for all $s \ge a_n/a_{n-1}$ and $p \in Y \setminus Z$.

iv) $\mathcal{E}xt_{\mathcal{D}}^i(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s)) = 0$ for $i \ge 1$ and $1 \le s \le \infty$.

The main ingredients in the proof of Theorem 4.1 are: Corollary 4.4, the corresponding results for the case of monomial plane curves [7], Cauchy-Kovalevskaya Theorem for Gevrey series (see [20, Cor. 2.2.4]), the perversity of $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ [25, Th. 6.3.3] and Kashiwara's constructibility theorem [13].

We will also describe a basis of the solution vector space in part *iii)* of Theorem 4.1 (see Theorem 4.23).

4.1 Reduction of the number of variables by restriction

In the sequel we will use some results concerning restriction of hypergeometric systems.

Theorem 4.2. *[5, Th. 4.4] Let $A = (1 \, a_2 \, \cdots \, a_n)$ be an integer row matrix with $1 < a_2 < \cdots < a_n$ and $\beta \in \mathbb{C}$. Then for $i = 2, \ldots, n$, the restriction of $\mathcal{M}_A(\beta)$ to $(x_i = 0)$ is isomorphic to the $\mathcal{D}'$-module*

$$\mathcal{M}_A(\beta)_{|(x_i=0)} := \frac{\mathcal{D}}{\mathcal{D} H_A(\beta) + x_i \mathcal{D}} \cong \frac{\mathcal{D}'}{\mathcal{D}' H_{A'}(\beta)}$$

where $A' = (1 \, a_2 \, \cdots \, a_{i-1} \, a_{i+1} \, \cdots \, a_n)$ and $\mathcal{D}'$ is the sheaf of linear differential operators with holomorphic coefficients on $\mathbb{C}^{n-1}$ (with coordinates $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$).

Theorem 4.3. *Let $A = (1\ ka\ kb)$ be an integer row matrix with $1 \leq a < b$, $1 < ka < kb$ and a, b relatively prime. Then for all $\beta \in \mathbb{C}$ there exist $\beta_0, \ldots, \beta_{k-1} \in \mathbb{C}$ such that the restriction of $\mathcal{M}_A(\beta)$ to $(x_1 = 0)$ is isomorphic to the $\mathcal{D}'$-module*

$$\mathcal{M}_A(\beta)_{|(x_1=0)} := \frac{\mathcal{D}}{\mathcal{D}H_A(\beta) + x_1\mathcal{D}} \simeq \bigoplus_{i=0}^{k-1} \mathcal{M}_{A'}(\beta_i)$$

where $\mathcal{D}'$ is the sheaf of linear differential operators on the plane $(x_1 = 0)$ and $A' = (a\ b)$. Moreover, for all but finitely many $\beta \in \mathbb{C}$ we can take $\beta_i = \frac{\beta - i}{k}$, $i = 0, 1, \ldots, k - 1$.

An ingredient in the proof of Theorem 4.1 is the following

Corollary 4.4. *Let $A = (1\ a_2\ \cdots\ a_n)$ be an integer row matrix with $1 < a_2 < \cdots < a_n$ and $\beta \in \mathbb{C}$. Then there exist $\beta_i \in \mathbb{C}$, $i = 0, \ldots, k - 1$ such that the restriction of $\mathcal{M}_A(\beta)$ to $X' = (x_1 = x_2 = \cdots = x_{n-2} = 0)$ is isomorphic to the $\mathcal{D}'$-module*

$$\mathcal{M}_A(\beta)_{|X'} := \frac{\mathcal{D}}{\mathcal{D}H_A(\beta) + (x_1, x_2, \cdots, x_{n-2})\mathcal{D}} \simeq \bigoplus_{i=0}^{k-1} \mathcal{M}_{A'}(\beta_i)$$

where $\mathcal{D}'$ is the sheaf of linear differential operators on X', $A' = \frac{1}{k}(a_{n-1}\ a_n)$ and $k = \gcd(a_{n-1}, a_n)$. Moreover, for all but finitely many $\beta \in \mathbb{C}$ we can take $\beta_i = \frac{\beta - i}{k}$, $i = 0, 1, \ldots, k - 1$.

Let us fix some notations and state some preliminaries results in order to prove Theorem 4.3.

Notation 4.5. *Let A be an integer $d \times n$–matrix of rank d and $\beta \in \mathbb{C}^n$. For any weight vector $\omega \in \mathbb{R}^n$ and any ideal $J \subset \mathbb{C}[\partial] = \mathbb{C}[\partial_1, \ldots, \partial_n]$ we denote by $\mathrm{in}_\omega(J)$ the initial ideal of J with respect to the graduation on $\mathbb{C}[\partial]$ induced by w. According to [30, p. 106] the fake initial ideal of $H_A(\beta)$ is the ideal $\mathrm{fin}_\omega(H_A(\beta)) = A_n\mathrm{in}_\omega(I_A) + A_n(A\theta - \beta)$ where $\theta = (\theta_1, \ldots, \theta_n)$ and $\theta_i = x_i\partial_i$.*

Assume now $A = (1\ ka\ kb)$ is an integer row matrix with $1 \leq a < b$, $1 < ka < kb$ and a, b relatively prime.

Let us write $P_1 = \partial_2^b - \partial_3^a$, $P_2 = \partial_1^{ka} - \partial_2$, $P_3 = \partial_1^{kb} - \partial_3$ and $E = \theta_1 + ka\theta_2 + kb\theta_3 - \beta$. It is clear that $P_1 \in H_A(\beta) = \langle P_2, P_3, E \rangle \subset A_3$.

Let us consider $\prec$ a monomial order on the monomials in A_3 satisfying:

$$\left.\begin{array}{l} \gamma_1 + a\gamma_2 + b\gamma_3 < \gamma_1' + a\gamma_2' + b\gamma_3' \\ \text{or} \\ \gamma_1 + a\gamma_2 + b\gamma_3 = \gamma_1' + a\gamma_2' + b\gamma_3' \text{ and } 3a\gamma_2 + 2b\gamma_3 < 3a\gamma_2' + 2b\gamma_3' \end{array}\right\} \Rightarrow x^\alpha\partial^\gamma \prec x^{\alpha'}\partial^{\gamma'}$$

Write $\omega = (1, 0, 0)$ and let us denote by $\prec_\omega$ the monomial order on the monomials in A_3 defined as

$$x^\alpha\partial^\gamma \prec_\omega x^{\alpha'}\partial^{\gamma'} \overset{Def.}{\Longleftrightarrow} \left\{\begin{array}{l} \gamma_1 - \alpha_1 < \gamma_1' - \alpha_1' \\ \text{or} \\ \gamma_1 - \alpha_1 = \gamma_1' - \alpha_1' \text{ and } x^\alpha\partial^\gamma \prec x^{\alpha'}\partial^{\gamma'} \end{array}\right.$$

Lemma 4.6. *Let $A = (1\ ka\ kb)$ be an integer row matrix with $1 \leq a < b$, $1 < ka < kb$ and a, b relatively prime. Then*

$$\mathrm{fin}_\omega(H_A(\beta)) = A_3\mathrm{in}_\omega(I_A) + A_3 E = A_3(P_1, E, \partial_1^k)$$

for $\beta \notin \mathbb{N}^* := \mathbb{N} \setminus \{0\}$ and for all $\beta \in \mathbb{N}^*$ big enough.

Proof. The proof follows from the application of Buchberger's algorithm in the Weyl algebra A_3 and the fact that $\partial_1^k \in \mathrm{fin}_\omega(H_A(\beta))$ for β as in the statement. □

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Definition 4.7. [30, Def. 5.1.1] *Let $I \subseteq A_n$ be a holonomic ideal (i.e. A_n/I is holonomic) and $\widetilde{\omega} \in \mathbb{R}^n \setminus \{0\}$. The b-function I with respect to $\widetilde{\omega}$ is the monic generator of the ideal*

$$\mathrm{in}_{(-\widetilde{\omega},\widetilde{\omega})}(I) \cap \mathbb{C}[\tau]$$

where $\tau = \widetilde{\omega}_1 \theta_1 + \cdots + \widetilde{\omega}_n \theta_n$ and $\mathrm{in}_{(-\widetilde{\omega},\widetilde{\omega})}(I)$ is the initial ideal of I with respect to the weight vector $(-\widetilde{\omega}, \widetilde{\omega})$.

Corollary 4.8. *Let $A = (1 \; ka \; kb)$ be an integer row matrix with $1 \leq a < b$, $1 < ka < kb$ and a, b relatively prime. Then the b-function of $H_A(\beta)$ with respect to $\omega = (1,0,0)$ is*

$$b(\tau) = \tau(\tau - 1)\cdots(\tau - (k-1))$$

for all but finitely many $\beta \in \mathbb{C}$.

Proof. From [30, Th. 3.1.3] for all but finitely many $\beta \in \mathbb{C}$ we have

$$\mathrm{in}_{(-\omega,\omega)}(H_A(\beta)) = \mathrm{fin}_\omega(H_A(\beta)).$$

Then by using Lemma 4.6 we get

$$\mathrm{in}_{(-\omega,\omega)}(H_A(\beta)) = A_3(P_1, E, \partial_1^k)$$

for all but finitely many $\beta \in \mathbb{C}$. An easy computation shows that $\{P_1, E, \partial_1^k\}$ is a Groebner basis of the ideal $\mathrm{in}_{(-\omega,\omega)}(H_A(\beta))$ with respect to any monomial order $>$ satisfying $\theta_3 > \theta_1, \theta_2$ and $\partial_2^b > \partial_3^a$. In particular we can consider the lexicographic order

$$x_3 > x_2 > \partial_2 > \partial_3 > x_1 > \partial_1$$

which is an elimination order for x_1 and ∂_1. So we get

$$\mathrm{in}_{(-\omega,\omega)}(H_A(\beta)) \cap \mathbb{C}[x_1]\langle\partial_1\rangle = \langle\partial_1^k\rangle$$

and since $x_1^k \partial_1^k = \theta_1(\theta_1 - 1)\cdots(\theta_1 - (k-1))$, we have

$$\mathrm{in}_{(-\omega,\omega)}(H_A(\beta)) \cap \mathbb{C}[\theta_1] = \langle\theta_1(\theta_1 - 1)\cdots(\theta_1 - (k-1))\rangle$$

This proves the corollary. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 4.9. *Corollary 4.8 can be related to [5, Th. 4.3] which proves that for $A = (1 \; a_2 \; \cdots \; a_n)$ with $1 < a_2 < \cdots < a_n$, the b-function of $H_A(\beta)$ with respect to e_i is $b(\tau) = \tau$, for $i = 2, \ldots, n$. Here $e_i \in \mathbb{R}^n$ is the vector with a 1 in the i-th coordinate and 0 elsewhere.*

Recall (see e.g. [30, Def. 1.1.3]) that a Groebner basis of a left ideal $I \subset A_n$ with respect to $(-\omega, \omega) \in \mathbb{R}^{2n}$ (or simply with respect to $\omega \in \mathbb{R}^n$) is a finite subset $G \subset I$ such that $I = A_n G$ and $\mathrm{in}_{(-\omega,\omega)}(I) = A_n \mathrm{in}_{(-\omega,\omega)}(G)$ where $\mathrm{in}_{(-\omega,\omega)}(G) = \{\mathrm{in}_{(-\omega,\omega)}(P) \,|\, P \in G\}$.

Lemma 4.10. *Let $A = (1 \; ka \; kb)$ be an integer row matrix with $1 \leq a < b$, $1 < ka < kb$ and a, b relatively prime. For all but finitely many $\beta \in \mathbb{C}$, a Groebner basis of $H_A(\beta) \subset A_3$ with respect to $\omega = (1,0,0)$ is*

$$\{P_1, P_2, P_3, E, R\}$$

for some $R \in A_3$ satisfying $\mathrm{in}_{(-\omega,\omega)}(R) = \partial_1^k$.

The following Proposition is a particular case of [4, Th. 6.5.] (see also [29, Th. 2.1]) and it will be used later.

Proposition 4.11. *Assume $A = (a_1 \ a_2 \cdots a_n)$ is an integer row matrix with $0 < a_1 < a_2 < \cdots < a_n$. For $\beta, \beta' \in \mathbb{C}^d$ we have that $\mathcal{M}_A(\beta) \simeq \mathcal{M}_A(\beta')$ if and only if one of the following conditions holds:*

i) $\beta, \beta' \in \mathbb{N}A$.

ii) $\beta, \beta' \in \mathbb{Z} \setminus \mathbb{N}A$.

iii) $\beta, \beta' \notin \mathbb{Z}$ but $\beta - \beta' \in \mathbb{Z}$.

Proof. (**Theorem 4.3**) We have $A = (1 \ ka \ kb)$ with $1 \leq a < b$, $1 < ka < kb$ and a, b relatively prime. From Proposition 4.11 it is enough to compute the restriction for all but finitely many $\beta \in \mathbb{C}$. We will compute the restriction of $\mathcal{M}_A(\beta)$ to $(x_1 = 0)$ by using an algorithm by T. Oaku and N. Takayama [30, Algorithm 5.2.8].

Let $r = k - 1$ be the biggest integer root of the Bernstein polynomial $b(\tau)$ of $H_A(\beta)$ with respect to $\omega = (1, 0, 0)$ (see Corollary 4.8). Recall that $\mathcal{D}' = \mathcal{D}_{\mathbb{C}^2}$ and that we are using (x_2, x_3) as coordinates in $\mathbb{C}^2$. We consider the free $\mathcal{D}'$-module with basis $\mathcal{B}_{k-1} := \{\partial_1^i : i = 0, 1, \ldots, k - 1\}$:

$$(\mathcal{D}')^{r+1} = (\mathcal{D}')^k \simeq \bigoplus_{i=0}^{k-1} \mathcal{D}'\partial_1^i.$$

To apply [30, Algorithm 5.2.8] to our case we will use the elements with ω–order less than or equal to $k - 1$, in the Groebner basis $\mathcal{G} := \{P_1, P_2, P_3, E, R\}$ of $H_A(\beta)$. Recall that $\mathcal{G}$ is given by Lemma 4.10 for all but finitely many $\beta \in \mathbb{C}$. Each operator $\partial_1^i P_1$, $\partial_1^i E$, $i = 0, \ldots, k - 1$, must be written as a $\mathbb{C}$–linear combination of monomials $x^u \partial^v$ and then substitute $x_1 = 0$ into this expression. The result is an element of $(\mathcal{D}')^k = \mathcal{D}'\mathcal{B}_k$. In this case we get:

$$(\partial_1^i P_1)_{|x_1=0} = P_1 \partial_1^i, \quad (\partial_1^i E)_{|x_1=0} = (kax_2\partial_2 + kbx_3\partial_3 - \beta + i)\partial_1^i, \quad i = 0, \ldots, k - 1$$

and this proves the theorem. $\qquad\square$

Remark 4.12. *Let us consider $A = (1 \ a_2 \ \cdots \ a_n)$, $1 < a_2 < \cdots < a_n$, $k = \gcd(a_{n-1}, a_n)$ and $A' = \frac{1}{k}(a_{n-1}, a_n)$. We can apply Cauchy-Kovalevskaya Theorem for Gevrey series (see [20, Cor. 2.2.4]), Corollary 4.4 and [5, Prop. 4.2] to the hypergeometric system $\mathcal{M}_A(\beta)$ to prove that for each $\beta \in \mathbb{C}$ there exist $\beta_i \in \mathbb{C}$, $i = 0, \ldots, k - 1$, and a quasi-isomorphism*

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s))_{|X'} \xrightarrow{\simeq} \bigoplus_{i=0}^{k-1} \mathbb{R}\mathcal{H}om_{\mathcal{D}_{X'}}(\mathcal{M}_{A'}(\beta_i), \mathcal{O}_{\widehat{X'|Y'}}(s))$$

for all $1 \leq s \leq \infty$ where $Y = (x_n = 0)$, $X' = (x_1 = x_2 = \cdots = x_{n-2} = 0)$ and $Y' = Y \cap X'$. Moreover, for all but finitely many β we can take $\beta_i = \frac{\beta - i}{k}$. Notice that coordinates in X, Y, X', Y' are $x = (x_1, \ldots, x_n)$, $y = (x_1, \ldots, x_{n-1})$, $x' = (x_{n-1}, x_n)$ and $y' = (x_{n-1})$ respectively.

The last quasi-isomorphism induces a $\mathbb{C}$–linear isomorphism

$$\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s))_{(0,\ldots,0,\epsilon_{n-1},0)} \xrightarrow{\simeq} \bigoplus_{i=0}^{k-1} \mathcal{E}xt^j_{\mathcal{D}_{X'}}(\mathcal{M}_{A'}(\beta_i), \mathcal{O}_{\widehat{X'|Y'}}(s)))_{(\epsilon_{n-1},0)}$$

for all $\epsilon_{n-1} \in \mathbb{C}$, $s \geq 1$ and $j \in \mathbb{N}$ and we also have equivalent results for $\mathcal{Q}_Y(s)$ and $\mathcal{Q}_{Y'}(s)$ instead of $\mathcal{O}_{\widehat{X|Y}}(s)$ and $\mathcal{O}_{\widehat{X'|Y'}}(s)$.

In particular, using [7, Proposition 5.9], we have:

Proposition 4.13. *Let $A = (1\ a_2\ \cdots\ a_n)$ be an integer row matrix with $1 < a_2 < \cdots < a_n$. Then for all $\beta \in \mathbb{C}$*

$$\dim_{\mathbb{C}}(\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_{(0,\ldots,0,\epsilon_{n-1},0)}) = \begin{cases} a_{n-1} & \text{if } s \geq a_n/a_{n-1},\ j = 0 \text{ and } \epsilon_{n-1} \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Corollary 4.14. *Let $A = (1\ a_2\ \cdots\ a_n)$ be an integer row matrix with $1 < a_2 < \cdots < a_n$. Then for all $\beta \in \mathbb{C}$*

$$\mathrm{Ch}(\mathrm{Irr}^{(s)}_Y(\mathcal{M}_A(\beta))) \subseteq T^*_Y Y \bigcup T^*_{Z \cap Y} Y$$

for $s \geq \frac{a_n}{a_{n-1}}$.

Proof. Here $\mathrm{Ch}(\mathrm{Irr}^{(s)}_Y(\mathcal{M}_A(\beta)))$ is the characteristic cycle of the perverse sheaf $\mathrm{Irr}^{(s)}_Y(\mathcal{M}_A(\beta))$ (see e.g. [19, Sec. 2.4]). The Corollary follows from the inclusion

$$\mathrm{Ch}^{(s)}(\mathcal{M}_A(\beta)) \subset T^*_X X \cup T^*_Y X \cup T^*_Z X$$

for $s \geq a_n/a_{n-1}$ and then by applying [19, Prop. 2.4.1]. $\qquad\square$

Proof. (**Theorem 4.1**) Let us consider the Whitney stratification $Y = Y_1 \cup Y_2$ of $Y = (x_n = 0) \subset \mathbb{C}^n$ defined as

$$Y_1 := Y \setminus (Y \cap Z) \equiv \mathbb{C}^{n-2} \times \mathbb{C}^*.$$

$$Y_2 := Y \cap Z \equiv \mathbb{C}^{n-2} \times \{0\}.$$

As $\mathrm{Irr}^{(s)}_Y(\mathcal{M}_A(\beta))$ is a perverse sheaf for all $\beta \in \mathbb{C}$ and $1 \leq s \leq \infty$ [25, Th. 6.3.3] we can apply Kashiwara's constructibility Theorem [13], the Riemann-Hilbert correspondence (see [24] and [15], [14]) and Corollary 4.14, to prove that

$$\mathcal{E}xt^i_{\mathcal{D}}(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_{|Y_j}$$

is a locally constant sheaf of finite rank for all $i \in \mathbb{N}$, $j = 1, 2$. To finish the proof it is enough to apply Proposition 4.13. $\qquad\square$

Remark 4.15. *Last proof uses Mebkhout's result on the perversity of the irregularity, Kashiwara's constructibility Theorem and the Riemann-Hilbert correspondence, all of them being deep results in $\mathcal{D}$–module theory. It would be interesting to give a more elementary proof of Theorem 4.1. In [7] such an elementary proof is given for the case $n = 2$.*

4.2 Gevrey solutions of $\mathcal{M}_A(\beta)$

We will compute a basis of the vector spaces $\mathcal{E}xt^i(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p$ for all $p \in Y$, $1 \leq s \leq \infty$, $\beta \in \mathbb{N}$, $i \in \mathbb{N}$ and $A = (1\ a_2\ \cdots\ a_n)$ is an integer row matrix with $1 < a_2 < \cdots < a_n$. In fact it is enough to achieve the computation for $i = 0$ and $p \in Y \setminus Z$, otherwise the corresponding germ is zero by Theorem 4.1.

Lemma 4.16. *Let $A = (1\ a_2\ \cdots\ a_n)$ be an integer row matrix with $1 < a_2 < \cdots < a_n$ and $\omega \in \mathbb{R}^n_{>0}$ satisfying*

    a) $w_i > a_i \omega_1$ for $2 \leq i \leq n-2$ or $i = n$

    b) $a_{n-1}\omega_1 > \omega_{n-1}$

c) $\omega_{n-1} > \omega_1, \ldots, \omega_{n-2}$

Then $H_A(\beta)$ has a_{n-1} exponents with respect to ω and they have the form

$$v^j = (j, 0, \ldots, 0, \frac{\beta - j}{a_{n-1}}, 0) \in \mathbb{C}^n$$

$j = 0, 1, \ldots, a_{n-1} - 1$.

Proof. The notion of *exponent* is given in [30, page 92]. The toric ideal I_A is generated by $P_{1,i} = \partial_1^{a_i} - \partial_i \in \mathbb{C}[\partial]$, $i = 2, \ldots, n$.

Let $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}_{>0}^n$ be a weight vector satisfying the statement of the lemma. We have:

$$\text{in}_{(-\omega,\omega)} P_{1,i} = \begin{cases} \partial_i & \text{if } i = 2, \ldots, n-2, n \\ \partial_1^{a_{n-1}} & \text{if } i = n-1 \end{cases}$$

In particular $\{P_{1,i} : i = 2, \ldots, n\}$ is a Groebner basis of I_A with respect to $(-\omega, \omega)$ and then

$$\text{in}_\omega I_A = \langle \partial_2, \ldots, \partial_{n-2}, \partial_1^{a_{n-1}}, \partial_n \rangle.$$

The standard pairs of $\text{in}_\omega(I_A)$ are ([30, Sec. 3.2]):

$$\mathcal{S}(\text{in}_\omega(I_A)) = \{(\partial_1^j, \{n-1\}) : j = 0, 1, \ldots, a_{n-1} - 1\}$$

To the standard pair $(\partial_1^j, \{n-1\})$ we associate, following [30, page 108], the *fake exponent*

$$v^j = (j, 0, \ldots, 0, \frac{\beta - j}{a_{n-1}}, 0)$$

of the module $\mathcal{M}_A(\beta)$ with respect to ω. It is easy to prove that these fake exponents are in fact exponents since they have minimal negative support [30, Th. 3.4.13]. $\square$

Remark 4.17. *With the above notation, the Γ–series ϕ_{v^j} associated with v^j for $j = 0, \ldots, a_{n-1} - 1$, is defined as*

$$\phi_{v^j} = x^{v^j} \sum_{u \in L_A} \Gamma[v^j; u] x^u$$

where $L_A = \ker_{\mathbb{Z}}(A)$ is the lattice generated by the vectors $\{u^2, \ldots, u^n\}$ and u^i is the $(i-1)$-th row of the matrix

$$\begin{pmatrix} -a_2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_{n-2} & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ a_{n-1} & 0 & 0 & \cdots & 0 & 0 & -1 & 0 \\ -a_n & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For any $\mathbf{m} = (m_2, \ldots, m_n) \in \mathbb{Z}^{n-1}$ let us denote $u(\mathbf{m}) := \sum_{i=2}^n m_i u^i \in L_A$. We can write

$$\phi_{v^j} = x^{v^j} \sum_{\substack{m_2, \ldots, m_{n-1}, m_n \geq 0 \\ \sum_{i \neq n-1} a_i m_i \leq j + a_{n-1} m_{n-1}}} \Gamma[v^j; u(\mathbf{m})] x^{u(\mathbf{m})}$$

for $j = 0, 1, \ldots, a_{n-1} - 1$. We have for $\mathbf{m} = (m_2, \ldots, m_n) \in \mathbb{N}^{n-1}$ such that $j - \sum_{i \neq n-1} a_i m_i + a_{n-1} m_{n-1} \geq 0$

$$\Gamma[v^j; u(\mathbf{m})] = \frac{(\frac{\beta-j}{a_{n-1}})_{m_{n-1}} j!}{m_2! \cdots m_{n-2}! m_n! (j - \sum_{i \neq n-1} a_i m_i + a_{n-1} m_{n-1})!}$$

and

$$x^{u(\mathbf{m})} = x_1^{-\sum_{i \neq n-1} a_i m_i + a_{n-1} m_{n-1}} x_2^{m_2} \cdots x_{n-2}^{m_{n-2}} x_{n-1}^{-m_{n-1}} x_n^{m_n}$$

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Proposition 4.18. *Let $A = (1 \; a_2 \; \cdots \; a_n)$ be an integer row matrix with $1 < a_2 < \cdots < a_n$, $Y = (x_n = 0) \subset X$ and $Z = (x_{n-1} = 0) \subset X$. Then we have:*

$$\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}})_p = \sum_{j=0}^{a_{n-1}-1} \mathbb{C}\phi_{v^j,p}$$

for all $\beta \in \mathbb{C}$, $p \in Y \setminus Z$.

Proof. Step 1.- Using [10] and [30] we will describe a_{n-1} linearly independent solutions of $\mathcal{M}_A(\beta)_p$, living in some Nilsson series ring. Then, using initial ideals, we will prove that an upper bound of the dimension of $\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}})_p$ is a_{n-1} for p in $Y \setminus Z$.

First of all, the series

$$\{\phi_{v^j} \mid j = 0, \ldots, a_{n-1} - 1\} \subset x^{v^j} \mathbb{C}[[x_1^{\pm 1}, x_2, \ldots, x_{n-2}, x_{n-1}^{-1}, x_n]]$$

described in Remark 4.17, are linearly independent since $\mathrm{in}_\omega(\phi_{v^j}) = x^{v^j}$ for $0 \le j \le a_{n-1} - 1$. They are solutions of the system $\mathcal{M}_A(\beta)$ (see [8], [10, Sec. 1],[30, Sec. 3.4]).

On the other hand

$$\dim_{\mathbb{C}} \mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}})_p \le a_{n-1} \tag{3}$$

for $p = (\epsilon_1, \ldots, \epsilon_{n-1}, 0)$, $\epsilon_{n-1} \ne 0$. This follows from the following facts:

a) The initial ideal $\mathrm{in}_\omega(I_A)$ equals $\langle \partial_2, \ldots, \partial_{n-2}, \partial_1^{a_{n-1}}, \partial_n \rangle$ for ω as in Lemma 4.16.

b) The germ of E at p is nothing but $E_p := E + \sum_{i=1}^{n-1} a_i \epsilon_i \partial_i$ (here $a_1 = 1$) and satisfies

$$\mathrm{in}_{(-\omega,\omega)}(E_p) = a_{n-1} \epsilon_{n-1} \partial_{n-1}.$$

c) By [30, Th. 2.5.5] if $f \in \mathcal{O}_{\widehat{X|Y},p}$ is a solution of the ideal $H_A(\beta)$ then $\mathrm{in}_\omega(f)$ must the annihilated by $\mathrm{in}_{(-\omega,\omega)}(H_A(\beta))$. That proves inequality (3).

Step 2.- It is easy to prove, using standard estimates, that the series $\phi_{v^j,p}$ are in fact in $\mathcal{O}_{\widehat{X|Y},p}$ for all $p \in Y \setminus Z$. Then by step 1, they form a basis of the the vector space $\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}})_p$ for $p \in Y \setminus Z$. $\square$

Remark 4.19. *We will prove (see Theorem 4.21) that the Gevrey index of $\phi_{v^j,p}$ is a_n/a_{n-1} for $\beta \in \mathbb{C}$ and $p \in Y \setminus Z$ except for $\beta \in \mathbb{N}$ and $j = q$ the unique integer $0 \le q \le a_{n-1} - 1$ such that $\beta - q \in a_{n-1}\mathbb{N}$. In that case ϕ_{v^q} is a polynomial (see details in Remark 4.22).*

Proposition 4.20. *Let $A = (1 \; a_2 \; \cdots \; a_n)$ be an integer row matrix with $1 < a_2 < \cdots < a_n$, $Y = (x_n = 0) \subset X$. Then we have:*

i) *For each $\beta \in \mathbb{C} \setminus \mathbb{N}$ we have $\mathcal{E}xt^j(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y}) = 0$ for all $j \in \mathbb{N}$.*

ii) *For each $\beta \in \mathbb{N}$ the sheaf $\mathcal{E}xt^j(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y})$ is locally constant on Y of rank 1 for $j = 0, 1$ and it is zero for $j \ge 2$.*

Proof. For all $\beta \in \mathbb{C}$ the characteristic variety of $\mathcal{M}_A(\beta)$ is $\mathrm{Ch}(\mathcal{M}_A(\beta)) = T_X^* X \cup T_Y^* X$ (see e.g. [1]). Then from Kashiwara's constructibility Theorem [13] we have that, for all $j \in \mathbb{N}$, the sheaf

$$\mathcal{E}xt^j(\mathcal{M}_A(\beta), \mathcal{O}_X)_{|Y} = \mathcal{E}xt^j(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y})_{|Y} \tag{4}$$

is locally constant.

Assume $\beta \notin \mathbb{N}$. From Corollary 4.4 and Proposition 4.11 we have that there exists $m \in \mathbb{N}$ such that $\mathcal{M}_A(\beta) \simeq \mathcal{M}_A(\beta - m)$ and

$$\mathcal{M}_A(\beta)_{|X'} \simeq \mathcal{M}_A(\beta - m)_{|X'} \simeq \bigoplus_{i=0}^{k-1} \mathcal{M}_{(a'_{n-1} \ a'_n)}\left(\frac{\beta - m - i}{k}\right)$$

with $X' = (x_1 = \cdots = x_{n-2} = 0) \subset X$, $k = \gcd(a_{n-1}, a_n)$ and $a'_\ell = \frac{a_\ell}{k}$ for $\ell = n - 1, n$. Then by applying Cauchy-Kovalevskaya Theorem (see Remark 4.12) we get:

$$\mathcal{E}xt^j(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y})_{|X'} \simeq \bigoplus_{i=0}^{k-1} \mathcal{E}xt^j(\mathcal{M}_{(a'_{n-1} \ a'_n)}\left(\frac{\beta - m - i}{k}\right), \mathcal{O}_{X'|Y'})$$

with $Y' = X' \cap Y$.

As $\beta \notin \mathbb{N}$ then $\frac{\beta - m - i}{k} \notin a'_{n-1}\mathbb{N} + a'_n\mathbb{N}$ for $i = 0, \ldots, k-1$. Then part $i)$ follows from [7, Proposition 4.5].

Assume now $\beta \in \mathbb{N}$. From Corollary 4.4 and Proposition 4.11 we have that there exists $m \in \mathbb{N}$ such that $\mathcal{M}_A(\beta) \simeq \mathcal{M}_A(\beta + m)$ and

$$\mathcal{M}_A(\beta)_{|X'} \simeq \mathcal{M}_A(\beta + m)_{|X'} \simeq \bigoplus_{i=0}^{k-1} \mathcal{M}_{(a'_{n-1} \ a'_n)}\left(\frac{\beta + m - i}{k}\right).$$

Then by applying Cauchy-Kovalevskaya Theorem (see Remark 4.12) we get:

$$\mathcal{E}xt^j(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y})_{|X'} \simeq \bigoplus_{i=0}^{k-1} \mathcal{E}xt^j(\mathcal{M}_{(a'_{n-1} \ a'_n)}\left(\frac{\beta + m - i}{k}\right), \mathcal{O}_{X'|Y'}).$$

By [7] this last module is in fact equal to

$$\mathcal{E}xt^j(\mathcal{M}_{(a'_{n-1} \ a'_n)}\left(\frac{\beta + m - i_0}{k}\right), \mathcal{O}_{X'|Y'})$$

where i_0 is the unique integer number such that $0 \leq i_0 \leq k - 1$ and $\beta + m - i_0 \in k\mathbb{N}$. Then part $ii)$ follows from [7, Proposition 4.6]. □

Theorem 4.21. *Let* $A = (1 \ a_2 \ \cdots \ a_n)$ *be an integer row matrix with* $1 < a_2 < \cdots < a_n$, $Y = (x_n = 0) \subset X$ *and* $Z = (x_{n-1} = 0) \subset X$. *Then we have:*

$i)$

$$\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s))_p = \bigoplus_{j=0}^{a_{n-1}-1} \mathbb{C}\phi_{v^j, p}$$

for all $\beta \in \mathbb{C}$, $p \in Y \setminus Z$ *and* $s \geq a_n/a_{n-1}$.

$ii)$

$$\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s))_p = \begin{cases} 0 & \text{if } \beta \notin \mathbb{N} \\ \mathbb{C}\phi_{v^q} & \text{if } \beta \in \mathbb{N} \end{cases}$$

for all $p \in Y \setminus Z$ *and* $1 \leq s < a_n/a_{n-1}$, *where* q *is the unique element in* $\{0, 1, \ldots, a_{n-1} - 1\}$ *satisfying* $\frac{\beta - q}{a_{n-1}} \in \mathbb{N}$ *and* ϕ_{v^q} *is a polynomial.*

Proof. *i)* Let us consider $a_n/a_{n-1} \leq s \leq \infty$ and $p \in Y \setminus Z$. Assume first that $\beta \notin \mathbb{N}$. By Proposition 4.20 and the long exact sequence of cohomology associated with the short exact sequence (1) we have that

$$\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s))_p \simeq \mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p$$

and by Theorem 4.1 this last vector space has dimension a_{n-1}. As

$$\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s))_p \subset \mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}})_p$$

part *i)* follows from Proposition 4.18 if $\beta \notin \mathbb{N}$.

Assume now $\beta \in \mathbb{N}$. Applying the long exact sequence of cohomology associated with the short exact sequence (1), Theorem 4.1 and Proposition 4.20 we get an exact sequence of vector spaces

$$0 \to \mathbb{C} \to \mathcal{L}_1 \to \mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p \to \mathbb{C} \to \mathcal{L}_2 \to 0$$

where $\mathcal{L}_1 = \mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s))_p$ and $\mathcal{L}_2 = \mathcal{E}xt^1(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s))_p$. Let us write $\nu_i = \dim_{\mathbb{C}}(\mathcal{L}_i)$. By Theorem 4.1 we also have $\nu_1 = a_{n-1} + \nu_2$. On the other hand, by Proposition 4.18, we know that $\nu_1 \leq a_{n-1}$. This implies $\nu_1 = a_{n-1}$ and $\mathcal{L}_2 = \{0\}$. In particular we have the equality

$$\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s))_p \subset \mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}})_p$$

part *i)* also follows from Proposition 4.18 if $\beta \in \mathbb{N}$.

Let us prove part *ii)*. First of all, by Theorem 4.1, $\mathcal{E}xt^j(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p = 0$ for all $j \in \mathbb{N}$. Then the result follows from the long exact sequence of cohomology associated with the short exact sequence (1) and Proposition 4.20. $\square$

Remark 4.22. *Let us recall here the notations introduced in Lemma 4.16. For $A = (1 \; a_2 \; \cdots \; a_n)$ an integer row matrix with $1 < a_2 < \cdots < a_n$ and $\omega \in \mathbb{R}_{>0}^n$ satisfying*

1. $w_i > a_i\omega_1$, for $2 \leq i \leq n-2$ or $i = n$

2. $a_{n-1}\omega_1 > \omega_{n-1}$

3. $\omega_{n-1} > \omega_1, \ldots, \omega_{n-2}$

we have proved (see Lemma 4.16) that $H_A(\beta)$ has a_{n-1} exponents with respect to ω and that they have the form:

$$v^j = (j, 0, \ldots, 0, \frac{\beta - j}{a_{n-1}}, 0) \in \mathbb{C}^n$$

$j = 0, 1, \ldots, a_{n-1} - 1$.

The corresponding Γ–series ϕ_{v^j} is defined as:

$$\phi_{v^j} = x^{v^j} \sum_{\substack{m_2, \ldots, m_{n-1}, m_n \geq 0 \\ \sum_{i \neq n-1} a_i m_i \leq j + a_{n-1} m_{n-1}}} \Gamma[v^j; u(\mathbf{m})] x^{u(\mathbf{m})}$$

for $j = 0, 1, \ldots, a_{n-1} - 1$, where for any $\mathbf{m} = (m_2, \ldots, m_n) \in \mathbb{Z}^{n-1}$ we denote $u(\mathbf{m}) := \sum_{i=2}^n m_i u^i \in L_A$.

For $\mathbf{m} = (m_2, \ldots, m_n) \in \mathbb{N}^{n-1}$ such that $j - \sum_{i \neq n-1} a_i m_i + a_{n-1} m_{n-1} \geq 0$, we have

$$\Gamma[v^j; u(\mathbf{m})] = \frac{(\frac{\beta-j}{a_{n-1}})_{m_{n-1}} j!}{m_2! \cdots m_{n-2}! m_n! (j - \sum_{i \neq n-1} a_i m_i + a_{n-1} m_{n-1})!}$$

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and

$$x^{u(\mathbf{m})} = x_1^{-\sum_{i\neq n-1} a_i m_i + a_{n-1} m_{n-1}} x_2^{m_2} \cdots x_{n-2}^{m_{n-2}} x_{n-1}^{-m_{n-1}} x_n^{m_n}.$$

If $\beta \in \mathbb{N}$ then there exists a unique $0 \le q \le a_{n-1} - 1$ such that $\beta - q \in a_{n-1}\mathbb{N}$. Let us write $m_0 = \frac{\beta-q}{a_{n-1}}$.

Then for $m \in \mathbb{N}$ big enough $m_0 - a_n m$ is a negative integer and the coefficient $\Gamma[v^q; u(\mathbf{m})]$ is zero and then ϕ_{v^q} is a polynomial in $\mathbb{C}[x]$.

Recall that $u^{n-1} = (a_{n-1}, 0, \dots, -1, 0) \in L_A$ and let us write

$$\widetilde{v^q} = v^q + (m_0 + 1)u^{n-1} = (q + (m_0 + 1)a_{n-1}, 0, \dots, 0, -1, 0) = (\beta + a_{n-1}, 0, \dots, 0, -1, 0).$$

We have $A\widetilde{v^q} = \beta$ an the corresponding Γ–series is

$$\phi_{\widetilde{v^q}} = x^{\widetilde{v^q}} \sum_{\mathbf{m} \in M(q)} \Gamma[\widetilde{v^q}; u(\mathbf{m})] x^{u(\mathbf{m})}$$

where for $\mathbf{m} = (m_2, \dots, m_n) \in \mathbb{Z}^n$ one has $u(\mathbf{m}) = \sum_{i=2}^n m_i u^i$ and

$$M(q) := \{(m_2, \dots, m_n) \in \mathbb{N}^{n-1} \mid q + (m_0 + m_{n-1} + 1)a_{n-1} - \sum_{i\neq n-1} a_i m_i \ge 0\}.$$

Let us notice that $\widetilde{v^q}$ does not have minimal negative support (see [30, p. 132-133]) and then the Γ–series $\phi_{\widetilde{v^q}}$ is not a solution of $H_A(\beta)$. We will prove in Theorem 4.23 that $H_A(\beta)_p(\phi_{\widetilde{v^q},p}) \subset \mathcal{O}_{X,p}$ for all $p \in Y \setminus Z$ and that $\phi_{\widetilde{v^q},p}$ is a Gevrey series of index a_n/a_{n-1}.

The second main result of this Section is the following

Theorem 4.23. *Let $A = (1\ a_2\ \cdots\ a_n)$ be an integer row matrix with $1 < a_2 < \cdots < a_n$, $Y = (x_n = 0) \subset X$ and $Z = (x_{n-1} = 0) \subset X$. Then for all $p \in Y \setminus Z$ and $s \ge a_n/a_{n-1}$ we have:*

i) If $\beta \notin \mathbb{N}$, then:

$$\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p = \bigoplus_{j=0}^{a_{n-1}-1} \mathbb{C}\overline{\phi_{v^j,p}}.$$

ii) If $\beta \in \mathbb{N}$, then there exists a unique $q \in \{0, \dots, a_{n-1} - 1\}$ such that $\frac{\beta-q}{a_{n-1}} \in \mathbb{N}$ and we have:

$$\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p = \bigoplus_{q\neq j=0}^{a_{n-1}-1} \mathbb{C}\overline{\phi_{v^j,p}} \oplus \mathbb{C}\overline{\phi_{\widetilde{v^q},p}}.$$

Here $\overline{\phi}$ stands for the class modulo $\mathcal{O}_{X|Y,p}$ of $\phi \in \mathcal{O}_{\widehat{X|Y},p}(s)$.

Proof. Part *i)* follows from Theorem 4.21 and Proposition 4.20 using the long exact sequence of cohomology.

Let us prove *ii)*. Since $\mathcal{E}xt^1(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s)) = 0$ (see Theorem 4.1) and applying Theorem 4.21, Proposition 4.20 and the long exact sequence in cohomology we get that

$$\mathcal{E}xt^1(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s))_{|Y\setminus Z}$$

is zero for $s \ge a_n/a_{n-1}$ and locally constant of rank 1 for $1 \le s < a_n/a_{n-1}$. We also have that

$$\mathcal{E}xt^1(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s))_{|Y\cap Z}$$

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is locally constant of rank 1 for all $s \geq 1$.

Assume $s \geq a_n/a_{n-1}$. We consider the following long exact sequence associated with the short exact sequence 1 (with $p \in Y \setminus Z$ and $\mathcal{M} = \mathcal{M}_A(\beta)$)

$$0 \to \mathcal{E}xt^0(\mathcal{M}, \mathcal{O}_{X|Y})_p \to \mathcal{E}xt^0(\mathcal{M}, \mathcal{O}_{\widehat{X|Y}}(s))_p \xrightarrow{\rho} \mathcal{E}xt^0(\mathcal{M}, \mathcal{Q}_Y(s))_p \to \mathcal{E}xt^1(\mathcal{M}, \mathcal{O}_{X|Y})_p \to 0$$

We also have

$$\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y})_p \simeq \mathbb{C}$$

$$\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s))_p \simeq \mathbb{C}^{a_{n-1}}$$

$$\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p \simeq \mathbb{C}^{a_{n-1}}$$

$$\mathcal{E}xt^1(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y})_p \simeq \mathbb{C}$$

Since $\beta \in \mathbb{N}$ there exists a unique $q = 0, 1, \ldots, a_{n-1} - 1$ such that $\frac{\beta - q}{a_{n-1}} \in \mathbb{N}$ and then $\phi_{v^q} \in \mathbb{C}[x]$ generates $\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y})_p = \text{Ker}(\rho)$.

Using the above exact sequence and the first isomorphism theorem we get that the family

$$\{\overline{\phi_{v^j,p}} : \ 0 \leq j \leq a_{n-1} - 1, j \neq q\}$$

is linearly independent in $\mathcal{Q}_Y(s)_p$ for all $p \in Y \setminus Z$.

In a similar way to the proof of Theorem 4.1 it can be proved that $\phi_{\widetilde{v^q},p} \in \mathcal{O}_{\widehat{X|Y}}(s)_p$ for all $p \in Y \setminus Z$ and $s \geq a_n/a_{n-1}$.

Writing $t_{n-1} = x_{n-1}^{-1}$ and defining:

$$\psi_{\widetilde{v^q}}(x_1, \ldots, x_{n-2}, t_{n-1}, x_n) := x^{-\widetilde{v^q}} \phi_{\widetilde{v^q}}(x_1, \ldots, x_{n-2}, \frac{1}{t_{n-1}}, x_n)$$

we have that

$$\psi_{\widetilde{v^q}} \in \mathbb{C}[[x_1, \ldots, x_{n-2}, t_{n-1}, x_n]]$$

Taking the subsum of $\psi_{\widetilde{v^q}}$ for $m_2 = \cdots = m_{n-2} = 0$, $m_n = a_{n-1}m$, $m_{n-1} = a_n m, m \in \mathbb{N}$, we get the power series

$$\sum_{m \geq 0} c_m \left(t_{n-1}^{a_n} x_n^{a_{n-1}} \right)^m$$

where

$$c_m = \frac{(-1)^{a_n m}(a_n m)!}{(a_{n-1}m)!}$$

This power series has Gevrey index a_n/a_{n-1} with respect to $x_n = 0$. Then $\phi_{\widetilde{v^q}}$ has Gevrey index a_n/a_{n-1}.

We have $E(\phi_{\widetilde{v^q}}) = P_i(\phi_{\widetilde{v^q}}) = 0$, for all $i = 1, 2, \ldots, n-2, n$ and $P_{n-1}(\phi_{\widetilde{v^q}})$ is a meromorphic function with poles along Z (and holomorphic on $X \setminus Z$):

$$P_{n-1}(\phi_{\widetilde{v^q}}) = \sum_{\underline{m} \in \widetilde{M}(q)} \frac{(\beta + a_{n-1})! x_1^{q - \sum_{i \neq n-1} a_i m_i + a_{n-1}(m_0 + 1)} x_2^{m_2} \cdots x_{n-2}^{m_{n-2}} x_{n-1}^{-1} x_n^{m_n}}{m_2! \cdots m_{n-2}! m_n! (q - \sum_{i \neq n-1} a_i m_i + a_{n-1}(m_0 + 1))!}$$

where

$$\widetilde{M}(q) = \{(m_2, \ldots, m_{n-2}, m_n) \in \mathbb{N}^{n-2} | \sum a_i m_i \leq q + a_{n-1}(m_0 + 1) = \beta + a_{n-1}\}$$

is a finite set (recall that $m_0 = \frac{\beta - q}{a_{n-1}} \in \mathbb{N}$).

In particular, $H_A(\beta)(\phi_{\widetilde{v^q}}) \subseteq \mathcal{O}_X(X \setminus Z)$.

So,
$$\overline{\phi_{\widetilde{v^q},p}} \in \mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p$$

for all $p \in Y \setminus Z$ and $s \geq a_n/a_{n-1}$.

In order to finish the proof we will see that for all $\lambda_j \in \mathbb{C}$ $(j = 0, \ldots, a_{n-1} - 1; j \neq q)$ and for all $p \in Y \setminus Z$ we have

$$\phi_{\widetilde{v^q},p} - \sum_{j \neq q} \lambda_j \phi_{v^j,p} \notin \mathcal{O}_{X|Y,p}.$$

Let us write

$$\psi_{v^j}(x_1, \ldots, x_{n-2}, t_{n-1}, x_n) := \phi_{v^j}\left(x_1, \ldots, x_{n-2}, \frac{1}{t_{n-1}}, x_n\right)$$

Assume to the contrary that there exist $p \in Y \setminus Z$ and $\lambda_j \in \mathbb{C}$ such that:

$$\phi_{\widetilde{v^q},p} - \sum_{j \neq q} \lambda_j \phi_{v^j,p} \in \mathcal{O}_{X|Y,p}$$

Let us consider the holomorphic function at p defined as

$$f := x^{\widetilde{v^q}} \psi_{\widetilde{v^q},p} - \sum_{j \neq q} \lambda_j \psi_{v^j,p}$$

We have the following equality of holomorphic functions at p:

$$\rho_s\left(f + \sum_{j \neq q} \lambda_j \psi_{v^j}\right) = \rho_s(x^{\widetilde{v^q}} \psi_{\widetilde{v^q}})$$

for $s > a_n$.

The function $\rho_s(x^{\widetilde{v^q}} \psi_{\widetilde{v^q}})$ is holomorphic in $\mathbb{C}^n$ while each $\rho_s(\psi_{v^j})$ has the form $t_{n-1}^{-\frac{\beta-j}{a_{n-1}}} \psi_j$ with ψ_j holomorphic in $\mathbb{C}^n$.

Making a loop around the t_{n-1} axis $(\log t_{n-1} \mapsto \log t_{n-1} + 2\pi i)$ we get the equality:

$$\rho_s\left(\widehat{f} + \sum_{j \neq q} c_j \lambda_j \psi_{v^j}\right) = \rho_s(x^{\widetilde{v^q}} \psi_{\widetilde{v^q}})$$

where $c_j = e^{-\frac{\beta-j}{a_{n-1}} 2\pi i} \neq 1$ (since $\frac{\beta-j}{a_{n-1}} \notin \mathbb{Z}$ for all $j \neq q$) and $\widehat{f}$ is obtained from f after the loop. Since f is holomorphic at p then $\widehat{f}$ also is. Subtracting both equalities we get:

$$\rho_s\left(\widehat{f} - f + \sum_{j \neq q} (c_j - 1) \lambda_j \psi_{v^j}\right) = 0$$

and then

$$\sum_{j \neq q} (c_j - 1) \lambda_j \psi_{v^j} = f - \widehat{f}$$

in the neighborhood of p. This contradicts the fact that the power series $\{\phi_{v^j} : j \neq q, 0 \leq j \leq a_{n-1} - 1\}$ are linearly independent modulo $\mathcal{O}_{X|Y,p}$ (here we have $c_j - 1 \neq 0$). This proves the theorem. $\square$

Corollary 4.24. *If $\beta \in \mathbb{N}$ then for all $p \in Y \setminus Z$ the vector space $\mathcal{E}xt^1(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y})_p$ is generated by the class of:*

$$(P_2(\phi_{\widetilde{v^q}}), \ldots, P_{n-1}(\phi_{\widetilde{v^q}}), P_n(\phi_{\widetilde{v^q}}), E(\phi_{\widetilde{v^q}})) =$$

$$= (0, \ldots, 0, \sum_{\underline{m} \in \widetilde{M}(q)} \frac{(\beta + a_{n-1})! \, x_1^{q - \sum_{i \neq n-1} a_i m_i + a_{n-1}(m_0+1)} x_2^{m_2} \cdots x_{n-2}^{m_{n-2}} x_{n-1}^{-1} x_n^{m_n}}{m_2! \cdots m_{n-2}! m_n! (q - \sum_{i \neq n-1} a_i m_i + a_{n-1}(m_0+1))!}, 0, 0)$$

in

$$\frac{(\mathcal{O}_{X|Y})_p^n}{\mathrm{Im}(\psi_0^*, \mathcal{O}_{X|Y})_p}$$

where

$$\widetilde{M}(q) = \{(m_2, \ldots, m_{n-2}, m_n) \in \mathbb{N}^{n-2} \mid \sum a_i m_i \leq q + a_{n-1}(m_0+1) = \beta + a_{n-1}\}$$

is a finite set (with $m_0 = \frac{\beta - q}{a_{n-1}} \in \mathbb{N}$) and ψ_0^* being the dual map of

$$\begin{aligned} \psi_0 : \mathcal{D}^n &\longrightarrow \mathcal{D} \\ (Q_1, \ldots, Q_n) &\mapsto \sum_{j=2}^n Q_j P_j + Q_n E \end{aligned}$$

Proof. It follows from the proof of Theorem 4.23 since $\mathcal{E}xt^1(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y})_p \simeq \mathbb{C}$ for all $p \in Y \setminus Z$ and moreover

$$(P_2(\phi_{v^j}), \ldots, P_n(\phi_{v^j}), E(\phi_{v^j})) = \underline{0}$$

for $0 \leq j \leq a_{n-1} - 1$, $j \neq q$. $\qquad\square$

Remark 4.25. *We can also compute the holomorphic solutions of $\mathcal{M}_A(\beta)$ at any point in $X \setminus Y$ for $A = (1 \ a_2 \ \ldots \ a_n)$ with $1 < a_2 < \ldots < a_n$ and for any $\beta \in \mathbb{C}$, where $Y = (x_n = 0) \subset X = \mathbb{C}^n$. We consider the vectors $w^j = (j, 0, \ldots, 0, \frac{\beta - j}{a_n}) \in \mathbb{C}^n$, $j = 0, 1, \ldots, a_n - 1$ then the germs at $p \in X \setminus Y$ of the series solutions $\{\phi_{w^j} : j = 0, 1, \ldots, a_n - 1\}$ is a basis of $\mathcal{E}xt_{\mathcal{D}}^0(\mathcal{M}_A(\beta), \mathcal{O}_X)_p$.*

We have summarized the main results of this Section in Figure 1. Here $A = (1 \ a_2 \ \cdots \ a_n)$, $s \geq \frac{a_n}{a_{n-1}}$, $p \in Y \setminus Z$, $z \in Y \cap Z$, $\beta_{\mathrm{esp}} \in \mathbb{N}$ and $\beta_{\mathrm{gen}} \notin \mathbb{N}$.

$(z, \beta_{\mathrm{esp}})$	$(p, \beta_{\mathrm{esp}})$		
$(z, \beta_{\mathrm{gen}})$	$(p, \beta_{\mathrm{gen}})$	$\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{F})$	$\mathcal{E}xt^1(\mathcal{M}_A(\beta), \mathcal{F})$
$\mathcal{F} = \mathcal{O}_{X|Y}$		1 \quad\quad 1	1 \quad\quad 1
		0 \quad\quad 0	0 \quad\quad 0
$\mathcal{F} = \mathcal{O}_{\widehat{X|Y}}(s)$		1 \quad\quad a_{n-1}	1 \quad\quad 0
		0 \quad\quad a_{n-1}	0 \quad\quad 0
$\mathcal{F} = \mathcal{Q}_Y(s)$		0 \quad\quad a_{n-1}	0 \quad\quad 0
		0 \quad\quad a_{n-1}	0 \quad\quad 0

Figure 1: Dimension of the germs of $\mathcal{E}xt_{\mathcal{D}_X}^i(\mathcal{M}_A(\beta), \mathcal{F})$

5 The case of a monomial curve

Let $A = (a_1 \ a_2 \ \cdots \ a_n)$ be an integer row matrix with $1 < a_1 < a_2 < \cdots < a_n$ and assume without loss of generality $\gcd(a_1, \ldots, a_n) = 1$.

In this Section we will compute de dimension of the germs of the cohomology of $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ at any point in $Y = \{x_n = 0\} \subseteq X = \mathbb{C}^n$ for all $\beta \in \mathbb{C}$ and $1 \leq s \leq \infty$.

We will consider the auxiliary matrix $A' = (1 \ a_1 \ \cdots \ a_n)$ and the corresponding hypergeometric ideal $H_{A'}(\beta) \subset A_{n+1}$ where A_{n+1} is the Weyl algebra of linear differential operators with coefficients in the polynomial ring $\mathbb{C}[x_0, x_1, \ldots, x_n]$. We denote ∂_0 the partial derivative with respect to x_0.

In this Section we denote $X' = \mathbb{C}^{n+1}$ and we identify $X = \mathbb{C}^n$ with the hyperplane $(x_0 = 0)$ in X'. If $\mathcal{D}_{X'}$ is the sheaf of linear differential operators with holomorphic coefficients in X' then the analytic hypergeometric system associated with (A', β), denoted by $\mathcal{M}_{A'}(\beta)$, is by definition the quotient of $\mathcal{D}_{X'}$ by the sheaf of ideals generated by the hypergeometric ideal $H_{A'}(\beta) \subset A_{n+1}$ (see Section 3).

One of the main results in this Section is the following

Theorem 5.1. *Let* $A' = (1\ a_1\ \cdots\ a_n)$ *an integer row matrix with* $1 < a_1 < \cdots < a_n$ *and* $\gcd(a_1, \ldots, a_n) = 1$. *For each* $\beta \in \mathbb{C}$ *there exists* $\beta' \in \mathbb{C}$ *such that the restriction of* $\mathcal{M}_{A'}(\beta)$ *to* $X = \{x_0 = 0\} \subset X'$ *is the* $\mathcal{D}_X$*–module*

$$\mathcal{M}_{A'}(\beta)_{|X} := \frac{\mathcal{D}_{X'}}{\mathcal{D}_{X'} H_{A'}(\beta) + x_0 \mathcal{D}_{X'}} \simeq \mathcal{M}_A(\beta')$$

where $A = (a_1\ a_2\ \cdots\ a_n)$. *Moreover, for all but finitely many* β *we have* $\beta' = \beta$.

Proof. Following [30], we will use the notations defined in Notation 4.5 and Definition 4.7. For $i = 1, 2, \ldots, n$ let us consider $\delta_i \in \mathbb{N}$ the smallest integer satisfying $1 + \delta_i a_i \in \sum_{j \neq i} a_j \mathbb{N}$. Such a δ_i exists because $\gcd(a_1, \ldots, a_n) = 1$.

Let us consider $\rho_{ij} \in \mathbb{N}$ such that

$$1 + \delta_i a_i = \sum_{j \neq i} \rho_{ij} a_j.$$

Then the operator $Q_i := \partial_0 \partial_i^{\delta_i} - \partial^{\rho_i}$ belongs to $I_{A'}$ where $\partial^{\rho_i} = \prod_{j \neq 0, i} \partial_j^{\rho_{ij}}$. Moreover, for $\omega = (1, 0, \ldots, 0) \in \mathbb{N}^{n+1}$ we have $\mathrm{in}_{(-\omega, \omega)}(Q_i) = \partial_0 \partial_i^{\delta_i} \in \mathrm{in}_\omega I_{A'}$ for $i = 1, \ldots, n$.

We also have that $P_1 = \partial_0^{a_1} - \partial_1 \in I_{A'}$ and $\mathrm{in}_{(-\omega, \omega)} P_1 = \partial_0^{a_1} \in \mathrm{in}_\omega I_{A'}$. Then

$$\mathrm{in}_\omega I_{A'} \supseteq \langle \partial_0^{a_1}, \partial_0 \partial_1^{\delta_1} \ldots, \partial_0 \partial_n^{\delta_n}, T_1, \ldots, T_r \rangle \tag{5}$$

for any binomial generating system $\{T_1, \ldots, T_r\} \subseteq \mathbb{C}[\partial_1, \ldots, \partial_n]$ of the ideal $I_A = I_{A'} \cap \mathbb{C}[\partial_1, \ldots, \partial_n]$ (notice that $u \in L_A = \ker_{\mathbb{Z}}(A) \subset \mathbb{Z}^n \iff (0, u) \in L_{A'} = \ker_{\mathbb{Z}}(A') \subset \mathbb{Z}^{n+1}$).

Using (5) we can prove (similarly to the proof of Lemma 4.6 for $k = 1$) that for $\beta \notin \mathbb{N}^*$ or $\beta \in \mathbb{N}^*$ big enough, we have

$$\partial_0 \in \mathrm{fin}_\omega(H_{A'}(\beta)) = \mathrm{in}_\omega I_{A'} + \langle E' \rangle \tag{6}$$

where $E' = E + x_0 \partial_0$ and $E := E(\beta) = \sum_{i=1}^n a_i x_i \partial_i - \beta$. So there exists $R \in H_{A'}(\beta)$ such that $\partial_0 = \mathrm{in}_{(-\omega, \omega)}(R)$. In particular we have

$$\langle H_A(\beta), \partial_0 \rangle \subseteq \mathrm{fin}_\omega(H_{A'}(\beta)) \subseteq \mathrm{in}_{(-\omega, \omega)}(H_{A'}(\beta))$$

and the b–function of $H_{A'}(\beta)$ with respect to ω is $b(\tau) = \tau$. So the restriction of $\mathcal{M}_{A'}(\beta)$ to $(x_0 = 0)$ is a cyclic $\mathcal{D}_X$-module (see [30, Algorithm 5.2.8]).

In order to compute $\mathcal{M}_{A'}(\beta)_{|(x_0=0)}$ we will follow [30, Algorithm 5.2.8]. First of all we need to describe the form of a Groebner basis of $H_{A'}(\beta)$ with respect to ω. Let $\{T_1, \ldots, T_r, R_1, \ldots, R_\ell\}$ be a Groebner basis of $I_{A'}$ with respect to ω. So we have

$$I_{A'} = \langle T_1, \ldots, T_r, R_1, \ldots, R_\ell \rangle$$

and

$$\mathrm{in}_\omega I_{A'} = \langle T_1, \ldots, T_r, \mathrm{in}_{(-\omega, \omega)} R_1, \ldots, \mathrm{in}_{(-\omega, \omega)} R_\ell \rangle.$$

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If, for some $i = 0, \ldots, \ell$, the ω-order of $\mathrm{in}_{(-\omega,\omega)} R_i$ is 0, then $\mathrm{in}_{(-\omega,\omega)} R_i = R_i \in I_{A'} \cap \mathbb{C}[\partial_1, \ldots, \partial_n] = I_A$ and then $\mathrm{in}_{(-\omega,\omega)} R_i = R_i \in \langle T_1, \ldots, T_r \rangle$.

If the ω-order of $\mathrm{in}_{(-\omega,\omega)} R_i$ is greater than or equal to 1, then ∂_0 divide $\mathrm{in}_{(-\omega,\omega)} R_i$. Then, according to (6), for $\beta \notin \mathbb{N}^*$ or $\beta \in \mathbb{N}^*$ big enough, we have

$$\mathrm{fin}_\omega(H_{A'}(\beta)) = \langle \partial_0, E, T_1, \ldots, T_r \rangle = \langle \partial_0 \rangle + A_{n+1} H_A(\beta) \subseteq \mathrm{in}_{(-\omega,\omega)}(H_{A'}(\beta)) \tag{7}$$

From [30, Th. 3.1.3], for all but finitely many $\beta \in \mathbb{C}$, we have

$$\mathrm{in}_{(-\omega,\omega)}(H_{A'}(\beta)) = \langle \partial_0, E, T_1, \ldots, T_r \rangle = \langle \partial_0 \rangle + A_{n+1} H_A(\beta). \tag{8}$$

So, for all but finitely many $\beta \in \mathbb{C}$, the set

$$\mathcal{G} = \{R, R_1, \ldots, R_\ell, E', T_1, \ldots, T_r\}$$

is a Groebner basis of $H_{A'}(\beta)$ with respect to ω, since first of all $\mathcal{G}$ is a generating system of $H_{A'}(\beta)$ and on the other hand $\mathrm{in}_{(-\omega,\omega)}(H_{A'}(\beta)) = A_{n+1} \mathrm{in}_{(-\omega,\omega)}(\mathcal{G})$.

We can now follow [30, Algorithm 5.2.8], as in the proof of Theorem 4.3, to prove the result for all but finitely many $\beta \in \mathbb{C}$. Then, to finish the proof it is enough to apply Proposition 4.11 for A'. $\square$

Remark 5.2. *Recall that* $Y = (x_n = 0) \subset X = \mathbb{C}^n$ *and* $Z = (x_{n-1} = 0) \subset X$. *Let us denote* $Y' = \{x_n = 0\} \subset X'$, $Z' = \{x_{n-1} = 0\} \subset X'$. *Notice that* $Y = Y' \cap X$ *and* $Z = Z' \cap X$.

By using Cauchy-Kovalevskaya Theorem for Gevrey series (see [20, Cor. 2.2.4]), [5, Proposition 4.2] and Theorem 5.1, we get, for all but finitely many $\beta \in \mathbb{C}$ *and for all* $1 \leq s \leq \infty$, *the following isomorphism*

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_{X'}}(\mathcal{M}_{A'}(\beta), \mathcal{O}_{\widehat{X'|Y'}}(s))_{|X} \overset{\sim}{\to} \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s)).$$

We also have the following

Theorem 5.3. *Let* $A = (a_1 \; a_2 \; \cdots \; a_n)$ *be an integer row matrix with* $1 < a_1 < a_2 < \cdots < a_n$ *and* $\gcd(a_1, \ldots, a_n) = 1$. *Then for all* $\beta \in \mathbb{C}$ *we have*

i) $\mathcal{E}xt^0_{\mathcal{D}_X}(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s)) = 0$ *for* $1 \leq s < a_n/a_{n-1}$.

ii) $\mathcal{E}xt^0_{\mathcal{D}_X}(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_{|Y \cap Z} = 0$ *for* $1 \leq s \leq \infty$.

iii) $\dim_\mathbb{C}(\mathcal{E}xt^0_{\mathcal{D}_X}(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p) = a_{n-1}$ *for* $a_n/a_{n-1} \leq s \leq \infty$ *and* $p \in Y \setminus Z$.

iv) $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s)) = 0$, *for* $i \geq 1$ *and* $1 \leq s \leq \infty$.

Here $Y = (x_n = 0) \subset \mathbb{C}^n$ *and* $Z = (x_{n-1} = 0) \subset \mathbb{C}^n$.

Proof. It follows from Remark 5.2, Theorem 4.1 and Proposition 4.11. $\square$

Remark 5.4. *With the notations of Theorem 5.3, a basis of the* $\mathbb{C}$–*vector space* $\mathcal{E}xt^0_\mathcal{D}(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p$ *for any* $\frac{a_n}{a_{n-1}} \leq s \leq \infty$, $p \in Y \setminus Z$ *and* $\beta \in \mathbb{C}$ *is given by the "substitution" (in a sense to be precised) of* $x_0 = 0$ *in the basis of* $\mathcal{E}xt^0_{\mathcal{D}'}(\mathcal{M}_{A'}(\beta), \mathcal{Q}_{Y'}(s))_{(0,p)}$ *described in Theorem 4.23.*

Remind that for $A' = (1 \; a_1 \; \ldots \; a_n)$ *and* $\beta \in \mathbb{C}$ *the* Γ–*series described in Section 4 are*

$$\phi_{v^j} = (x')^{v^j} \sum_{\substack{m_1, \ldots, m_{n-1}, m_n \geq 0 \\ \sum_{i \neq n-1} a_i m_i \leq j + a_{n-1} m_{n-1}}} \Gamma[v^j; u(\mathbf{m})](x')^{u(\mathbf{m})}$$

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where $x' = (x_0, x_1, \ldots, x_n)$, $v^j = (j, 0, \ldots, 0, \frac{\beta-j}{a_{n-1}}, 0) \in \mathbb{C}^{n+1}$ for $j = 0, 1, \ldots, a_{n-1} - 1$ and for $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ we have

$$(x')^{u(\mathbf{m})} = x_0^{-\sum_{i \neq n-1} a_i m_i + a_{n-1} m_{n-1}} x_1^{m_1} \cdots x_{n-2}^{m_{n-2}} x_{n-1}^{-m_{n-1}} x_n^{m_n}.$$

For $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n$ such that $j - \sum_{i \neq n-1} a_i m_i + a_{n-1} m_{n-1} \geq 0$ we have

$$\Gamma[v^j; u(\mathbf{m})] = \frac{(\frac{\beta-j}{a_{n-1}})_{m_{n-1}} j!}{m_1! \cdots m_{n-2}! m_n! (j - \sum_{i \neq n-1} a_i m_i + a_{n-1} m_{n-1})!}.$$

Since $I_A = I_{A'} \cap \mathbb{C}[\partial_1, \ldots, \partial_n]$, if $I_{A'}(f) = 0$ then $I_A(f_{|x_0=0}) = 0$ for every formal power series $f \in \mathcal{O}_{\widehat{X'|Y'}, p'}$ where $p' = (0, p) \in Y' \cap \{x_0 = 0\} = Y$.

Furthermore, a Laurent monomial $(x')^{w'}$ is annihilated by the Euler operator associated with (A', β) if and only if $A'w' = \beta$ and after the substitution $x_0 = 0$ this monomial becomes zero or x^w (in the case $w' = (0, w)$) which are both annihilated by the Euler operator associated with (A, β), since $Aw = A'w' = \beta$ in the case $w' = (0, w)$.

Hence, for $p \in Y$, every formal series solution $f \in \mathcal{O}_{\widehat{X'|Y'}, (0,p)}$ of $\mathcal{M}_{A'}(\beta)$ becomes, after the substitution $x_0 = 0$, a formal series solution $f_{|x_0=0} \in \mathcal{O}_{\widehat{X|Y}, p}$ of $\mathcal{M}_A(\beta)$. The analogous result is also true for convergent series solutions at a point of $x_0 = 0$.

After the substitution $x_0 = 0$ in the series ϕ_{v^j} we get

$$\phi_{v^j|x_0=0} = \sum_{\substack{m_1, \ldots, m_{n-1}, m_n \geq 0 \\ \sum a_i m_i = j + a_{n-1} m_{n-1}}} \frac{(\frac{\beta-j}{a_{n-1}})_{m_{n-1}} j! x_1^{m_1} \cdots x_{n-2}^{m_{n-2}} x_{n-1}^{\frac{\beta-j}{a_{n-1}} - m_{n-1}} x_n^{m_n}}{m_1! \cdots m_{n-2}! m_n!}$$

for $j = 0, 1, \ldots, a_{n-1} - 1$.

The summation before is taken over the set

$$\Delta_j := \{(m_1, \ldots, m_n) \in \mathbb{N}^n : \sum_{i \neq n-1} a_i m_i = j + a_{n-1} m_{n-1}\}.$$

It is clear that $(0, \ldots, 0) \in \Delta_0$ and for $j \geq 1$, Δ_j is a non empty set since $\gcd(a_1, \ldots, a_n) = 1$. Moreover Δ_j is a countably infinite set for $j \geq 0$. To this end take some $\underline{\lambda} := (\lambda_1, \ldots, \lambda_n) \in \Delta_j$. Then $\underline{\lambda} + \mu(0, \ldots, 0, a_n, a_{n-1})$ is also in Δ_j for all $\mu \in \mathbb{N}$.

The series $\phi_{v^j|x_0=0}$ is a Gevrey series of order $s = \frac{a_n}{a_{n-1}}$ since ϕ_{v^j} also is. We will see that in fact the Gevrey index of $\phi_{v^j|x_0=0}$ is $\frac{a_n}{a_{n-1}}$ for $j = 0, \ldots, a_{n-1} - 1$ such that $\frac{\beta-j}{a_{n-1}} \notin \mathbb{N}$. To this end let us consider the subsum of $\phi_{v^j|x_0=0}$ over the set of $(m_1, \ldots, m_n) \in \mathbb{N}^n$ of the form $\underline{\lambda}^{(j)} + \mathbb{N}(0, \ldots, 0, a_n, a_{n-1})$ for some fixed $\underline{\lambda}^{(j)} \in \Delta_j$. Then we get the series:

$$\frac{j! (\frac{\beta-j}{a_{n-1}})_{\lambda_{n-1}^{(j)}} x_1^{\lambda_1^{(j)}} \cdots x_{n-2}^{\lambda_{n-2}^{(j)}} x_{n-1}^{\frac{\beta-j}{a_{n-1}} - \lambda_{n-1}^{(j)}}}{\lambda_1^{(j)}! \cdots \lambda_{n-2}^{(j)}!} \sum_{m \geq 0} \frac{(\frac{\beta-j}{a_{n-1}} - \lambda_{n-1}^{(j)})_{a_n m} x_{n-1}^{-a_n m} x_n^{\lambda_n^{(j)} + a_{n-1} m}}{(\lambda_n^{(j)} + a_{n-1} m)!}$$

and it can be proven, by using d'Alembert ratio test, that its Gevrey index equals $\frac{a_n}{a_{n-1}}$ at any point in $Y \setminus Z$, for any $j = 0, \ldots, a_{n-1} - 1$ such that $\frac{\beta-j}{a_{n-1}} \notin \mathbb{N}$.

For all $j = 0, \ldots, a_{n-1} - 1$ we have

$$\phi_{v^j|x_0=0} \in x_{n-1}^{\frac{\beta-j}{a_{n-1}}} \mathbb{C}[[x_1, \ldots, x_{n-2}, x_{n-1}^{-1}, x_n]]$$

and in particular these a_{n-1} series are linearly independent and hence a basis of $\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s))_p$ for $s \geq a_n/a_{n-1}$ and $p \in Y \setminus Z$) if all of them are nonzero (see Theorem 4.21).

If there exists $0 \leq j \leq a_{n-1} - 1$ such that $\phi_{v^j|x_0=0} = 0$ then we have that ϕ_{v^j} is a polynomial divisible by x_0 and this happens if and only if $\beta \in \mathbb{N} \setminus \mathbb{N}A$.

In this last case we do not get a basis of $\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s))_p$ by the previous procedure. We will proceed as follows. Let us consider $w' = (0, \omega) \in \mathbb{N}^{n+1}$ such that $\beta' := \beta - A'w' \in \mathbb{Z}_{<0}$. Then taking the basis $\{\phi_{v^j} \mid j = 0, \ldots, a_{n-1} - 1\}$ of $\mathcal{E}xt^0(\mathcal{M}_{A'}(\beta'), \mathcal{O}_{\widehat{X'|Y'}}(s))_{(0,p)}$ given by Theorem 4.21 and after the substitution $x_0 = 0$, we get a basis of $\mathcal{E}xt^0(\mathcal{M}_A(\beta'), \mathcal{O}_{\widehat{X|Y}}(s))_p$ for $s \geq a_n/a_{n-1}$ and $p \in Y \setminus Z$.

Since $\beta, \beta' \in \mathbb{Z} \setminus \mathbb{N}A$ then $\partial^w : \mathcal{M}_A(\beta') \to \mathcal{M}_A(\beta)$ is an isomorphism (see [31, Remark 3.6] and [4, Lemma 6.2]) and we can use this isomorphism to obtain a basis of $\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s))_p$.

Using previous discussion and similar ideas to the ones of Section 4 (we will use the notations therein) we can prove the following Theorem.

Theorem 5.5. Let $A = (a_1 \ a_2 \ \cdots \ a_n)$ be an integer row matrix with $0 < a_1 < a_2 < \cdots < a_n$, $Y = (x_n = 0) \subset X$ and $Z = (x_{n-1} = 0) \subset X$. Then for all $p \in Y \setminus Z$, $\beta \in \mathbb{C}$ and $s \geq a_n/a_{n-1}$ we have:

i) If $\beta \notin \mathbb{N}$, then:

$$\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p = \bigoplus_{j=0}^{a_{n-1}-1} \mathbb{C}\overline{(\phi_{v^j|x_0=0})_p}.$$

ii) If $\beta \in \mathbb{N}$, then there exists a unique $q \in \{0, \ldots, a_{n-1} - 1\}$ such that $\frac{\beta-q}{a_{n-1}} \in \mathbb{N}$ and we have:

$$\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p = \bigoplus_{q \neq j=0}^{a_{n-1}-1} \mathbb{C}\overline{(\phi_{v^j|x_0=0})_p} \oplus \mathbb{C}\overline{(\phi_{\widetilde{v^q}|x_0=0})_p}.$$

Here $\overline{\phi}$ stands for the class modulo $\mathcal{O}_{X|Y,p}$ of $\phi \in \mathcal{O}_{\widehat{X|Y},p}(s)$.

Remark 5.6. We can also compute the holomorphic solutions of $\mathcal{M}_A(\beta)$ at any point in $X \setminus Y$ for $A = (a_1 \ a_2 \ \ldots \ a_n)$ with $0 < a_1 < a_2 < \ldots < a_n$ and for any $\beta \in \mathbb{C}$, where $Y = (x_n = 0) \subset X = \mathbb{C}^n$ (see [7, Sec. 2.1] and Remark 4.25). As in the beginning of Section 5 let us consider the auxiliary matrix $A' = (1 \ a_1 \ a_2 \ \ldots \ a_n)$ and the notation therein.

Let us consider the vectors $w^j = (j, 0, \ldots, 0, \frac{\beta-j}{a_n}) \in \mathbb{C}^{n+1}$, $j = 0, 1, \ldots, a_n - 1$. Then the germs at $p' = (0, p) \in X' \setminus Y'$ (with $p \in X \setminus Y$) of the series solutions $\{\phi_{w^j} : j = 0, 1, \ldots, a_n - 1\}$ is a basis of $\mathcal{E}xt^0_{\mathcal{D}'}(\mathcal{M}_{A'}(\beta), \mathcal{O}_{X'})_{p'}$. Taking

$$\{\phi_{w^j|x_0=0} : j = 0, 1, \ldots, a_n - 1\}$$

we get a basis of $\mathcal{E}xt^0_{\mathcal{D}}(\mathcal{M}_A(\beta), \mathcal{O}_X)_p$ for $\beta \in \mathbb{C}$ such that $\beta \notin \mathbb{N} \setminus \mathbb{N}A$ at any point $p \in X \setminus Y$. When $\beta \in \mathbb{N} \setminus \mathbb{N}A$ we can proceed as in Remark 5.4.

Conclusions

1) In Sections 4 and 5 we have proved that the irregularity complex $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ is zero for $1 \leq s < a_n/a_{n-1}$ and concentrated in degree 0 for $a_n/a_{n-1} \leq s \leq \infty$ (see Theorems 4.1 and 5.3). Here A is a row integer matrix $(a_1 \ a_2 \ \cdots \ a_n)$ with $0 < a_1 < a_2 < \cdots < a_n$ and β is a parameter in $\mathbb{C}$. We have reduced the case $a_1 > 1$ to the one where $a_1 = 1$ and then to the two dimensional case treated in [7].

2) We have described a basis of $\mathcal{E}xt^0_{\mathcal{D}_X}(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p$ for $p \in Y \setminus Z$ and $a_n/a_{n-1} \leq s \leq \infty$ (see Theorems 4.23 and 5.5). Here $Y = (x_n = 0) \subset X = \mathbb{C}^n$ and $Z = (x_{n-1} = 0) \subset X$. From the form of the basis it is easy to see that the eigenvalues of the corresponding monodromy, with respect to Z, are simply $\exp(\frac{2\pi i(\beta-k)}{a_{n-1}})$ for $k = 0, \ldots, a_{n-1} - 1$. Notice that for $\beta \in \mathbb{Z}$ one eigenvalue (the one corresponding to the unique $k = 0, \ldots, a_{n-1} - 1$ such that $\frac{\beta-k}{a_{n-1}} \in \mathbb{Z}$) is just 1. See Remark 5.4 for notations.

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