RESTRICTION OF HYPERGEOMETRIC $\mathscr{D}$-MODULES WITH RESPECT TO COORDINATE SUBSPACES

MARÍA-CRUZ FERNÁNDEZ-FERNÁNDEZ AND ULI WALTHER

Abstract. We compute the restriction of an A-hypergeometric $\mathscr{D}$-module with respect to a coordinate subspace under certain genericity conditions on the parameter.

1. Introduction

Let $A \in \mathbb{Z}^{d\times n}$ be a matrix of rank d with integer entries a_{ij}, $i = 1,\dots,d$, $j = 1,\dots,n$. For $\boldsymbol{u} \in \mathbb{Z}^n$, let $\boldsymbol{u}_+, \boldsymbol{u}_- \in \mathbb{N}^n$ be such that $\boldsymbol{u} = \boldsymbol{u}_+ - \boldsymbol{u}_-$, and write $\Box_{\boldsymbol{u}}$ for the element $\partial^{\boldsymbol{u}_+} - \partial^{\boldsymbol{u}_-}$ of $R_A = \mathbb{C}[\partial_1,\dots,\partial_n]$. Here and elsewhere we use multi-index notation: $\partial^{\boldsymbol{v}} = \prod_{i=1}^n v_i$ for $\boldsymbol{v} \in \mathbb{N}^n$. The *toric ideal* associated with A is the prime binomial ideal

$$I_A = \langle \Box_{\boldsymbol{u}} \colon \boldsymbol{u} \in \mathbb{Z}^n,\ A\boldsymbol{u} = 0\rangle \subseteq R_A.$$

Identifying ∂_j with the partial derivation operator $\frac{\partial}{\partial x_j}$, let $D \supseteq R_A$ be the Weyl algebra of order n, i.e. the $\mathbb{C}$-algebra of linear partial differential operators with coefficients in the polynomial ring $\mathbb{C}[x_1,\dots,x_n]$. Following [GGZ87] and [GZK89], the *hypergeometric ideal* associated with a matrix A as above and a vector of complex parameters $\beta \in \mathbb{C}^d$ is the left ideal

$$H_A(\beta) := DI_A + D\langle E_A - \beta\rangle \subseteq D,$$

where $E_A - \beta$ is the sequence of Euler operators

$$E_i - \beta_i = \sum_{j=1}^n a_{ij}x_j\partial_j - \beta_i, \qquad i = 1,\dots,d.$$

The (global) hypergeometric D-module associated with (A,β) is by definition the quotient

$$M_A(\beta) := D/H_A(\beta).$$

Let $\mathscr{D}$ be the sheaf of linear partial differential operators with holomorphic coefficients in $\mathbb{C}^n$. To the pair (A,β) one may associate the corresponding analytic hypergeometric $\mathscr{D}$-module, denoted by $\mathscr{M}_A(\beta)$, which is the quotient of $\mathscr{D}$ modulo the left $\mathscr{D}$-ideal generated by $H_A(\beta)$.

The restriction functor is a useful tool for the study of the irregularity of a holonomic $\mathscr{D}$-module $\mathscr{M}$ (see, for example, the Cauchy–Kovalevskaya theorem for Gevrey series [LM02, Cor. 2.2.4]). There is an algorithm due to Oaku–Takayama

Date: October 18, 2010.

Key words and phrases. Euler–Koszul functor, hypergeometric $\mathscr{D}$-modules, restriction.

M.F. supported by the FPU-Grant AP2005-2360, Ministry of Education (Spain), and partially supported by MTM2007-64509 and FQM333.

U.W. supported by NSF grant DMS 0901123.

(see [Oak97, OT01]) for the effective computation of the restriction of holonomic D-modules to linear subspaces by means of Gröbner basis calculations in the Weyl algebra. This algorithm is employed in the proof of the explicit restriction formulas in [CJT03, Thm. 4.4] and [FFCJ11, Thm. 4.2] that hold for certain classes of hypergeometric systems. The purpose of our paper is, for sufficiently general parameter $\beta \in \mathbb{C}^d$, to generalize these formulas to the case of arbitrary A by using the Euler–Koszul functor developed in [MMW05]. We also show by example that there are parameters for which our formula does not hold (see Example 2.8).

2. Explicit restriction formula for $\mathscr{M}_A(\beta)$.

Notation 2.1. We denote by $\boldsymbol{a}_j \in \mathbb{Z}^d$ the j-th column of A, $j = 1, \ldots, n$. For any subset $\tau \subseteq \{1, \ldots, n\}$ we shall write $x_\tau = (x_i)_{i\in\tau}$, $\partial_\tau = (\partial_i)_{i\in\tau}$, $R_\tau = \mathbb{C}[\partial_\tau]$, and $D_\tau = \mathbb{C}[x_\tau]\langle\partial_\tau\rangle$. If I_τ is the toric ideal associated with the submatrix A_τ consisting of the columns indexed by τ then $R_\tau/I_\tau = S_\tau$ is isomorphic to the semigroup ring $S_\tau = \mathbb{C}[t^{\boldsymbol{a}_i} : i \in \tau] \subseteq \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$.

Consider for $Y_\tau = \{x_i = 0 : \ i \notin \tau\}$ the natural inclusion

$$i_\tau \colon Y_\tau \hookrightarrow X = \mathbb{C}^n.$$

We say that $\beta \in \mathbb{C}^d$ is *generic* if it is outside a hyperplane arrangement depending on A, and that β is *very generic* if it is outside a locally finite arrangement of countably many hyperplanes.

With this notation we will prove:

Theorem 2.2. *Suppose one of the following conditions holds:*

(i) *$\beta \in \mathbb{C}^d$ is generic and $\mathbb{Q}_{\geq 0}A_\tau = \mathbb{Q}_{\geq 0}A$ (i.e., the real positive cones spanned by the columns of A and A_τ respectively agree);*

(ii) *$\beta \in \mathbb{C}^d$ is very generic and* $\operatorname{rank}(A_\tau) = d$.

Choose a collection $\Omega \subseteq \mathbb{N}A$ of coset representatives for $\mathbb{Z}A/\mathbb{Z}\tau$. Then the (derived) restriction of $\mathscr{M}_A(\beta)$ with respect to Y_τ is given by

$$\mathbb{L}^{-k} i_\tau^* \mathscr{M}_A(\beta) \simeq \begin{cases} \bigoplus_{\lambda\in\Omega} \mathscr{M}_{A_\tau}(\beta - \lambda) & \text{if } k = 0; \\ 0 & \text{if } k > 0. \end{cases} \tag{2.1}$$

Proof. The left-multiplications by x_i, $i \notin \tau$, form commuting endomorphisms of left $\mathscr{D}_\tau$-modules (but not of left $\mathscr{D}_A$-modules). By [MT04, Prop. 3.1]), $\mathbb{L}i_\tau^*\mathscr{M}_A(\beta)$ is quasi-isomorphic to the Koszul complex $K_\bullet(x_i\cdot :\ i \notin \tau; \mathscr{M}_A(\beta))$ as a complex of left $\mathscr{D}_\tau$-modules. Moreover, because of the flatness of the fibers of $\mathscr{D}_A$ (resp. $\mathscr{D}_\tau$) over their algebraic counterparts and since affine spaces are D-affine, it suffices to prove Theorem 2.2 using the global sections D_A (resp. D_τ) instead of the sheaves $\mathscr{D}_A$ (resp. $\mathscr{D}_\tau$).

Define a $\mathbb{Z}^d$-grading on $R_A \subseteq D_A$ by

$$\deg(\partial_i) = -\boldsymbol{a}_i = -\deg(x_i), \qquad i = 1, \ldots, n.$$

For any $\mathbb{Z}^d$-graded R_A-module N, the Euler–Koszul complex $\mathcal{K}_\bullet(E_A - \beta, N)$ (see [MMW05, Def. 4.2]) is the Koszul complex of left D_A-modules defined by the sequence $E_A - \beta$ of commuting endomorphisms on the left D_A-module $D_A \otimes_{R_A} N$. This complex is concentrated in homological degrees d to 0. The i-th Euler–Koszul homology of N is

$$\mathcal{H}_i(E_A - \beta; N) = H_i(\mathcal{K}_\bullet(E_A - \beta, N)).$$

By [MMW05, Theorem 6.6], $\mathcal{K}_\bullet(E_A-\beta,S_A)$ is a resolution of $M_A(\beta)$ for generic β.

Set $C_{\bullet,\bullet}=K_\bullet(x_i\cdot:\ i\notin\tau;\mathcal{K}_\bullet(E_A-\beta,S_A))$, a double complex as follows from [MMW05, Lem. 4.3] and

- $x_i\cdot(D_A\otimes_{R_A}S_A)_\alpha\subseteq(D_A\otimes_{R_A}S_A)_{\alpha+a_i}$,
- $x_i\cdot(E_A-\beta-\alpha)=(E_A-\beta-\alpha-a_i)x_i$, and
- $\mathcal{K}_\bullet(E_A-\beta,S_A)=\bigoplus_{\alpha\in\mathbb{Z}^d}K_\bullet(E_A-\beta-\alpha,(D_A\otimes_{R_A}S_A)_\alpha)$,

showing that all the squares in $C_{\bullet,\bullet}$ are commutative.

Denote by $\overline{\tau}$ the complement $A\smallsetminus\tau$. Let π be the natural projection of $C_{\bullet,\bullet}$ to $K_\bullet(x_i\cdot:i\notin\tau;M_A(\beta))$ and let η be the natural projection of $C_{\bullet,\bullet}$ to $D_A/x_{\overline{\tau}}D_A\otimes_{D_A}\mathcal{K}_\bullet(E_A-\beta,S_A)$. Consider the induced morphisms:

$$\mathrm{Tot}(C_{\bullet,\bullet})\xrightarrow{\pi}\mathrm{Tot}(K_\bullet(x_i\cdot:i\notin\tau;M_A(\beta)))=K_\bullet(x_i\cdot:i\notin\tau;M_A(\beta))\tag{2.2}$$

and

$$\mathrm{Tot}(C_{\bullet,\bullet})\xrightarrow{\eta}\mathrm{Tot}(D_A/x_{\overline{\tau}}D_A\otimes_{D_A}\mathcal{K}_\bullet(E_A-\beta,S_A))\tag{2.3}$$

Remark 2.3. By [MMW05, Thm. 6.6], $\mathcal{K}_\bullet(E_A-\beta,S_A)$ is a resolution of $\mathcal{M}_A(\beta)$ if and only if β is not rank-jumping. Thus, for such β, π and η are a quasi-isomorphisms and $H_\bullet\,\mathrm{Tot}(C_{\bullet,\bullet})=\mathrm{Tor}_\bullet^{D_A}(D_A/x_{\overline{\tau}}D_A,M_A(\beta))$.

Notation 2.4. For a toric S_A-module N we write $\mathcal{K}_\bullet(E_\tau-\beta,N)$ for the Koszul complex of left D_τ-modules defined by the sequence $E_\tau-\beta$ on the (*a fortiori* weakly toric) S_τ-module N.

Using the isomorphism (see [MMW05])

$$\begin{array}{rcl}(D_A\otimes_{R_A}S_A) & \longrightarrow & \mathbb{C}[x_{\overline{\tau}}]\otimes_{\mathbb{C}}(D_\tau\otimes_{R_\tau}S_A),\\ x_\tau^{\mu_\tau}x_{\overline{\tau}}^{\mu_{\overline{\tau}}}\partial_\tau^{\nu_\tau}\partial_{\overline{\tau}}^{\nu_{\overline{\tau}}}\otimes m & \mapsto & x_{\overline{\tau}}^{\mu_{\overline{\tau}}}\otimes(x_\tau^{\mu_\tau}\partial_\tau^{\nu_\tau})\otimes\partial_{\overline{\tau}}^{\nu_{\overline{\tau}}}m.\end{array}\tag{2.4}$$

one may now identify $D_A/x_{\overline{\tau}}D_A\otimes_{D_A}\mathcal{K}_\bullet(E-\beta,S_A)$ with $\mathcal{K}_\bullet(E_\tau-\beta,S_A)$ as complexes of D_τ-modules. We have thus proved

Lemma 2.5. *If β is not rank-jumping for A and* $\mathrm{rank}(A_\tau)=\mathrm{rank}(A)$ *then*

$$\mathbb{L}i_\tau^*M_A(\beta)\simeq\mathcal{K}_\bullet(E_\tau-\beta,S_A)$$

as complexes of left D_τ-modules. □

Remark 2.6. For all $\beta\in\mathbb{C}^d$, $i_\tau^*M_A(\beta)\simeq\mathcal{H}_0(E_\tau-\beta,S_A)$ because i_τ^* is right exact.

It is clear that $\mathbb{Q}_{\geq0}A=\mathbb{Q}_{\geq0}A_\tau$ implies that $\mathrm{rank}(A_\tau)=\mathrm{rank}(A)=d$ and this last condition is equivalent to $[\mathbb{Z}A:\mathbb{Z}A_\tau]<+\infty$.

Consider $S_A=\mathbb{C}[\mathbb{N}A]$ and $S_\tau=\mathbb{C}[\mathbb{N}A_\tau]$. Then the assumption $\mathbb{Q}_{\geq0}A=\mathbb{Q}_{\geq0}A_\tau$ guarantees that S_A is a finitely generated $\mathbb{Z}^d$-graded S_τ-module and so it is a toric R_τ-module (see Definition 4.5. and Example 4.7 in [MMW05]). If we don't assume $\mathbb{Q}_{\geq0}A=\mathbb{Q}_{\geq0}A_\tau$ but only $\mathrm{rank}(A_\tau)=\mathrm{rank}(A)=d$ then S_A is a weakly toric R_τ-module (see [SW09]).

Choose a collection of coset representatives $\Omega\subseteq\mathbb{N}A$ for $\mathbb{Z}A/\mathbb{Z}A_\tau$. Then, with $S_\tau(\lambda)=t^\lambda S_\tau$, there is a short exact sequence

$$0\to\bigoplus_{\lambda\in\Omega}S_\tau(\lambda)\hookrightarrow S_A\to Q\to0,\tag{2.5}$$

where no shifted copy of $\mathbb{N}\tau$ is contained in $\mathbb{N}A\smallsetminus\bigcup_{\lambda\in\Omega}(\lambda+\mathbb{N}\tau)$. In particular, $\dim(Q)<d$.

Assume from now on that condition (i) is in force. As S_A is then a finite integral extension of S_τ, S_τ-modules are toric over A precisely when they are toric over τ. We consider the long exact sequence of Euler–Koszul homology over D_τ associated with the sequence (2.5).

By [MMW05, Proposition 5.3] vanishing of $\mathcal{H}_i(E_\tau - \beta, Q)$ for all $i \geq 0$ is equivalent to $-\beta \notin \operatorname{qdeg}(Q)$, where $\operatorname{qdeg}(Q)$ is the Zariski closure in $\mathbb{C}^d$ of the set of the $\mathbb{Z}^d$-degrees of Q. As $\dim(Q) < d$, generic β will be outside $-\operatorname{qdeg}(Q)$. Hence for generic β we have

$$\begin{aligned}\mathcal{H}_i(E_\tau - \beta, S_A) &\simeq \mathcal{H}_i(E_\tau - \beta, \bigoplus_{\lambda\in\Omega} S_\tau(\lambda))\\ &= \bigoplus_{\lambda\in\Omega} \mathcal{H}_i(E_\tau - \beta, S_\tau(\lambda))\\ &\simeq \bigoplus_{\lambda\in\Omega} \mathcal{H}_i(E_\tau - \beta + \lambda, S_\tau)(\lambda)\end{aligned}$$

for all $i \geq 0$, and $\mathcal{H}_0(E_\tau - \beta + \lambda, S_\tau) = M_{A_\tau}(\beta - \lambda)$. Furthermore, for generic β, $\beta - \lambda$ is not rank-jumping for A_τ and so $\mathcal{H}_k(E_\tau - \beta + \lambda, S_\tau) = 0$ for all $k > 0$ and for all $\lambda \in \Omega$; this implies that $\mathbb{L}^{-k} i_\tau^* M_A(\beta) = 0$ for all $k > 0$.

In case (ii) is in force, the argument is similar, using weakly toric modules, which arise since S_A is not a finite S_τ-module in this case. The main vanishing tool is then [SW09, Thm. 5.4] instead of [MMW05, Prop. 5.3]. This concludes the proof of Theorem 2.2. ☐

Remark 2.7. If $\mathscr{M}$ is a holonomic $\mathscr{D}$-module and Y_τ is non-characteristic for $\mathscr{M}$ then $\mathbb{L}^{-k} i_\tau^* \mathscr{M} = 0$ for all $k > 0$ and the holonomic rank of $\mathscr{M}$ coincides with the holonomic rank of $i_\tau^* \mathscr{M}$ (see for example [MT04]). In our case, Y_τ is non-characteristic for $\mathscr{M}_A(\beta)$ if and only if τ contains all the nonzero vertices of the convex hull Δ_A of $\{0, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_n\} \subseteq \mathbb{R}^d$.

Let us show that formula (2.1) may fail if β is not sufficiently generic.

Example 2.8. (1) Let $A = (\boldsymbol{a}_0\ \boldsymbol{a}_1\ \boldsymbol{a}_3\ \boldsymbol{a}_4)$ with $\boldsymbol{a}_i = \binom{1}{i}$ for $i = 0, 1, 3, 4$. In this case, by [ST98].

$$\operatorname{rank} M_A(\beta) = \begin{cases} \operatorname{vol}_{\mathbb{Z}A}(\Delta_A) = 4 & \text{if} \quad \beta \in \mathbb{C}^2 \smallsetminus \binom{1}{2}, \\ 5 & \text{if} \quad \beta = \binom{1}{2} \end{cases}.$$

Consider $\tau = \{0, 1, 4\}$ and $Y_\tau = \{x_3 = 0\}$, so $[\mathbb{Z}A : \mathbb{Z}\tau] = 1$ and $\mathbb{Q}_+ A = \mathbb{Q}_+ \tau$. According to Remark 2.7 the holonomic rank of $i_\tau^* \mathscr{M}_A(\beta)$ is 5 for $\beta = \binom{1}{2}$.

On the other hand, the toric ideal I_τ associated with A_τ is principal and thus Cohen–Macaulay. It follows that the holonomic rank of $\mathscr{M}_{A_\tau}(\beta)$ is $\operatorname{vol}_{\mathbb{Z}\tau}(\Delta_\tau) = 4$ for all $\beta \in \mathbb{C}^2$ (see [MMW05, Corollary 9.2]). This implies that for $\beta = \binom{1}{2}$, $i_\tau^* \mathscr{M}_A(\beta)$ cannot be isomorphic to $\mathscr{M}_{A_\tau}(\beta')$ for any $\beta' \in \mathbb{C}^2$.

(2) Considering $A = (\boldsymbol{a}_0\ \boldsymbol{a}_1\ \boldsymbol{a}_2\ \boldsymbol{a}_3\ \boldsymbol{a}_4)$ with notation as above, and restricting to $x_2 = 0$, one has $\mathbb{Q}_+ A = \mathbb{Q}_+ \tau$ and $\mathbb{Z}A = \mathbb{Z}\tau$, but the restriction of $\mathscr{M}_A(\binom{1}{2})$ is not $\mathscr{M}_\tau(\binom{1}{2})$ since once again the ranks disagree, this time the original GKZ-system having smaller rank (the rational quartic is arithmetically Cohen–Macaulay). However, by [Sai01, Thm. 6.3], $\mathscr{M}_A(\beta) \simeq \mathscr{M}(\binom{1}{2})$ for $\beta \in \mathbb{N}A$ and restricting $\mathscr{M}(\beta)$ to τ for such β leads to $\mathscr{M}_\tau(\beta)$ unless $\beta = \binom{1}{2}$.

Remark 2.9. We do not know a pair (A, τ) for which the conclusion of our theorem fails on a set that is not a finite subspace arrangement. Indeed, while it seems clear that β not rank-jumping for both A and τ is relevant, we believe that the question whether $\beta \in -\operatorname{qdeg}(Q)$ is a red herring. The situation appears alike to the duality question discussed in [Wal07]

References

[CJT03] Francisco Jesús Castro-Jiménez and Nobuki Takayama, *Singularities of the hypergeometric system associated with a monomial curve*, Trans. Amer. Math. Soc. **355** (2003), no. 9, 3761–3775 (electronic). MR 1990172 (2004j:32009)

[FFCJ11] María Fernández-Fernández and Francisco Jesús Castro-Jiménez, *Gevrey solutions of the irregular hypergeometric system associated with an affine monomial curve*, Trans. Amer. Math. Soc. **363** (2011), no. 2, 923–948.

[GGZ87] I. M. Gel′fand, M. I. Graev, and A. V. Zelevinskiĭ, *Holonomic systems of equations and series of hypergeometric type*, Dokl. Akad. Nauk SSSR **295** (1987), no. 1, 14–19. MR 902936 (88j:58118)

[GZK89] I. M. Gel′fand, A. V. Zelevinskiĭ, and M. M. Kapranov, *Hypergeometric functions and toric varieties*, Funktsional. Anal. i Prilozhen. **23** (1989), no. 2, 12–26. MR 1011353 (90m:22025)

[LM02] Yves Laurent and Zoghman Mebkhout, *Image inverse d'un $\mathcal{D}$-module et polygone de Newton*, Compositio Math. **131** (2002), no. 1, 97–119. MR 1895922 (2003g:32021)

[MMW05] Laura Felicia Matusevich, Ezra Miller, and Uli Walther, *Homological methods for hypergeometric families*, J. Amer. Math. Soc. **18** (2005), no. 4, 919–941 (electronic). MR 2163866 (2007d:13027)

[MT04] Philippe Maisonobe and Tristan Torrelli, *Image inverse en théorie des $\mathcal{D}$-modules*, Éléments de la théorie des systèmes différentiels géométriques, Sémin. Congr., vol. 8, Soc. Math. France, Paris, 2004, pp. 1–57. MR 2077644 (2006a:32009)

[Oak97] Toshinori Oaku, *Algorithms for b-functions, restrictions, and algebraic local cohomology groups of D-modules*, Adv. in Appl. Math. **19** (1997), no. 1, 61–105. MR 1453406 (98d:14031)

[OT01] Toshinori Oaku and Nobuki Takayama, *Algorithms for D-modules—restriction, tensor product, localization, and local cohomology groups*, J. Pure Appl. Algebra **156** (2001), no. 2-3, 267–308. MR 1808827 (2002i:13036)

[Sai01] Mutsumi Saito, *Isomorphism classes of A-hypergeometric systems*, Compositio Math. **128** (2001), no. 3, 323–338. MR 1858340 (2003f:33019)

[ST98] Bernd Sturmfels and Nobuki Takayama, *Gröbner bases and hypergeometric functions*, Gröbner bases and applications (Linz, 1998), London Math. Soc. Lecture Note Ser., vol. 251, Cambridge Univ. Press, Cambridge, 1998, pp. 246–258. MR 1708882 (2001c:33026)

[SW09] Mathias Schulze and Uli Walther, *Hypergeometric D-modules and twisted Gauß-Manin systems*, J. Algebra **322** (2009), no. 9, 3392–3409. MR 2567427

[Wal07] Uli Walther, *Duality and monodromy reducibility of A-hypergeometric systems*, Math. Ann. **338** (2007), no. 1, 55–74. MR 2295504 (2008e:32043)

Universidad de Sevilla. Departamento de Álgebra. Apdo. 1160. E–41080–Sevilla (Spain)

E-mail address: `mcferfer@algebra.us.es`

Purdue University, Department of Mathematics, 150 N. University St., West Lafayette, IN 47907, USA

E-mail address: `walther@math.purdue.edu`