ON IRREGULAR BINOMIAL D–MODULES

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ABSTRACT. We prove that a holonomic binomial D–module $M_A(I,\beta)$ is regular if and only if certain associated primes of I determined by the parameter vector $\beta \in \mathbb{C}^d$ are homogeneous. We further describe the slopes of $M_A(I,\beta)$ along a coordinate subspace in terms of the known slopes of some related hypergeometric D–modules that also depend on β. When the parameter β is generic, we also compute the dimension of the generic stalk of the irregularity of $M_A(I,\beta)$ along a coordinate hyperplane and provide some remarks about the construction of its Gevrey solutions.

1. INTRODUCTION

Binomial D-modules have been introduced by A. Dickenstein, L.F. Matusevich and E. Miller in [DMM10]. These objects generalize both GKZ hypergeometric D-modules [GGZ87, GZK89] and (binomial) Horn systems, as treated in [DMM10] and [Sai02].

Here D stands for the complex Weyl algebra of order n, where $n \geq 0$ is an integer. Elements in D are linear partial differential operators; such an operator P can be written as a finite sum

$$P = \sum_{\alpha,\gamma} p_{\alpha\gamma} x^\alpha \partial^\gamma$$

where $p_{\alpha\gamma} \in \mathbb{C}$, $\alpha = (\alpha_1,\ldots,\alpha_n), \gamma = (\gamma_1,\ldots,\gamma_n) \in \mathbb{N}^n$ and $x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, $\partial^\gamma = \partial_1^{\gamma_1}\cdots\partial_n^{\gamma_n}$. The partial derivative $\frac{\partial}{\partial x_i}$ is just denoted by ∂_i.

Our input is a pair (A,β) where β is a vector in $\mathbb{C}^d$ and $A = (a_{ij}) \in \mathbb{Z}^{d\times n}$ is a matrix whose columns $a_1,\ldots,a_n$ span the $\mathbb{Z}$-module $\mathbb{Z}^d$. We also assume that all $a_i \neq 0$ and that the cone generated by the columns in $\mathbb{R}^n$ contains no lines (one says in this case that this cone is *pointed*).

The polynomial ring $\mathbb{C}[\partial] := \mathbb{C}[\partial_1,\ldots,\partial_n]$ is a subring of the Weyl algebra D. The matrix A induces a $\mathbb{Z}^d$-grading on $\mathbb{C}[\partial]$ (also called the A-grading) by defining $\deg(\partial_i) = -a_i$.

A binomial in $\mathbb{C}[\partial]$ is a polynomial with at most two monomial terms. An ideal I in $\mathbb{C}[\partial]$ is said to be binomial is it is generated by binomials. We also say that the ideal I is an A-graded ideal if it is generated by A-homogenous elements (equivalently if for every polynomial in I all its A-graded components are also in I).

The matrix A also induces a $\mathbb{Z}^d$-grading on the Weyl algebra D (also called the A-grading) by defining $\deg(\partial_i) = -a_i$ and $\deg(x_i) = a_i$.

To the matrix A one associates the toric ideal $I_A \subset \mathbb{C}[\partial]$ generated by the family of binomials $\partial^u - \partial^v$ where $u,v \in \mathbb{N}^n$ and $Au = Av$. The ideal I_A is a prime A-graded ideal.

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Recall that to the pair (A, β) one can associate the GKZ hypergeometric ideal

$$H_A(\beta) = DI_A + D(E_1 - \beta_1, \ldots, E_d - \beta_d)$$

where $E_i = \sum_{j=1}^n a_{ij} x_j \partial_j$ is the i^{th} Euler operator associated with A. The corresponding GKZ hypergeometric D–module is nothing but the quotient (left) D–module $M_A(\beta) := \frac{D}{H_A(\beta)}$, [GGZ87], [GZK89].

Following [DMM10], for any A–graded binomial ideal $I \subset \mathbb{C}[\partial]$ we denote by $H_A(I, \beta)$ the A-graded left ideal in D defined by

$$H_A(I, \beta) = DI + D(E_1 - \beta_1, \ldots, E_d - \beta_d).$$

The binomial D–module associated with the triple (A, β, I) is, by definition, the quotient $M_A(I, \beta) := \frac{D}{H_A(I,\beta)}$. Notice that the ideal $H_A(I_A, \beta)$ is nothing but the GKZ hypergeometric ideal $H_A(\beta)$.

In [DMM10] the authors have answered essential questions about binomial D–modules. The main treated questions are related to the holonomicity of the systems and to the dimension of their holomorphic solution space around a non singular point. In particular, in [DMM10, Theorem 6.3] they prove that the holonomicity of $M_A(I, \beta)$ is equivalent to regular holonomicity when I is standard $\mathbb{Z}$-graded (i.e., the row-span of A contains the vector $(1, \ldots, 1)$). However, it turns out that the final sentence in [DMM10, Theorem 6.3], stating that the regular holonomicity of $M_A(I, \beta)$ for a given parameter β implies standard homogeneity of the ideal I, is true for binomial Horn systems but it is not for general binomial D–modules. This is shown by Examples 3.10 and 3.11.

These two Examples are different in nature. More precisely, the system $M_A(I, \beta)$ considered in Example 3.10 is regular holonomic for parameters β outside a certain line in the affine complex plane and irregular otherwise, while the system considered in Example 3.11 is regular holonomic for all parameters despite the fact that the binomial ideal I is not homogeneous with respect to the standard $\mathbb{Z}$–grading. This is a surprising phenomenon since it is not allowed neither for GKZ hypergeometric systems nor for binomial Horn systems.

We further provide, in Theorem 3.7, a characterization of the regular holonomicity of a system $M_A(I, \beta)$ that improves the above mentioned result of [DMM10, Th. 6.3].

A central question in the study of the irregularity of a holonomic D-module M is the computation of its slopes along smooth hypersurfaces (see [Meb90] and [LM99]). On the other hand, the Gevrey solutions of M along smooth hypersurfaces are closely related with the irregularity and the slopes of M. More precisely, the classes of these Gevrey series solutions of M modulo convergent series define the 0-th cohomology group of the irregularity of M [Meb90, Définition 6.3.1].

In Section 4 we describe the L–characteristic variety and the slopes of $M_A(I, \beta)$ along coordinate subspaces in terms of the same objects of the binomial D–modules associated with some of the *toral* primes of the ideal I determined by β (see Theorem 4.3). The binomial D–module associated with a toral prime is essentially a GKZ hypergeometric system and the L–characteristic variety and the slopes along coordinate subspaces of such a system are completely described in [SW08] in a combinatorial way (see also [CT03] and [Har03, Har04] for the cases $d = 1$ and $n = d + 1$).

Gevrey solutions of hypergeometric systems along coordinate subspaces are described in [Fer10] (see also [FC11], [FC08]). In Section 5 we compute the dimension of the generic stalk of the irregularity of binomial D-modules when the parameter is generic (see Theorem 5.1). We finally give a procedure to compute Gevrey solutions of $M_A(I,\beta)$ by using known results in the hypergeometric case ([GZK89], [SST00] and [Fer10]).

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2. Preliminaries on Euler–Koszul homology, binomial primary decomposition and toral and Andean modules

We review here some definitions, notations and results of [ES96], [MMW05], [DMM10] and [DMM$_2$10] that will be used in the sequel.

We will denote $R = \mathbb{C}[\partial]$. Recall that the A–grading on the ring R is defined by $\deg(\partial_j) = -a_j$ where a_j is the j^{th}-column of A. This A–grading on R can be extended to the ring D by setting $\deg(x_j) = a_j$.

Definition 2.1. [DMM10, Definition 2.4] Let $V = \oplus_{\alpha\in\mathbb{Z}^d} V_\alpha$ be an A-graded R-module. The set of true degrees of V is

$$\operatorname{tdeg}(V) = \{\alpha \in \mathbb{Z}^d : \ V_\alpha \neq 0\}$$

The set of quasidegrees of V is the Zariski closure in $\mathbb{C}^d$ of $\operatorname{tdeg}(V)$.

Euler-Koszul complex $\mathcal{K}_\bullet(E-\beta; V)$ associated with an A-graded R–module V.

For any A–graded left D–module $N = \oplus_{\alpha\in\mathbb{Z}^d} N_\alpha$ we denote $\deg_i(y) = \alpha_i$ if $y \in N_\alpha$.

The map $E_i - \beta_i : N_\alpha \to N_\alpha$ defined by $(E_i - \beta_i)(y) = (E_i - \beta_i - \alpha_i)y$ can be extended (by $\mathbb{C}$–linearity) to a morphism of left D–modules $E_i - \beta_i : N \to N$. We denote by $E - \beta$ the sequence of commuting endomorphisms $E_1 - \beta_1, \ldots, E_d - \beta_d$. This allows us to consider the Koszul complex $K_\bullet(E-\beta, N)$ which is concentrated in homological degrees d to 0.

Definition 2.2. [MMW05, Definition 4.2] For any $\beta \in \mathbb{C}^d$ and any A-graded R–module V, the Euler-Koszul complex $\mathcal{K}_\bullet(E-\beta, V)$ is the Koszul complex $K_\bullet(E-\beta, D\otimes_R V)$. The i^{th} Euler-Kozsul homology of V, denoted by $\mathcal{H}_i(E-\beta, V)$, is the homology $H_i(\mathcal{K}_\bullet(E-\beta, V))$.

Remark 2.3. Recall that we have the A–graded isomorphism $\mathcal{H}_i(E-\beta, V)(\alpha) \simeq \mathcal{H}_i(E-\beta+\alpha, V)(\alpha)$ for all $\alpha \in \mathbb{Z}^d$ [MMW05]. Here $V(\alpha)$ is nothing but V with the shifted A–grading $V(\alpha)_\gamma = V_{\alpha+\gamma}$ for all $\gamma \in \mathbb{Z}^d$.

Binomial primary decomposition for binomial ideals.

We recall from [ES96] that for any sublattice $\Lambda \subset \mathbb{Z}^n$ and any partial character $\rho : \Lambda \to \mathbb{C}^*$, the corresponding associated binomial ideal is

$$I_\rho = \langle \partial^{u_+} - \rho(u)\partial^{u_-} \mid u = u_+ - u_- \in \Lambda\rangle$$

where u_+ and u_- are in $\mathbb{N}^n$ and they have disjoint supports. The ideal I_ρ is prime if and only if Λ is a saturated sublattice of $\mathbb{Z}^n$ (*i.e.* $\Lambda = \mathbb{Q}\Lambda \cap \mathbb{Z}^n$). We know from [ES96, Corollary 2.6] that any binomial prime ideal in R has the form $I_{\rho,J} := I_\rho + \mathfrak{m}_J$ (where $\mathfrak{m}_J = \langle \partial_j \mid j \notin J\rangle$) for some partial character ρ whose domain is a saturated sublattice of $\mathbb{Z}^J$ and some $J \subset \{1,\ldots,n\}$.

For any $J \subset \{1,\ldots,n\}$ we denote by ∂_J the monomial $\prod_{j\in J}\partial_j$.

Theorem 2.4. [DMM$_2$10, Theorem 3.2] *Fix a binomial ideal I in R. Each associated binomial prime $I_{\rho,J}$ has an explicitly defined monomial ideal $U_{\rho,J}$ such that*

$$I = \bigcap_{I_{\rho,J}\in Ass(I)} \mathcal{C}_{\rho,J}$$

for $\mathcal{C}_{\rho,J} = ((I + I_\rho) : \partial_J^\infty) + U_{\rho,J}$*, is a primary decomposition of* I *as an intersection of* A*–graded primary binomial ideals.*

Toral and Andean modules.
In [DMM$_2$10, Definition 4.3] a finitely generated A-graded R–module $V = \oplus V_\alpha$ is said to be *toral* if its Hilbert function H_V (defined by $H_V(\alpha) = \dim_{\mathbb{C}} V_\alpha$ for $\alpha \in \mathbb{Z}^d$) is bounded above.
With the notations above, a R–module of type $R/I_{\rho,J}$ is toral if and only if its Krull dimension equals the rank of the matrix A_J (see [DMM10, Lemma 3.4]). Here A_J is the submatrix of A whose columns are indexed by J. In this case the module $R/\mathcal{C}_{\rho,J}$ is toral and we say that the ideal $I_{\rho,J}$ is a toral prime and $\mathcal{C}_{\rho,J}$ is a toral primary component.
If $\dim(R/I_{\rho,J}) \neq \operatorname{rank}(A_J)$ then the module $R/\mathcal{C}_{\rho,J}$ is said to be *Andean*, the ideal $I_{\rho,J}$ is an *Andean* prime and $\mathcal{C}_{\rho,J}$ is an *Andean* primary component.
An A–graded R–module V is said to be *natively toral* if there exist a binomial toral prime ideal $I_{\rho,J}$ and an element $\alpha \in \mathbb{Z}^d$ such that $V(\alpha)$ is isomorphic to $R/I_{\rho,J}$ as A–graded modules (see [DMM10, Definition 4.1]).

Proposition 2.5. [DMM10, Proposition 4.2] *An* A*–graded* R*–module* V *is toral if and only if it has a filtration*

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{\ell-1} \subset V_\ell = V$$

whose successive quotients V_k/V_{k-1} *are all natively toral.*

Such a filtration on V is called a *toral filtration*.
Following [DMM10, Definition 5.1] an A-graded R-module V is said to be *natively Andean* if there is an $\alpha \in \mathbb{Z}^d$ and an Andean quotient ring $R/I_{\rho,J}$ over which $V(\alpha)$ is torsion-free of rank 1 and admits a $\mathbb{Z}^J/\Lambda$-grading that refines the A-grading via $\mathbb{Z}^J/\Lambda \to \mathbb{Z}^d = \mathbb{Z}A$, where ρ is defined on $\Lambda \subset \mathbb{Z}^J$. Moreover, if V has a finite filtration

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{\ell-1} \subset V_\ell = V$$

whose successive quotients V_k/V_{k-1} are all natively Andean, then V is Andean (see [DMM10, Section 5]).
In [DMM$_2$10, Example 4.6] it is proven that the quotient $R/C_{\rho,J}$ is Andean for any Andean primary component $C_{\rho,J}$ of any A-graded binomial ideal.
We finish this section with the definition and a result about the so-called *Andean arrangement* associated with an A-graded binomial ideal I in R. Let us fix an irredundant primary decomposition

$$I = \bigcap_{I_{\rho,J} \in Ass(I)} \mathcal{C}_{\rho,J}$$

as in Theorem 2.4.

Definition 2.6. [DMM10, Definition 6.1] The Andean arrangement $\mathcal{Z}_{\text{Andean}}(I)$ is the union of the quasidegree sets $\operatorname{qdeg}(R/C_{\rho,J})$ for the Andean primary components $C_{\rho,J}$ of I.

From [DMM10, Lemma 6.2] the Andean arrangement $\mathcal{Z}_{\text{Andean}}(I)$ is a union of finitely many integer translates of the subspaces $\mathbb{C}A_J \subset \mathbb{C}^n$ for which there is an Andean associated prime $I_{\rho,J}$.
From [DMM10, Theorem 6.3] we have that the binomial D–module $M_A(I, \beta)$ is holonomic if and only if $-\beta \notin \mathcal{Z}_{\text{Andean}}(I)$.

3. CHARACTERIZING REGULAR HOLONOMIC BINOMIAL D–MODULES

Let I be an A–graded binomial ideal and fix a binomial primary decomposition $I = \cap_{\rho,J} C_{\rho,J}$ where $C_{\rho,J}$ is a $I_{\rho,J}$–primary binomial ideal.
Let us consider the ideal

$$I_\beta := \bigcap_{-\beta\in \mathrm{qdeg}(R/C_{\rho,J})} C_{\rho,J}$$

i.e., the intersection of all the primary components $C_{\rho,J}$ of I such that $-\beta$ lies in the quasidegrees set of the module $R/C_{\rho,J}$.

Remark 3.1. Notice that if $-\beta \notin \mathcal{Z}_{\mathrm{Andean}}(I)$ then R/I_β is contained in the toral direct sum

$$\bigoplus_{-\beta\in \mathrm{qdeg}(R/C_{\rho,J})} R/C_{\rho,J}$$

and so it is a toral module.

The following result generalizes [DMM10, Proposition 6.4].

Proposition 3.2. *If $-\beta \notin \mathcal{Z}_{\mathrm{Andean}}(I)$ then the natural surjection $R/I \twoheadrightarrow R/I_\beta$ induces a isomorphism in Euler–Koszul homology*

$$\mathcal{H}_i(E-\beta, R/I) \simeq \mathcal{H}_i(E-\beta, R/I_\beta)$$

for all i. In particular, $M_A(I,\beta) \simeq M_A(I_\beta, \beta)$.

Proof. By [DMM10, Proposition 6.4] we have that

$$\mathcal{H}_i(E-\beta, R/I) \simeq \mathcal{H}_i(E-\beta, R/I_{\mathrm{toral}})$$

for all i, where I_{toral} denotes the intersection of all the toral primary components of I. Thus, we can assume without loss of generality that all the primary components of I are toral. The rest of the proof is now analogous to the proof of [DMM10, Proposition 6.4] if we substitute the ideals I_{toral} and I_{Andean} there by the ideals I_β and $\overline{I_\beta}$ respectively, where

$$\overline{I_\beta} = \bigcap_{-\beta\notin \mathrm{qdeg}(R/C_{\rho,J})} C_{\rho,J},$$

and the Andean direct sum $\bigoplus_{I_{\rho,J}\mathrm{Andean}} R/C_{\rho,J}$ there by the toral direct sum

$$\bigoplus_{-\beta\notin \mathrm{qdeg}(R/C_{\rho,J})} R/C_{\rho,J}$$

Finally, we can use Lemma 4.3 and Theorem 4.5 in [DMM10] in a similar way as [DMM10, Lemma 5.4] is used in the proof of Proposition 6.4 of [DMM10]. □

The following Lemma gives a description of the quasidegrees set of a toral module of type $R/C_{\rho,J}$. E. Miller has pointed out that this result follows from Proposition 2.13 and Theorem 2.15 in [DMM$_2$10]. We will include here a slightly different proof of this Lemma.

Lemma 3.3. *For any $I_{\rho,J}$–primary toral ideal $C_{\rho,J}$ the quasidegrees set of $M = R/C_{\rho,J}$ equals the union of at most $\mu_{\rho,J}$ $\mathbb{Z}^d$–graded translates of $\mathbb{C}A_J$, where $\mu_{\rho,J}$ is the multiplicity of $I_{\rho,J}$ in $C_{\rho,J}$. More precisely, for any toral filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M$ we have that* *the quasidegrees set of M is the union of the quasidegrees set of all the successive quotients M_i/M_{i-1} that are isomorphic to $\mathbb{Z}^d$–graded translates of $R/I_{\rho,J}$.*

Proof. Since M is toral we have by [DMM10, Lemma 4.7] that $\dim(\mathrm{qdeg}(M)) = \dim M = \mathrm{rank}\, A_J$. Since $C_{\rho,J}$ is primary, any zero-divisor of M is nilpotent. For all $j \in J$ we have that $\partial_j^m \notin C_{\rho,J} \subseteq I_\rho + \mathfrak{m}_J$ and so ∂_j is not a zero-divisor in M for all $j \in J$. Thus, the true degrees set of M verifies $\mathrm{tdeg}(M) = \mathrm{tdeg}(M) - \mathbb{N}A_J$. This and the fact that $\dim(\mathrm{qdeg}(M)) = \mathrm{rank}\, A_J$ imply that there exists $\alpha_1, \ldots, \alpha_r \in \mathbb{Z}^d$ such that $\mathrm{tdeg}(M) = \cup_{i=1}^r (\alpha_i - \mathbb{N}A_J)$ and

$$\mathrm{qdeg}(M) = \bigcup_{i=1}^{r} (\alpha_i + \mathbb{C}A_J) \tag{3.1}$$

Consider now a toral filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M$. We know that there are exactly $\mu_{\rho,J}$ different values of i such that $M_i/M_{i-1} \simeq R/I_{\rho,J}(\gamma_i)$ for some $\gamma_i \in \mathbb{Z}^d$. For the other successive quotients $M_l/M_{l-1} \simeq R/I_{\rho_l,J_l}(\gamma_l)$ we have that I_{ρ_l,J_l} is a toral prime which properly contains $I_{\rho,J}$. In particular, we have that $\mathrm{rank}\, A_{J_l} = \dim R/I_{\rho_l,J_l} < \dim R/I_{\rho,J} = \mathrm{rank}\, A_J$. Since $\mathrm{qdeg}(R/I_{\rho_l,J_l}) = \mathbb{C}A_{J_l}$ has dimension $\mathrm{rank}\, A_{J_l} < \mathrm{rank}\, A_J$ and $\mathrm{qdeg}(M) = \bigcup_i \mathrm{qdeg}(M_i/M_{i-1})$ we have by (3.1) that the quasidegrees set of any M_i/M_{i-1} is contained in the quasidegrees set of some $M_j/M_{j-1} \simeq R/I_{\rho,J}(\gamma_j)$. In particular $r \leq \mu_{\rho,J}$ and each affine subspace $(\alpha_i + \mathbb{C}A_J)$ in (3.1) is the quasidegrees set of some $M_j/M_{j-1} \simeq R/I_{\rho,J}(\gamma_j)$. □

Remark 3.4. Notice that $H_A(I_{\rho,J}, \beta) = DH_{A_J}(I_\rho, \beta) + D(\partial_j : \; j \notin J)$. In addition, if $I_{\rho,J}$ is toral then the D_J–module $M_{A_J}(I_\rho, \beta)$ is isomorphic to the hypergeometric system $M_{A_J}(\beta)$ via an A–graded isomorphism of D_J–modules induced by rescaling the variables x_j, $j \in J$, using the character ρ. Thus we can apply most of the well-knows results for hypergeometric systems to $M_A(I_{\rho,J}, \beta)$ (with $I_{\rho,J}$ a toral prime) in an appropriated form.

Lemma 3.5. *If $I_{\rho,J}$ is toral and $-\beta \in \mathrm{qdeg}(R/I_{\rho,J})$ the following conditions are equivalent:*

i) *$\mathcal{H}_i(E-\beta, R/I_{\rho,J})$ is regular holonomic for all i.*
ii) *$\mathcal{H}_0(E-\beta, R/I_{\rho,J})$ is regular holonomic.*
iii) *$I_{\rho,J}$ is homogeneous (equivalently A_J is homogeneous).*

Proof. $i) \Rightarrow ii)$ is obvious, $ii) \Rightarrow iii)$ follows straightforward from [SW08, Corollary 3.16] and $iii) \Rightarrow i)$ is a particular case of the last statement in [DMM10, Theorem 4.5] and it also follows from [Hot98, Ch. II, 6.2, Thm.]. □

Remark 3.6. Recall from [DMM10, Theorem 4.5] that for any toral module V we have that $-\beta \notin \mathrm{qdeg} V$ if and only if $\mathcal{H}_0(E-\beta, V) = 0$ if and only if $\mathcal{H}_i(E-\beta, V) = 0$ for all i. In particular, since the D–module 0 is regular holonomic it follows that conditions i) and ii) in Lemma 3.5 are also equivalent without the condition $-\beta \in \mathrm{qdeg}(R/I_{\rho,J})$.

Theorem 3.7. *Let $I \subseteq R$ be an A-graded binomial ideal such that $M_A(I, \beta)$ is holonomic (equivalently, $-\beta \notin \mathcal{Z}_{\mathrm{Andean}}(I)$). The following conditions are equivalent:*

i) *$\mathcal{H}_i(E-\beta, R/I)$ is regular holonomic for all i.*
ii) *$M_A(I, \beta)$ is regular holonomic.*
iii) *All the associated toral primes $I_{\rho,J}$ of I such that $-\beta \in \mathrm{qdeg}(R/C_{\rho,J})$ are homogeneous.*

Proof. The implication $i) \Rightarrow ii)$ is obvious. Let us prove $ii) \Rightarrow iii)$. For any toral primary component $C_{\rho,J}$ of I we have $I \subseteq C_{\rho,J}$ and so there is a natural epimorphism $M_A(I,\beta) \twoheadrightarrow M_A(C_{\rho,J},\beta)$. Since $M_A(I,\beta)$ is regular holonomic then $M_A(C_{\rho,J},\beta)$ is also regular holonomic. Take a toral filtration of $M = R/C_{\rho,J}$, $0 \subseteq M_1 \subseteq \cdots \subseteq M_r = M$. We claim that

$$\mathcal{H}_j(E-\beta, M_i/M_{i-1}) \text{ and } \mathcal{H}_0(E-\beta, M_{i-1}) \text{ are regular holonomic} \tag{3.2}$$

for all i,j.

Let us prove (3.2) by decreasing induction on i. For $i=r$, we have a surjection from the regular holonomic D–module $\mathcal{H}_0(E-\beta, M_r) = M_A(C_{\rho,J},\beta)$ to $\mathcal{H}_0(E-\beta, M_r/M_{r-1})$ and so it is regular holonomic too. By Remark 2.3, Lemma 3.5 and Remark 3.6 we have that the D-module $\mathcal{H}_j(E-\beta, M_r/M_{r-1})$ is regular holonomic for all j. Since

$$\mathcal{H}_1(E-\beta, M_r/M_{r-1}) \longrightarrow \mathcal{H}_0(E-\beta, M_{r-1}) \longrightarrow \mathcal{H}_0(E-\beta, M_r)$$

is exact we have that $\mathcal{H}_0(E-\beta, M_{r-1})$ is regular holonomic.

Assume that (3.2) holds for some $i = k+1 \leq r$ and for all j. We consider the exact sequence

$$0 \longrightarrow M_{k-1} \longrightarrow M_k \longrightarrow M_k/M_{k-1} \longrightarrow 0$$

and the following part of the long exact sequence of Euler-Koszul homology

$$\cdots \mathcal{H}_1(E-\beta, M_k/M_{k-1}) \to \mathcal{H}_0(E-\beta, M_{k-1}) \to \mathcal{H}_0(E-\beta, M_k) \twoheadrightarrow \mathcal{H}_0(E-\beta, M_k/M_{k-1}). \tag{3.3}$$

By induction hypothesis $\mathcal{H}_0(E-\beta, M_k)$ is regular holonomic. This implies that $\mathcal{H}_0(E-\beta, M_k/M_{k-1})$ is regular holonomic by (3.3). Applying Remark 2.3, Lemma 3.5 and Remark 3.6 we have that $\mathcal{H}_j(E-\beta, M_k/M_{k-1})$ is regular holonomic for all j. Thus, by (3.3) we have that $\mathcal{H}_0(E-\beta, M_{k-1})$ is regular holonomic too and we have finished the induction proof of (3.2).

Assume that $-\beta \in \operatorname{qdeg}(R/C_{\rho,J})$. By Lemma 3.3 there exists i such that $-\beta$ lies in the quasidegrees set of $M_i/M_{i-1} \simeq R/I_{\rho,J}(\gamma_i)$ and we also have by (3.2) that

$$\mathcal{H}_0(E-\beta, M_i/M_{i-1}) \simeq \mathcal{H}_0(E-\beta+\gamma_i, R/I_{\rho,J})(\gamma_i)$$

is a nonzero regular holonomic D-module. Thus, by Lemma 3.5 we have that $I_{\rho,J}$ is homogeneous.

Let us prove $iii) \Rightarrow i)$. By Proposition 3.2 we just need to prove that $M_A(I_\beta,\beta)$ is regular holonomic. We have that all the associated primes of I_β are toral and homogeneous. In particular $M = R/I_\beta$ is a toral module and for any toral filtration of M the successive quotients M_i/M_{i-1} are isomorphic to some $\mathbb{Z}^d$–graded translate of a quotient $R/I_{\rho_i,J_i}$ where I_{ρ_i,J_i} is toral and contains a minimal prime $I_{\rho,J}$ of I_β. Such minimal prime is homogeneous by assumption and so A_J is homogeneous. Since $J_i \subseteq J$ we have that A_{J_i} and I_{ρ_i,J_i} are homogeneous too. Now, we just point out that that the proof of the last statement in [DMM10, Theorem 4.5] still holds for $V = M$ if we don't require A to be homogenous but all the primes occurring in a toral filtration of M to be homogeneous.

□

Remark 3.8. Theorem 3.7 shows in particular that the property of a binomial D-module $M_A(I,\beta)$ of being regular (holonomic) can fail to be constant when $-\beta$ runs outside the Andean arrangement. This phenomenon is forbidden to binomial Horn systems $M_A(I(B),\beta)$ (see [DMM10, Definition 1.5]) since the inclusion $I(B) \subseteq I_A$ induces a surjective morphism

$$\mathcal{H}_0(E-\beta, I(B)) \twoheadrightarrow M_A(\beta)$$

and then regular holonomicity of $\mathcal{H}_0(E-\beta, R/I(B))$ implies regular holonomicity of $M_A(\beta)$, which is equivalent to the standard homogeneity of I_A by [Hot98, SST00, SW08].

Definition 3.9. The non-regular arrangement of I (denoted by $\mathcal{Z}_{\mathrm{non-regular}}(I)$) is the union of the Andean arrangement of I and the union of quasidegrees sets of the quotients of R by primary components $C_{\rho,J}$ of I such that $I_{\rho,J}$ is not homogeneous with respect to the standard grading. So, we have

$$\mathcal{Z}_{\mathrm{non-regular}}(I) = \mathcal{Z}_{\mathrm{Andean}}(I) \cup \left(\bigcup_{I_{\rho,J} \text{ non homogeneous}} \mathrm{qdeg}(R/C_{\rho,J}) \right).$$

Example 3.10. Consider the ideal $I = \langle \partial_1^2\partial_2 - \partial_2^2, \partial_2\partial_3, \partial_2\partial_4, \partial_1^2\partial_3 - \partial_3^2\partial_4, \partial_1^2\partial_4 - \partial_3\partial_4^2 \rangle$. It is A-graded for the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 & 0 \\ 1 & 2 & 0 & 2 \end{pmatrix}$$

but I is not standard $\mathbb{Z}$-graded. We have the prime decomposition $I = I_1 \cap I_2 \cap I_3$ where $I_1 = \langle \partial_2, \partial_3, \partial_4 \rangle$, $I_2 = \langle \partial_1^2 - \partial_2, \partial_3, \partial_4 \rangle$ and $I_3 = \langle \partial_2, \partial_1^2 - \partial_3\partial_4 \rangle$ are toral primes of I. In particular $\mathcal{Z}_{\mathrm{Andean}}(I) = \emptyset$) and by the proof of [DMM10, Proposition 6.6] we have that $\mathcal{Z}_{\mathrm{primary}}(I) = \{0\}$ (see [DMM10, Definition 6.5] for the definition of the primary arrangement $\mathcal{Z}_{\mathrm{primary}}(I)$).
Using [DMM10, Theorem 6.8] we have that $M_A(I,\beta)$ is isomorphic to the direct sum of $M_A(I_j,\beta)$ for $j = 1,2,3$ if $\beta \neq 0$. Moreover, $\mathrm{qdeg}(R/I_j) = \mathbb{C}\binom{1}{1}$ for $j = 1,2$ and $\mathrm{qdeg}(R/I_3) = \mathbb{C}^2$. Thus, for generic parameters (more precisely for $\beta \in \mathbb{C}^2 \setminus \mathbb{C}\binom{1}{1}$) we have that $M_A(I,\beta)$ is isomorphic to $M_A(I_3,\beta)$ that is a regular holonomic by Lemma 3.5.
On the other hand, there is a surjective morphism from $M_A(I,\beta)$ to $M_A(I_2,\beta)$ and if $\beta \in \mathbb{C}\binom{1}{1}$ we have that $M_A(I_2,\beta)$ is an irregular D-module because $s = 2$ is a slope along $x_2 = 0$. Thus we conclude that $M_A(I,\beta)$ is regular holonomic if $\beta \in \mathbb{C}^2 \setminus \mathbb{C}\binom{1}{1}$ and it is an irregular holonomic D-module when $\beta \in \mathbb{C}\binom{1}{1}$. In particular, $\mathcal{Z}_{\mathrm{non-regular}}(I) = \mathbb{C}\binom{1}{1} \subset \mathbb{C}^2$. It can also be checked that the singular locus of $M_A(I,\beta)$ is $\{x_1x_2x_3x_4(x_1^2 - 4x_3x_4) = 0\}$ when $\beta \in \mathbb{C}\binom{1}{1}$ and $\{x_3x_4(x_1^2 - 4x_3x_4) = 0\}$ otherwise.

Example 3.11. The primary binomial ideal $I = \langle \partial_1 - \partial_2, \partial_3^4, \partial_4^3, \partial_3^3 - \partial_4^2 \rangle$ is A–graded with respect to the matrix $A = (1\ 1\ 2\ 3)$. Note that I is not homogeneous with respect to the standard $\mathbb{Z}$-grading. However, its radical ideal $\sqrt{I} = \langle \partial_1 - \partial_2, \partial_3, \partial_4 \rangle$ is homogeneous. Thus, by Theorem 3.7 we have that $M_A(I,\beta)$ is regular holonomic.

4. L–Characteristic variety and slopes of binomial D–modules

Let L be the filtration on D defined by a weight vector $(u,v) \in \mathbb{R}^{2n}$ with $u_i + v_i = c > 0$ for some constant $c > 0$.
This includes in particular the intermediate filtrations $pF + qV$ between the filtration F by the order of the linear differential operators and the Kashiwara-Malgrange filtration V along

a coordinate subspace. The filtrations $pF + qV$ are the ones considered when studying the algebraic slopes of a coherent D–module along a coordinate subspace [LM99].
We will consider the L–characteristic variety $\mathrm{Ch}^L(N)$ of a finitely generated D–module N on $\mathbb{C}^n$ defined as the support of $\mathrm{gr}^L N$ in $T^*\mathbb{C}^n$ (see e.g. [Lau87], [SW08, Definition 3.1]). We recall that in fact for $L = pF + qV$ this is a global algebraic version of Laurent's microcharacteristic variety of type $s = p/q + 1$ [Lau87, §3.2] (see also [SW08, Remark 3.3]).
The L-characteristic variety and the slopes of a hypergeometric D-module $M_A(\beta)$ are controlled by the so-called (A, L)–umbrella [SW08]. Let us recall its definition in the special case when $v_i > 0$ for all i. We denote by Δ_A^L the convex hull of $\{0, a_1^L, \ldots, a_n^L\}$ where $a_j^L = \frac{1}{v_j} a_j$. The (A, L)-umbrella is the set Φ_A^L of faces of Δ_A^L which do not contain 0. The empty face is in Φ_A^L. One identifies $\tau \in \Phi_A^L$ with $\{j | a_j^L \in \tau\}$, or with $\{a_j | a_j^L \in \tau\}$, or with the corresponding submatrix A_τ of A.
By [SW08, Corollary 4.17] the L-characteristic variety of a hypergeometric D–module $M_A(\beta)$ is

$$\mathrm{Ch}^L(M_A(\beta)) = \bigcup_{\tau \in \Phi_A^L} \overline{C_A^\tau} \tag{4.1}$$

where $\overline{C_A^\tau}$ is the Zariski closure in $T^*\mathbb{C}^n$ of the conormal space to the orbit $O_A^\tau \subset T_0^*\mathbb{C}^n = \mathbb{C}^n$ corresponding to the face τ. In particular $\mathrm{Ch}^L(M_A(\beta))$ is independent of β. By definition we have the equality $O_A^\tau := (\mathbb{C}^*)^d \cdot \mathbf{1}_A^\tau$ where $\mathbf{1}_A^\tau \in \mathbb{N}^n$ is defined by $(\mathbf{1}_A^\tau)_j = 1$ if $j \in \tau$ and $(\mathbf{1}_A^\tau)_j = 0$ otherwise. The action of the torus is given with respect to the matrix A. If the filtration given by L equals the F-filtration (i.e. the order filtration) then this description of the F–characteristic variety coincides with a result of [Ado94, Lemmas 3.1 and 3.2].

Proposition 4.1. *If M is a $I_{\rho,J}$–coprimary toral module and $-\beta \in \mathrm{qdeg}(M)$ then the L–characteristic variety of $\mathcal{H}_0(E - \beta, M)$ is the L–characteristic variety of $M_A(I_{\rho,J}, 0)$. In particular, the set of slopes of $\mathcal{H}_0(E - \beta, M)$ along a coordinate subspace in $\mathbb{C}^n$ coincide with the ones of $M_A(I_{\rho,J}, 0)$.*

Proof. Since M is $I_{\rho,J}$–coprimary there exists $m \geq 0$ such that $I_{\rho,J}^m$ annihilates M. Consider a set of A–homogeneous elements $m_1, \ldots, m_k \in M$ generating M as R–module. This leads to a natural A–graded surjection $\bigoplus_{i=1}^k R/I_{\rho,J}^m(-\deg(m_i)) \twoheadrightarrow M$. In particular, there is a surjective morphism of D-modules

$$\bigoplus_{i=1}^k \mathcal{H}_0(E - \beta, R/I_{\rho,J}^m(-\deg(m_i))) \twoheadrightarrow \mathcal{H}_0(E - \beta, M)$$

inducing the inclusion:

$$\mathrm{Ch}^L(\mathcal{H}_0(E - \beta, M)) \subseteq \mathcal{V}(\mathrm{in}_L(I_{\rho,J}^m), Ax\xi) = \mathcal{V}(\mathrm{in}_L(I_\rho), A_J x_J \xi_J, \xi_j : j \notin J).$$

Here (x, ξ) stands for the coordinates in the cotangent space $T^*\mathbb{C}^n$, $x\xi = (x_1\xi_1, \ldots, x_n\xi_n)$ and $\mathcal{V}$ is the zero set in $T^*\mathbb{C}^n$ of the corresponding ideal.
The equality $\mathrm{Ch}^L(M_A(I_{\rho,J}, 0)) = \mathcal{V}(\mathrm{in}_L(I_\rho), A_J x_J \xi_J, \xi_j : j \notin J)$ follows from [SW08, (3.2.2) and Corollary 4.17]. Thus,

$$\mathrm{Ch}^L(\mathcal{H}_0(E - \beta, M)) \subseteq \mathrm{Ch}^L(M_A(I_{\rho,J}, 0)) \tag{4.2}$$

Let us now prove the equality

$$\mathrm{Ch}^L(\mathcal{H}_0(E-\beta,M))=\mathrm{Ch}^L(M_A(I_{\rho,J},0)) \tag{4.3}$$

by induction on the length r of a toral filtration $0=M_0\subsetneq M_1\subsetneq\cdots\subsetneq M_r=M$ of M.
If $r=1$ we have that $M\simeq R/I_{\rho,J}(\gamma)$ for some $\gamma\in\mathbb{Z}^d$ and $-\beta\in\mathrm{qdeg}(M)$ means that $-\beta+\gamma\in\mathrm{qdeg}(R/I_{\rho,J})=\mathbb{C}A_J$. Thus, $\mathcal{H}_0(E-\beta,M)\simeq M_A(I_{\rho,J},\beta-\gamma)$ and we have (4.3) because the L–characteristic variety of $M_A(I_{\rho,J},\beta')$ is independent of $\beta'\in-\mathrm{qdeg}(R/I_{\rho,J})$ by the results in [SW08].
Assume by induction that we have (4.3) for all toral $I_{\rho,J}$–coprimary modules M with a toral filtration of length r such that $-\beta\in\mathrm{qdeg}(M)$.
Let M be a $I_{\rho,J}$–coprimary toral module with toral filtration of length $r+1$, i.e. $0=M_0\subsetneq M_1\subseteq\cdots\subsetneq M_{r+1}=M$. From the exact sequence

$$0\longrightarrow M_r\longrightarrow M\longrightarrow M/M_r\longrightarrow 0$$

we obtain the long exact sequence of Euler–Koszul homology

$$\cdots\longrightarrow\mathcal{H}_1(E-\beta,M/M_r)\longrightarrow\mathcal{H}_0(E-\beta,M_r)\longrightarrow\mathcal{H}_0(E-\beta,M)\longrightarrow\mathcal{H}_0(E-\beta,M/M_r)\longrightarrow 0.$$

Now, we need to distinguish two cases.
Assume first that $-\beta\notin\mathrm{qdeg}(M/M_r)$. Thus, $\mathcal{H}_j(E-\beta,M/M_r)=0$ for all j by [DMM10, Theorem 4.5] and we have that $\mathcal{H}_0(E-\beta,M_r)\simeq\mathcal{H}_0(E-\beta,M)$ so they both have the same L–characteristic variety. Notice that the fact that $-\beta\in\mathrm{qdeg}M\setminus\mathrm{qdeg}(M/M_r)$ along with Lemma 3.3 guarantees that there exists some $i\leq r$ such that $M_i/M_{i-1}\simeq R/I_{\rho,J}(\gamma_i)$. This implies that M_r is also $I_{\rho,J}$–coprimary and we can apply the induction hypothesis.
Assume now that $-\beta\in\mathrm{qdeg}(M/M_r)$. In this case we still have that the L–characteristic variety of $\mathcal{H}_0(E-\beta,M/M_r)$ is contained in the L–characteristic variety of $\mathcal{H}_0(E-\beta,M)$. If $M/M_r\simeq R/I_{\rho,J}(\gamma)$ we have that $\mathrm{Ch}^L(M_A(I_{\rho,J},0))\subseteq\mathrm{Ch}^L(\mathcal{H}_0(E-\beta,M))$ and using (4.2) we get (4.3).
We are left with the case when $-\beta\in\mathrm{qdeg}(M/M_r)$ and $M/M_r\simeq R/I_{\rho',J'}(\gamma)$ with $I_{\rho,J}\subsetneq I_{\rho',J'}$. This implies that M_r is also $I_{\rho,J}$–coprimary. Moreover, it is clear that $-\beta\in\mathrm{qdeg}(M_r)$ by using Lemma 3.3. Thus, we have by induction hypothesis that the L–characteristic variety of $\mathcal{H}_0(E-\beta,M_r)$ is the L–characteristic variety of $M_A(I_{\rho,J},0)$.
Assume to the contrary that there exists an irreducible component C of the L–characteristic variety of $M_A(I_{\rho,J},0)$ that is not contained in the L–characteristic variety of $\mathcal{H}_0(E-\beta,M)$. This implies that C is not contained in $\mathrm{Ch}^L(\mathcal{H}_0(E-\beta,M/M_r))$, i.e. the multiplicity $\mu_{A,0}^{L,C}(M/M_r,\beta)$ of C in the L-characteristic cycle of $\mathcal{H}_0(E-\beta,M/M_r)$ is zero (see [SW08, Definition 4.7]). As a consequence, the multiplicity $\mu_{A,i}^{L,C}(M/M_r,\beta)$ of C in the L-characteristic cycle of $\mathcal{H}_i(E-\beta,M/M_r)$ is zero for all $i\geq 0$ because we can use an adapted version of [SW08, Theorems 4.11 and 4.16] since M/M_r is a module of the form $R/(I_{A_{J'}}+\mathfrak{m}_{J'})(\gamma)$ after rescaling the variables via ρ. Now, using the long exact sequence of Euler–Koszul homology and the additivity of the L–characteristic cycle we conclude that $\mu_{A,i}^{L,C}(M,\beta)=\mu_{A,i}^{L,C}(M_r,\beta)$ for all $i\geq 0$. In particular we have that $\mu_{A,0}^{L,C}(M,\beta)>0$ and thus C is contained in the L–characteristic variety of $\mathcal{H}_0(E-\beta,M)$. We conclude that the L–characteristic variety of $M_A(I_{\rho,J},0)$ is contained in the L–characteristic variety of $\mathcal{H}_0(E-\beta,M)$ and this finishes the induction proof. □

The following result is well known. We include a proof for the sake of completeness.

Lemma 4.2. *Let $I_1,\dots,I_r$ be a sequence of ideals in R and $\omega\in\mathbb{R}^n$ a weight vector. Then*

$$\cap_{j=1}^r \sqrt{\operatorname{in}_\omega(I_j)} = \sqrt{\operatorname{in}_\omega(\cap_j I_j)} \tag{4.4}$$

Proof. The inclusion $\operatorname{in}_\omega(\cap_j I_j)\subseteq \cap_{j=1}^r \operatorname{in}_\omega(I_j)$ is obvious and then we can take radicals. Let us see that $\cap_{j=1}^r \operatorname{in}_\omega(I_j)\subseteq \sqrt{\operatorname{in}_\omega(\cap_j I_j)}$. Let us consider an ω–homogeneous element f in $\cap_{j=1}^r \operatorname{in}_\omega(I_j)$; then for all $j=1,\dots,r$ there exists $g_j\in I_j$ such that $f=\operatorname{in}_\omega(g_j)$. Thus we have $\prod_j g_j\in\cap_j I_j$ and $f^r=\prod_j \operatorname{in}_\omega(g_j)=\operatorname{in}_\omega(\prod_j g_j)\in\operatorname{in}_\omega(\cap_j I_j)$. In particular, $f\in\sqrt{\operatorname{in}_\omega(\cap_j I_j)}$. This finishes the proof as the involved ideals are ω–homogeneous. □

The following result is a direct consequence of [DMM10, Theorem 6.8] and Proposition 4.1 when $-\beta\notin\mathcal{Z}_{\text{primary}}(I)$. However, we want to prove it when $-\beta\notin\mathcal{Z}_{\text{Andean}}(I)$ that is a weaker condition.

Theorem 4.3. *Let I be a A–graded binomial ideal and consider a binomial primary decomposition $I=\cap_{\rho,J}C_{\rho,J}$. If $M_A(I,\beta)$ is holonomic (equivalently, $-\beta$ lies outside the Andean arrangement) then the L-characteristic variety of $M_A(I,\beta)$ coincide with the union of the L-characteristic varieties of $M_A(I_{\rho,J},0)$ for all associated toral primes $I_{\rho,J}$ of I such that $-\beta\in\operatorname{qdeg}(R/C_{\rho,J})$. In particular, the slopes of $M_A(I,\beta)$ along a coordinate subspace in $\mathbb{C}^n$ coincide with the union of the set of slopes of $M_A(I_{\rho,J},0)$ along the same coordinate subspace for $I_{\rho,J}$ varying between all the associated toral primes of I such that $-\beta\in\operatorname{qdeg}(R/C_{\rho,J})$.*

Proof. By Proposition 3.2, we have that $M_A(I,\beta)$ is isomorphic to $M_A(I_\beta,\beta)$. We also have that

$$\bigcup_{-\beta\in\operatorname{qdeg}(R/C_{\rho,J})} \operatorname{Ch}^L(M_A(C_{\rho,J},\beta))\subseteq \operatorname{Ch}^L(M_A(I_\beta,\beta))\subseteq\mathcal{V}(\operatorname{in}_L(I_\beta),Ax\xi) \tag{4.5}$$

On the other hand, by Lemma 4.2 we have that $\mathcal{V}(\operatorname{in}_L(I_\beta))=\cup\mathcal{V}(\operatorname{in}_L(C_{\rho,J}))=\mathcal{V}(\operatorname{in}_L(I_{\rho,J}))$. Here $\mathcal{V}$ is the zero set of the corresponding ideal. The result in the statement follows from the last inclusion, the inclusions (4.5) and Proposition 4.1. □

Remark 4.4. Notice that Theorem 4.3 implies that the map from $\mathbb{C}^d\setminus\mathcal{Z}_{\text{Andean}}(I)$ to *Sets* sending β to the set of slopes of $M_A(I,\beta)$ along any fixed coordinate subspace is upper-semi-continuous in β. The Examples 4.5 and 4.6 illustrate Theorem 4.3. [ES96, Theorem 4.1] has been very useful in order to construct binomial A–graded ideals I starting from some toral primes that we wanted to be associated primes of I.

Example 4.5. The binomial ideal

$$I=\langle\partial_1\partial_3,\partial_1\partial_4,\partial_2\partial_3,\partial_2\partial_4,\partial_3\partial_4,\partial_1^4\partial_2^3-\partial_1\partial_5,\partial_1^3\partial_2^4-\partial_2\partial_5,\partial_3^4-\partial_3\partial_5,\partial_4^4-\partial_4\partial_5^2\rangle$$

is A–graded for the matrix

$$A=\begin{pmatrix}1&0&1&2&3\\0&1&1&2&3\end{pmatrix}$$

Its primary components are the toral primes $I_i:=I_{\rho_i,J_i}$, $i=1,2,3,4$, where $J_1=\{3,5\}$, $J_2=\{4,5\}$, $J_3=\{1,2,5\}$, $J_4=\{5\}$ and $\rho_i:\ker_{\mathbb{Z}}A_{J_i}\longrightarrow\mathbb{C}^*$ is the trivial character for $i=1,2,3,4$. Notice that $\operatorname{qdeg}(R/I_i)=\mathbb{C}A_{J_i}=\mathbb{C}\binom{1}{1}$ for $i=1,2,4$ and $\operatorname{qdeg}R/I_3=\mathbb{C}^2$. Using Theorem 4.3, Remark 3.4 and the results in [SW08] we have the following:

If $\beta \in \mathbb{C}^2 \setminus \mathbb{C}\binom{1}{1}$ then $M_A(I,\beta) \simeq M_A(I_3,\beta)$ has a unique slope $s=6$ along the hyperplane $\{x_5=0\}$ and it is regular along the other coordinate hyperplanes.
If $\beta \in \mathbb{C}\binom{1}{1}$ then $M_A(I,\beta)$ has the slopes $s_1=3/2$, $s_2=3$ and $s_3=6$ along $\{x_5=0\}$.

Example 4.6. The binomial ideal $I=\langle \partial_4\partial_5, \partial_3\partial_5, \partial_2\partial_5, \partial_1\partial_5, \partial_3\partial_4\partial_6, \partial_2\partial_4\partial_6, \partial_1\partial_3\partial_4, \partial_1\partial_2\partial_4, \partial_5^3 - \partial_6\partial_5, \partial_2\partial_3^2\partial_4 - \partial_3\partial_4^2, \partial_1^2\partial_4^3 - \partial_4\partial_6, \partial_1^2\partial_2^2\partial_3^3 - \partial_3\partial_6, \partial_1^2\partial_2^3\partial_3^2 - \partial_2\partial_6\rangle$ is A–graded for the matrix

$$A=\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix}$$

Using Macaulay2 we get a primary decomposition of I where the primary components are the toral primes $I_i = I_{\rho_i,J_i}$, $i=1,\ldots,6$, where $J_1=\{1,2,3,6\}$, $J_2=\{2,3,4\}$, $J_3=\{2,4\}$, $J_4=\{5,6\}$, $J_5=\{1,4,6\}$, $J_6=\{1,6\}$ and $\rho_i : \ker_{\mathbb{Z}} A_{J_i} \longrightarrow \mathbb{C}^*$ is the trivial character for $i=1,\ldots,6$.
We have $I_1=\langle \partial_1^2\partial_2^2\partial_3^2-\partial_6, \partial_4, \partial_5\rangle$, $I_2=\langle \partial_2\partial_3-\partial_4, \partial_1, \partial_5, \partial_6\rangle$, $I_3=\langle \partial_1, \partial_3, \partial_5, \partial_6\rangle$, $I_4=\langle \partial_5^2-\partial_6, \partial_1, \partial_2, \partial_3, \partial_4\rangle$, $I_5=\langle \partial_1^2\partial_4^2-\partial_6, \partial_2, \partial_3, \partial_5\rangle$ and $I_6=\langle \partial_2, \partial_3, \partial_4, \partial_5\rangle$.
We know that $\operatorname{qdeg} R/I_i = \mathbb{C}A_{J_i}$ for $i=1,\ldots,6$. In particular, R/I_1 is the unique component with Krull dimension $d=3$.
There are four components with Krull dimension $d-1=2$, namely $R/I_2, R/I_3$ have quasidegrees set $\mathbb{C}A_{J_2}=\mathbb{C}A_{J_3}=\{y_1=0\}\subseteq \mathbb{C}^3$ and $R/I_5, R/I_6$ have quasidegrees set $\mathbb{C}A_{J_5}=\mathbb{C}A_{J_6}=\{y_2=y_3\}\subseteq\mathbb{C}^3$. There is one component R/I_4 with Krull dimension one and quasidegrees set equal to the line $\mathbb{C}A_{J_4}=\{y_1=y_2=y_3\}\subseteq\mathbb{C}^3$.
Thus, in order to study the behavior of $M_A(I,\beta)$ when varying $\beta\in\mathbb{C}^3$ it will be useful to stratify the space of parameters $\mathbb{C}^3$ by the strata $\Lambda_1=\mathbb{C}^3\setminus\{y_1(y_2-y_3)=0\}$, $\Lambda_2=\{y_1=0\}\setminus\{y_2-y_3=0\}$, $\Lambda_3=\{y_2-y_3=0\}\setminus\{y_1=0\}$, $\Lambda_4=\{y_1=y_2=y_3\}\setminus\{0\}$, $\Lambda_5=\overline{\Lambda_2}\cap\overline{\Lambda_3}\setminus\{0\}=\{y_1=0=y_2-y_3\}\setminus\{0\}$ and $\Lambda_6=\{0\}$.
Let us compute the slopes of $M_A(I,\beta)$ along coordinate hyperplanes according with Theorem 4.3, Remark 3.4 and the results in [SW08]. Recall that $a_1\ldots,a_6$ stand for the columns of the matrix A. We have the following situations:

1) If $-\beta\in\Lambda_1$ then R/I_1 is the unique component whose quasidegrees set contains $-\beta$. Thus, $M_A(I,\beta)\simeq M_A(I_1,\beta)$ has a unique slope $s=6$ along the hyperplane $\{x_6=0\}$ because $a_6/s=[1/3,1/3,1/3]^t$ belongs to the plane passing through a_1,a_2,a_3.
2) If $-\beta\in\Lambda_2$, then $-\beta\in\operatorname{qdeg}(R/I_i)$ if and only if $i\in\{1,2,3\}$ so $M_A(I,\beta)$ has the slope $s=6$ along $\{x_6=0\}$ arising from I_1 and the slope $s=2$ along $\{x_4=0\}$ arising from I_2 (since $a_4/2$ lie in the line passing through a_2,a_3).
3) If $-\beta\in\Lambda_3$, then $-\beta\in\operatorname{qdeg}R/I_i$ if and only if $i\in\{1,5,6\}$. $M_A(I,\beta)$ has the slopes $s=4$ (arising from I_5) and $s=6$ (arising from I_1) along $\{x_6=0\}$.
4) If $-\beta\in\Lambda_4$, then $-\beta\in\operatorname{qdeg}R/I_i$ if and only if $i\in\{1,4,5,6\}$. $M_A(I,\beta)$ has the slopes $s=2$ (arising from I_4), $s=4$ (arising from I_5) and $s=6$ (arising from I_1) along $\{x_6=0\}$.
5) If $-\beta\in\Lambda_5=\cap_{i\neq 4}\operatorname{qdeg}R/I_i\setminus\operatorname{qdeg}R/I_4$ then $M_A(I,\beta)$ has the slopes $s=4$ (arising from I_5) and $s=6$ (arising from I_1) along $\{x_6=0\}$ and the slope $s=2$ (arising from I_2) along $\{x_4=0\}$.
6) If $-\beta\in\Lambda_6$ (i.e. $\beta=0$) we have that $-\beta$ is in the quasidegrees set of all the components R/I_i. Thus, $M_A(I,\beta)$ has the slopes $s=2$ (arising from I_4), $s=4$ (arising from I_5)

and $s=6$ (arising from I_1) along $\{x_6=0\}$ and the slope $s=2$ (arising from I_2) along $\{x_4=0\}$.

In all the cases there are no more slopes along coordinate hyperplanes. Notice that when we move $-\beta$ from one stratum Λ_i of dimension r, $1\leq r\leq d=3$, to another stratum $\Lambda_j\subseteq\overline{\Lambda_i}$ of dimension $r-1$ then $M_A(I,\beta)$ can have new slopes along a hyperplane but no slope disappears.

Remark 4.7. By [DMM10, Lemma 7.2], all toral primes of a lattice-basis ideal $I(B)$ have dimension exactly d and are minimal primes of $I(B)$. Thus, the L-characteristic varieties and the set of slopes of $M_A(I(B),\beta)$ are independent of $-\beta\notin\mathcal{Z}_{\text{Andean}}(I(B))$.

To finish this Section we are going to compute the multiplicities of the L–characteristic cycle of a holonomic binomial D–module $M_A(I,\beta)$ for β generic. Recall that the volume $\operatorname{vol}_\Lambda(B)$ of a matrix B with columns $b_1,\dots,b_k\in\mathbb{Z}^d$ with respect to a lattice $\Lambda\supseteq\mathbb{Z}B$ is nothing but the Euclidean volume of the convex hull of $\{0\}\cup\{b_1,\dots,b_k\}$ normalized so that the unit simplex in the lattice Λ has volume one.
From now on we assume that $\mathcal{Z}_{\text{Andean}}(I)\neq\mathbb{C}^d$ and that $\beta\in\mathbb{C}^d$ is generic. In particular we assume that all the quotients $R/C_{\rho,J}$ whose quasidegrees set contain $-\beta$ are toral and have Krull dimension d. The generic condition will also guarantee that β is not a rank–jumping parameter of any hypergeometric system $\mathcal{H}_0(E-\beta,I_{\rho,J})$.
Under this assumptions it is proved in [DMM10, Theorem 6.10] that the holonomic rank of $M_A(I,\beta)$ equals

$$\operatorname{rank}(M_A(I,\beta))=\sum_{R/I_{\rho,J}\text{ toral } d\text{–dimensional}}\mu_{\rho,J}\operatorname{vol}_{\mathbb{Z}A_J}(A_J)$$

We will use the same strategy in order to compute the multiplicities in the L-characteristic cycle $\operatorname{CCh}^L(M_A(I,\beta))$. It is enough to compute the multiplicities in the L-characteristic cycle of $M_A(C_{\rho,J},\beta)$ for each d–dimensional toral component $C_{\rho,J}$ of I and then apply [DMM10, Theorem 6.8].
In [SW08, Section 3.3] the authors give an index formula for the multiplicity $\mu_{A,0}^{L,\tau}(\beta)$ of the component $\overline{C_A^\tau}$ in the L–characteristic cycle $\operatorname{CCh}^L(M_A(\beta))$ of a hypergeometric D–module; see equality (4.1). They prove that these multiplicities are independent of β if β is generic (see [SW08, Theorem 4.28]). Let us denote by $\mu_A^{L,\tau}$ this constant value.
If M is a finitely generated R–module, we denote by $\mu_{A,0}^{L,\tau}(M,\beta)$ the multiplicity of the component $\overline{C_A^\tau}$ in the L–characteristic cycle $\operatorname{CCh}^L(\mathcal{H}_0(E-\beta,M))$ (see [SW08, Definition 4.7]).
For $J\subset\{1,\dots,n\}$ we denote A_J the submatrix whose columns are indexed by J, D_J the Weyl algebra with variables $\{x_j,\partial_j\,|\,j\in J$ and L_J the filtration on D_J induced by the weights (u_j,v_j) for $j\in J$. In particular, we can define the multiplicity $\mu_{A_J}^{L_J,\tau}$ for any face τ of the (A_J,L_J)–umbrella $\Phi_{A_J}^{L_J}$.

Theorem 4.8. *Let $R/C_{\rho,J}$ be a toral d–dimensional module and let β be generic. We have for all $\tau\in\Phi_{A_J}^{L_J}$ and for any filtration L on D that*

$$\mu_{A,0}^{L,\tau}(R/C_{\rho,J},\beta)=\mu_{\rho,J}\mu_{A_J}^{L_J,\tau}.$$

Proof. It follows the ideas of the last part of the proof of [DMM10, Theorem 6.10] (see also the proof of Theorem 3.7). We write $M=R/C_{\rho,J}$ and consider a toral filtration $M_0=(0)\subseteq$

$M_1 \subseteq \cdots \subseteq M_r = M$ each successive quotient M_i/M_{i-1} being isomorphic to $\frac{R}{I_{\rho_i,J_i}}(\gamma_i)$ for some $\gamma_i \in \mathbb{Z}^d$. The number of successive quotients of dimension d is the multiplicity $\mu_{\rho,J}$ of the ideal $I_{\rho,J}$ in $C_{\rho,J}$. From the assumption on β we can take $-\beta$ outside the union of the quasidegree sets of $\frac{R}{I_{\rho_i,J_i}}$ with Krull dimension $< d$. Then

$$\mathcal{H}_j(E-\beta, M_i/M_{i-1}) = \begin{cases} 0 & \text{if} \quad I_{\rho_i,J_i} \neq I_{\rho,J} \\ \mathcal{H}_j(E-\beta+\gamma_i, R/I_{\rho,J})(\gamma_i) & \text{otherwise.} \end{cases}$$

Again using that β is generic, we have that $\mathcal{H}_j(E-\beta, M_i/M_{i-1}) = 0$ for any i and any $j \geq 1$. The statement of the Theorem follows by applying decreasing induction on i and the additivity of $\mu_{A,0}^{L,\tau}$ with respect to the exact sequence

$$0 \longrightarrow \mathcal{H}_0(E-\beta, M_{i-1}) \longrightarrow \mathcal{H}_0(E-\beta, M_i) \longrightarrow \mathcal{H}_0(E-\beta, M_i/M_{i-1}) \longrightarrow 0.$$

We notice here that the multiplicity $\mu_A^{L,\tau}$ for $\mathcal{H}_0(E-\beta+\gamma_i, R/I_{\rho,J})$ equals $\mu_{A_J}^{L_J,\tau}$ for the hypergeometric D_J–module $M_{A_J}(\beta-\gamma_i)$ because β is generic. □

5. On the Gevrey solutions and the irregularity of binomial D–modules

Let us denote by Y_i the hyperplane $x_i = 0$ in $\mathbb{C}^n$. Again by [DMM10, Theorem 6.8], in order to study the Gevrey solutions and the irregularity of a holonomic binomial D–module $M_A(I,\beta)$ for generic parameters $\beta \in \mathbb{C}^d$ it is enough to study each binomial D–module $M_A(C_{\rho,J},\beta)$ arising from a d–dimensional toral primary component $R/C_{\rho,J}$. For any real number s with $s \geq 1$, we consider, the irregularity complex of order s, $\mathrm{Irr}_{Y_i}^{(s)}(M_A(C_{\rho,J},\beta))$ (see [Meb90, Definition 6.3.1]). Since $M_A(C_{\rho,J},\beta)$ is holonomic, by a result of Z. Mebkhout [Meb90, Theorem 6.3.3] this complex is a perverse sheaf and then for $p \in Y_i$ generic it is concentrated in degree 0.
For $r \in \mathbb{R}$ with $r \geq 1$ we denote by L_r the filtration on D induced by $L_r = F + (r-1)V$ and we will write simply Φ_A^r instead of $\Phi_A^{L_r}$ and $\mu_{A,0}^{r,\tau}$ instead of $\mu_{A,0}^{L_r,\tau}$.

Theorem 5.1. *Let $R/C_{\rho,J}$ be a toral d–dimensional module, β generic, $p \in Y_i$ generic, $i = 1,\ldots,n$ and s a real number with $s \geq 1$. We have that*

$$\dim_{\mathbb{C}} \mathcal{H}^0\left(\mathrm{Irr}_{Y_i}^s(M_A(C_{\rho,J},\beta))\right)_p = \mu_{\rho,J} \sum_{i\notin\tau\in\Phi_{A_J}^s\setminus\Phi_{A_J}^1} \mathrm{vol}_{\mathbb{Z}A_J}(A_\tau)$$

Proof. We follow the argument of the proof of Theorem 7.5 in [Fer10]. We apply results of Y. Laurent and Z. Mebkhout [LM99, Lemme 1.1.2 and Section 2.3] to get

$$\dim_{\mathbb{C}} \mathcal{H}^0\left(\mathrm{Irr}_{Y_i}^s(M_A(C_{\rho,J},\beta))\right)_p = \mu_{A,0}^{s+\epsilon,\emptyset} - \mu_{A,0}^{1+\epsilon,\emptyset} + \mu_{A,0}^{1+\epsilon,\{i\}} - \mu_{A,0}^{s+\epsilon,\{i\}}.$$

To finish the proof we apply Theorem 4.8 and Theorem 7.5 [Fer10]. □

Remark 5.2. Notice that the above formula for $\dim_{\mathbb{C}} \mathcal{H}^0(\mathrm{Irr}_{Y_i}^s(M_A(C_{\rho,J},\beta)))_p = 0$ yields zero if $i \notin J$ since in that case the induced filtration $(L_s)_J$ (denoted just by s by abuse of notation) is constant and so $\Phi_{A_J}^s \setminus \Phi_{A_J}^1 = \emptyset$.

Let us see how to compute Gevrey solutions of a binomial D–module $M_A(I,\beta)$. By (3.3) in [DMM$_2$10] the $I_{\rho,J}$–primary component $C_{\rho,J}$ of an irredundant primary decomposition of any

A–graded binomial ideal I (for some minimal associated prime $I_{\rho,J} = I_\rho + \mathfrak{m}_J$ of I) contains I_ρ. Thus,

$$I_\rho + \mathfrak{m}_J^r \subseteq C_{\rho,J} \subseteq \sqrt{C_{\rho,J}} = I_{\rho,J} = I_\rho + \mathfrak{m}_J \tag{5.1}$$

for sufficiently large integer r. In fact, it is not hard to check that $C_{\rho,J} = I_\rho + B_{\rho,J}$ for some binomial ideal $B_{\rho,J} \subseteq R$ such that $\mathfrak{m}_J^r \subseteq B_{\rho,J} \subseteq \mathfrak{m}_J$. Let us fix such an ideal $B_{\rho,J}$.
For any monomial ideal $\mathfrak{n} \subseteq C_{\rho,J}$ such that $\sqrt{\mathfrak{n}} = \mathfrak{m}_J$ we have that

$$H_A(I_\rho + \mathfrak{n}, \beta) \subseteq H_A(C_{\rho,J}, \beta) \subseteq H_A(I_{\rho,J}, \beta).$$

Let us fix such an ideal $\mathfrak{n}$. In particular, any formal solution of $M_A(I_{\rho,J}, \beta)$ is a solution of $M_A(C_{\rho,J}, \beta)$ and any solution of $M_A(C_{\rho,J}, \beta)$ is a solution of $M_A(I_\rho + \mathfrak{n}, \beta)$.
Let us assume that $C_{\rho,J}$ is toral (i.e. $R/I_{\rho,J}$ has Krull dimension equal to $\operatorname{rank} A_J$). We will also assume that $\operatorname{rank} A_J = \operatorname{rank} A$ in order to ensure that $\operatorname{qdeg}(R/C_{\rho,J}) = \mathbb{C}^d$.
On the one hand, both the solutions of $M_A(I_{\rho,J}, \beta)$ and the solutions of $M_A(I_\rho + \mathfrak{n}, \beta)$ can be described explicitly if the parameter vector β is generic enough. More precisely, a formal solution of the hypergeometric system $M_A(I_{\rho,J}, \beta)$ with very generic β is known to be of the form

$$\phi_v = \sum_{u \in \ker A_J \cap \mathbb{Z}^J} \rho(u) \frac{(v)_{u_-}}{(v+u)_{u_+}} x_J^{v+u}$$

where $v \in \mathbb{C}^J$ such that $A_J v = \beta$ and $(v)_w = \prod_{j\in J} \prod_{0 \le i \le w_j - 1} (v_j - i)$ is the Pochhammer symbol (see [GZK89, SST00]). Here, v needs to verify additional conditions in order to ensure that ϕ_v is a formal series along a coordinate subspace or a holomorphic solution.
The vectors v you need to consider to describe a basis of the space of Gevrey solutions of a given order along a coordinate subspace of $\mathbb{C}^n$ for the binomial D-module $M_A(I_{\rho,J}, \beta)$ are the same that are described in [Fer10] for the hypergeometric system $M_{A_J}(\beta)$.
On the other hand, for γ in $\mathbb{N}^{\overline{J}}$ let G_γ be either a basis of the space of holomorphic solutions near a non singular point or the space of Gevrey solutions of a given order along a coordinate hyperplane of $\mathbb{C}^J$ for the system $M_{A_J}(I_\rho, \beta - A_{\overline{J}}\gamma)$, where $\overline{J}$ denotes the complement of J in $\{1, \ldots, n\}$ and $x_{\overline{J}}^\gamma$ runs in the set $\mathrm{S}_{\overline{J}}(\mathfrak{n})$ of monomials in $\mathbb{C}[x_{\overline{J}}]$ annihilated by the monomial differential operators in $\mathfrak{n}$. Then a basis of the same class of solutions for the system $M_A(I_\rho + \mathfrak{n}, \beta)$ is given by

$$\mathcal{B} = \{x_{\overline{J}}^\gamma \varphi : \ x^\gamma \in \mathrm{S}_{\overline{J}}(\mathfrak{n}), \ \varphi \in G_\gamma\}$$

We conclude that any holomorphic or formal solution of $M_A(C_{\rho,J}, \beta)$ can be written as a linear combination of the series in $\mathcal{B}$. The coefficients in a linear combination of elements in $\mathcal{B}$ that provide a solution of $M_A(C_{\rho,J}, \beta)$ can be computed if we force a general linear combination to be annihilated by the binomial operators in a set of generators of $B_{\rho,J}$ that are not in $\mathfrak{n}$.
Thus, the main problem in order to compute formal or analytic solutions of $M_A(C_{\rho,J}, \beta)$ is that the ideal $B_{\rho,J}$ is not a monomial ideal in general and that a minimal set of generators may involve some variables x_j for $j \in J$. Let us illustrate this situation with the following example.

Example 5.3. Let us write $x = x_1, y = x_2, z = x_3, t = x_4$ and consider the binomial ideal $C_{\rho,J} = I_\rho + B_{\rho,J} \subseteq \mathbb{C}[\partial_x, \partial_y, \partial_z, \partial_t]$ where $J = \{1, 2\}$, $\rho : \ker(A_J) \cap \mathbb{Z}^2 \to \mathbb{C}^*$ is the trivial character, A is the row matrix $(2, 3, 2, 2)$, $I_\rho = \langle \partial_x^3 - \partial_y^2 \rangle$ and $B_{\rho,J} = \langle \partial_z^2 - \partial_x \partial_t, \partial_t^2 \rangle$.
Notice that $C_{\rho,J}$ is A-graded for the row matrix $A = (2\ 3\ 2\ 2)$ and that $C_{\rho,J}$ is toral and primary.

Since $C_{\rho,J}$ is primary and its radical ideal is $I_\rho + \mathfrak{m}_J = \langle \partial_x^3 - \partial_y^2, \partial_z, \partial_t \rangle$, we have that $M_A(C_{\rho,J}, \beta)$ is an irregular binomial D-module for all parameters $\beta \in \mathbb{C}$ (see Theorem 3.7) and that it has only one slope $s = 3/2$ along its singular locus $\{y = 0\}$.
We are going to compute the Gevrey solutions of $M_A(C_{\rho,J}, \beta)$ corresponding to this slope.
By the previous argument and using that $\mathfrak{n} = \langle \partial_z^4, \partial_t^2 \rangle \subseteq B_{\rho,J}$ we obtain that any Gevrey solution of $M_A(C_{\rho,J}, \beta)$ along $\{y = 0\}$ can be written as

$$f = \sum_{\gamma,k} \lambda_{\gamma,k} z^{\gamma_z} t^{\gamma_t} \phi_k(\beta - 2\gamma_z - 2\gamma_t)$$

where $\lambda_{\gamma,k} \in \mathbb{C}$, $\gamma = (\gamma_z, \gamma_t)$, $\gamma_z \in \{0, 1, 2, 3\}$, $\gamma_t, k \in \{0, 1\}$ and

$$\phi_k(\beta - 2\gamma_z - 2\gamma_t) = \sum_{m \geq 0} \frac{((\beta - 3k)/2 - \gamma_z - \gamma_t)_{3m}}{(k + 2m)_{2m}} x^{(\beta-3k)/2-\gamma_z-\gamma_t-3m} y^{k+2m}$$

is a Gevrey series of index $s = 3/2$ along $y = 0$ at any point $p \in \{y = 0\} \cap \{x \neq 0\}$ if $(\beta - 3k)/2 - \gamma_z - \gamma_t \notin \mathbb{N}$.
We just need to force the condition $\partial_x \partial_t(f) = \partial_z^2(f)$ in order to obtain the values of $\lambda_{\gamma,k}$ such that f is a solution of $M_A(C_{\rho,J}, \beta)$.
In this example, we obtain the conditions $\lambda_{(2,1),k} = \lambda_{(3,1),k} = 0$ for $k = 0, 1$ and

$$\lambda_{(\gamma_z+2,0),1} = \frac{((\beta - 3k)/2 - \gamma_z)}{(a + 1)(a + 2)} \lambda_{(\gamma_z,1),k}$$

for $k, \gamma_z = 0, 1$.
In particular we get an explicit basis of the space of Gevrey solutions of $M_A(C_{\rho,J}, \beta)$ along $y = 0$ with index equal to the slope $s = 3/2$ and we have that the dimension of this space is 8. Notice that $8 = 4 \cdot 2$ is the expected dimension (see Theorem 5.1) since $\mu_{\rho,J} = 4$ and the dimension of the corresponding space for $M_A(I_{\rho,J}, \beta)$ is 2 (see [FC11, FC08]).

References

[Ado94] A. Adolphson. *A-hypergeometric functions and rings generated by monomials*, Duke Math. J. 73 (2) (1994) 269- 290.

[CT03] F. J. Castro-Jiménez and N. Takayama. *Singularities of the hypergeometric system associated with a monomial curve*. Trans. Amer. Math. Soc., vol.355, no. 9, p. 3761- 3775 (2003).

[DMM10] A. Dickenstein, L. Matusevich, E. Miller. *Binomial $\mathcal{D}$-modules*. Duke Math. J. 151 no. 3 (2010), 385-429.

[DMM$_2$10] A. Dickenstein, L. Matusevich, E. Miller. *Combinatorics of binomial primary decomposition*. Math. Z. 264 (2010), no. 4, 745-763.

[ES96] D. Eisenbud and B. Sturmfels, *Binomial ideals*, Duke Math. J. 84 (1996), 1–45.

[Fer10] M.C. Fernández-Fernández, *Irregular hypergeometric $\mathcal{D}$-modules*. Adv. Math. 224 (2010) 1735-1764.

[FC11] M.C. Fernández-Fernández and F.J. Castro-Jiménez, *Gevrey solutions of the irregular hypergeometric system associated with an affine monomial curve*. Trans. Amer. Math. Soc. 363 (2011), 923-948.

[FC08] M.C. Fernández-Fernández and F.J. Castro-Jiménez, *Gevrey solutions of irregular hypergeometric systems in two variables*. *arXiv:0811.3390v1 [math.AG]*

[GGZ87] Gelfand, I.M., Graev, M.I. and Zelevinsky, A.V. *Holonomic systems of equations ans series of hypergeometric type*. Dokl. Akad. Nauk SSSR 295 (1987), no. 1, 14–19; translation in Soviet Math. Dokl. 36 (1988), no. 1, 5–10.

[GZK89] Gelfand, I.M., Zelevinsky, A.V. and Kapranov, M.M., *Hypergeometric functions and toric varieties (or Hypergeometric functions and toral manifolds)*. Translated from Funktsional. Anal. i Prilozhen. 23 (1989), no. 2, 12–26; translation in Funct. Anal. Appl. 23 (1989), no. 2, 94–106; and I.M. Gelfand, A.V. Zelevinskiĭand M.M. Kapranov, Correction to the paper: "Hypergeometric functions and toric varieties" [Funktsional. Anal. i Prilozhen. 23 (1989), no. 2, 12–26]; (Russian) Funktsional. Anal. i Prilozhen. 27 (1993), no. 4, 91; translation in Funct. Anal. Appl. 27 (1993), no. 4, 295 (1994).

[Har03] M. I. Hartillo. *Hypergeometric slopes of codimension 1*. Revista Matemática Iberoamericana, vol. 19, no. 2 (2003), 455-466.

[Har04] M. I. Hartillo. *Irregular hypergeometric sistems associated with a singular monomial curve*. Trans. Amer. Math. Soc., vol. 357, no. 11 (2004), 4633-4646.

[Hot98] R. Hotta. *Equivariant $\mathcal{D}$-modules*. `arXiv.org (1998), no. RT/9805021`.

[Lau87] Y. Laurent. *Polygône de Newton et b-fonctions pour les modules microdifférentiels*. Annales scientifiques de l'ENS 4^e série, tome 20, no. 3 (1987), 391-441.

[LM99] Y. Laurent and Z. Mebkhout. *Pentes algébriques et pentes analytiques d'un $\mathcal{D}$-module*. Annales Scientifiques de L'E.N.S. 4^e série, tome 32, n. 1 (1999) p.39-69.

[MMW05] L.F. Matusevich, E. Miller and U. Walther. *Homological methods for hypergeometric families*. J. Amer. Math. Soc. 18 (2005), 4, p.919-941.

[Meb90] Z. Mebkhout. *Le théorème de positivité de l'irrégularité pour les D_X-modules*, in: The Grothendieck Festschrift, in: Progr. Math., vol. 88 (3), Birkhäuser, 1990, pp. 83-131.

[Sai02] T.M. Sadikov. *On the Horn system of partial differential equations and series of hypergeometric type*. Math. Scand. 91 (2002), no. 1, 127-149.

[SST00] M. Saito, B. Sturmfels and N. Takayama, *Gröbner Deformations of Hypergeometric Differential Equations*, Springer–Verlag, Berlin, 2000.

[SW08] M. Schulze and U. Walther. *Irregularity of hypergeometric systems via slopes along coordinate subspaces*. Duke Math. J. 142, 3 (2008), 465-509.

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