Gevrey solutions of irregular hypergeometric systems in two variables

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20 November 2008

Abstract

We describe the Gevrey series solutions at singular points of the irregular hypergeometric system (GKZ system) associated with an affine plane monomial curve. We also describe the irregularity complex of such a system with respect to its singular support and in particular we prove, using elementary methods, that this irregularity complex is a perverse sheaf as assured by a theorem of Z. Mebkhout.

Introduction

To every row integer matrix $A = (a_1 \ a_2)$, with positive and relatively prime entries, and every complex parameter $\beta \in \mathbb{C}$ we can associate the hypergeometric system $\mathcal{M}_A(\beta)$ defined by the following two linear partial differential equations:

$$\left(\frac{\partial}{\partial x_1}\right)^{a_2}(\varphi) - \left(\frac{\partial}{\partial x_2}\right)^{a_1}(\varphi) = 0$$

$$a_1 x_1 \frac{\partial \varphi}{\partial x_1} + a_2 x_2 \frac{\partial \varphi}{\partial x_2} - \beta \varphi = 0.$$

General hypergeometric systems have been introduced by I.M. Gel'fand, M.I. Graev, M.M. Kapranov and A.V. Zelevinsky ([6], [7], [8]) and their analytic solutions, at a generic point in $\mathbb{C}^n$, have been widely studied (see e.g. [7], [8], [1], [19], [18]).

In this work we explicitly describe the Gevrey solutions –at singular points– of the hypergeometric system $\mathcal{M}_A(\beta)$ associated with $A = (a_1 \ a_2)$ and $\beta \in \mathbb{C}$. To this end we will use the Γ–series introduced in [8] and also used in [19] in a very useful and slightly different form. We use these Γ–series to describe the *Gevrey filtration of the irregularity complex* of $\mathcal{M}_A(\beta)$ with respect to the coordinate axes and in particular with respect to the singular support of the system.

Despite the simplicity of the equations defining these hypergeometric systems the study of its Gevrey solutions is quite involved. In [5] the study of the Gevrey solutions of the hypergeometric system associated with any affine monomial curve in $\mathbb{C}^n$ is reduced to the two dimensional case by using deep results in $\mathcal{D}$–module theory. This justifies our separated treatment for the two variables case.

The behavior of Gevrey solutions of a hypergeometric system (and more generally of any holonomic $\mathcal{D}$–module) is closely related to its irregularity complex as proved by Z. Mebkhout and by Y. Laurent and Z. Mebkhout ([16], [15], [12], [13]). For any hypergeometric system in two variables we will describe its irregularity complex without using any of the deep results in the above references. In particular we will prove (see Conclusions) that the irregularity complex is a perverse sheaf.

The paper has the following structure. In Section 1 we recall the definition of Gevrey series. In Section 2 we recall the definition of Γ–series and summarize the description of the holomorphic solutions of $\mathcal{M}_A(\beta)$ at a generic point of $\mathbb{C}^2$. Section 3 is devoted to the definition –due to Z. Mebkhout [16]– of $\mathrm{Irr}_Y(\mathcal{M}_A(\beta))$, the irregularity complex of the system $\mathcal{M}_A(\beta)$ with respect to its singular support Y in $\mathbb{C}^2$. In Section 4 we prove that the germ of the irregularity complex $\mathrm{Irr}_Y(\mathcal{M}_A(\beta))$ at the origin is zero. In Section 5 we first prove that the complex $\mathrm{Irr}_Y(\mathcal{M}_A(\beta))_p$ for $p \in Y$, $p \neq (0,0)$, is concentrated in degree 0 and then we describe a basis of its 0-th cohomology group. We also prove, by elementary methods, that the irregularity complex $\mathrm{Irr}_Y(\mathcal{M}_A(\beta))$ is a *perverse sheaf* on Y. This is a very particular case of a theorem of Z. Mebkhout [16, Th. 6.3.3].

Some related results can be found in [17], [20] and also in [14] and [10].

1 Gevrey series

Let us write $X = \mathbb{C}^2$ with its structure of complex manifold, $\mathcal{O}_X$ (or simply $\mathcal{O}$) the sheaf of holomorphic functions on X and $\mathcal{D}_X$ (or simply $\mathcal{D}$) the sheaf of linear differential operators with coefficients in $\mathcal{O}_X$. The sheaf $\mathcal{O}_X$ has a natural structure of left $\mathcal{D}_X$–module. Let Z be a hypersurface (i.e. a plane curve) in X with defining ideal $\mathcal{I}_Z$. We denote by $\mathcal{O}_{X|Z}$ the restriction to Z of the sheaf $\mathcal{O}_X$ (and we will also denote by $\mathcal{O}_{X|Z}$ its extension by 0 on X). Recall that the formal completion of $\mathcal{O}_X$ along Z is defined as

$$\mathcal{O}_{\widehat{X|Z}} := \varprojlim_k \mathcal{O}_X/\mathcal{I}_Z^k.$$

By definition $\mathcal{O}_{\widehat{X|Z}}$ is a sheaf on X supported on Z and it has a natural structure of left $\mathcal{D}_X$–module. We will also denote by $\mathcal{O}_{\widehat{X|Z}}$ the corresponding sheaf on Z. We denote by $\mathcal{Q}_Z$ the quotient sheaf defined by the following exact sequence

$$0 \to \mathcal{O}_{X|Z} \longrightarrow \mathcal{O}_{\widehat{X|Z}} \longrightarrow \mathcal{Q}_Z \to 0.$$

The sheaf $\mathcal{Q}_Z$ has then a natural structure of left $\mathcal{D}_X$–module.

Assume $Y \subset X$ is a smooth curve and that it is locally defined by $x_2 = 0$ for some system of local coordinates (x_1, x_2) around a point $p \in Y$. Let us consider a real number $s \geq 1$. A germ

$$f = \sum_{i \geq 0} f_i(x_1)x_2^i \in \mathcal{O}_{\widehat{X|Y},p}$$

is said to be a Gevrey series of order s (along Y at the point p) if the power series

$$\rho_s(f) := \sum_{i \geq 0} \frac{1}{i!^{s-1}} f_i(x_1)x_2^i$$

is convergent at p.

The sheaf $\mathcal{O}_{\widehat{X|Y}}$ admits a natural filtration by the sub-sheaves $\mathcal{O}_{\widehat{X|Y}}(s)$ of Gevrey series of order s, $1 \leq s \leq \infty$ where by definition $\mathcal{O}_{\widehat{X|Y}}(\infty) = \mathcal{O}_{\widehat{X|Y}}$. So we have $\mathcal{O}_{\widehat{X|Y}}(1) = \mathcal{O}_{X|Y}$. We can also consider the induced filtration on $\mathcal{Q}_Y$, i.e. the filtration by the sub-sheaves $\mathcal{Q}_Y(s)$ defined by the exact sequence:

$$0 \to \mathcal{O}_{X|Y} \longrightarrow \mathcal{O}_{\widehat{X|Y}}(s) \longrightarrow \mathcal{Q}_Y(s) \to 0 \tag{1.1}$$

Definition 1.1. *Let Y be a smooth curve in $X = \mathbb{C}^2$ and let p be a point in Y. The Gevrey index of a formal power series $f \in \mathcal{O}_{\widehat{X|Y},p}$ with respect to Y is the smallest $1 \leq s \leq \infty$ such that $f \in \mathcal{O}_{\widehat{X|Y}}(s)_p$.*

2 The hypergeometric system associated with an affine monomial plane curve

We denote by $A_2(\mathbb{C})$ or simply A_2 the complex Weyl algebra of order 2, i.e. the ring of linear differential operators with coefficients in the polynomial ring $\mathbb{C}[x] := \mathbb{C}[x_1, x_2]$. The partial derivative $\frac{\partial}{\partial x_i}$ will be denoted by ∂_i.

Let $A = (a\ b)$ be an integer nonzero row matrix and $\beta \in \mathbb{C}$. Let us denote by $E_A(\beta)$ the linear differential operator $E_A(\beta) := ax_1\partial_1 + bx_2\partial_2 - \beta$. The toric ideal $I_A \subset \mathbb{C}[\partial] := \mathbb{C}[\partial_1, \partial_2]$ associated with A is generated by the binomial $\partial_1^{b'} - \partial_2^{a'}$ where $a' = a/d$, $b' = b/d$ and $d = \gcd(a, b)$. The algebraic plane curve defined by I_A is then an affine monomial plane curve.

The left ideal $A_2 I_A + A_2 E_A(\beta) \subset A_2$ is denoted by $H_A(\beta)$ and it is called the *hypergeometric ideal* associated with (A, β). The (global) hypergeometric module associated with (A, β) is by definition (see [6], [8]) the quotient $M_A(\beta) := A_2/H_A(\beta)$.

To the pair (A, β) we can also associate the corresponding analytic hypergeometric $\mathcal{D}_X$–module, denoted by $\mathcal{M}_A(\beta)$, which is the quotient of $\mathcal{D}_X$ modulo the sheaf of left ideals in $\mathcal{D}_X$ generated by $H_A(\beta)$.

In this paper we will assume that $A = (a\ b)$ is an integer row matrix with $0 < a < b$ and $\beta \in \mathbb{C}$. We can assume without loss of generality that a, b are relatively prime. Nevertheless similar methods to the ones presented here can be applied to different kind of plane monomial curves (see Remark 5.15).

The module $\mathcal{M}_A(\beta)$ is the quotient of $\mathcal{D}_X$ modulo the sheaf of ideals generated by the operators $P := \partial_1^b - \partial_2^a$ and $E_A(\beta) = ax_1\partial_1 + bx_2\partial_2 - \beta$. Sometimes we will write $E = E(\beta) = E_A(\beta)$ if no confusion is possible.

Although it can be deduced from general results ([6] and [1, Th. 3.9]) a direct computation shows that the characteristic variety of $\mathcal{M}_A(\beta)$ is $T_X^* X \cup T_Y^* X$ where $Y = (x_2 = 0)$ and then the singular support of $\mathcal{M}_A(\beta)$ is the axis Y. The module $\mathcal{M}_A(\beta)$ is therefore holonomic.

2.1 Holomorphic solutions of $\mathcal{M}_A(\beta)$ at a generic point

By [8, Th. 2] and [1, Cor. 5.21] the dimension of the vector space of holomorphic solutions of $\mathcal{M}_A(\beta)$ at a point $p \in X \setminus Y$ equals b. A basis of such vector space of solutions can be described, using Γ–series ([6], [8, Sec. 1] and [19, Sec. 3.4]) as follows. For $v \in \mathbb{C}^2$ and $u \in \mathbb{Z}^2$ let us denote

$$\Gamma[v; u] := \frac{(v)_{u_-}}{(v + u)_{u_+}}$$

if $(v + u)_{u_+} \neq 0$ and $\Gamma[v; u] := 0$ otherwise. Here

$$(z)_\alpha = \prod_{i: \alpha_i > 0} \prod_{j=0}^{\alpha_i - 1} (z_i - j)$$

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is the Pochhammer symbol, for any $z \in \mathbb{C}^n$ and any $\alpha \in \mathbb{N}^n$.

For $j = 0, \ldots, b-1$ let us consider

$$v^j = (j, \frac{\beta - ja}{b}) \in \mathbb{C}^2$$

and the corresponding Γ–series

$$\phi_{v^j} = x^{v^j} \sum_{m \geq 0} \Gamma[v^j; u(m)] \left(\frac{x_1^b}{x_2^a}\right)^m \in x^{v^j} \mathbb{C}[[x_1, x_2^{-1}]]$$

with $u(m) = (bm, -am) \in L_A = \operatorname{Ker}_{\mathbb{Z}}(A)$, which defines a holomorphic function at any point $p \in X \setminus Y$. This can be easily proven by applying d'Alembert ratio test to the series in $\frac{x_1^b}{x_2^a}$

$$\psi := \sum_{m \geq 0} \Gamma[v^j; u(m)] \left(\frac{x_1^b}{x_2^a}\right)^m .$$

Writing $c_m := \Gamma[v^j; u(m)]$ we have

$$\lim_{m \to \infty} \left| \frac{c_{m+1}}{c_m} \right| = \lim_{m \to \infty} \frac{(am)^a}{(bm)^b} = 0.$$

Notice that $\phi_{v^j} \notin \mathcal{O}_X(X \setminus Y)$ if $\frac{\beta - ja}{b} \notin \mathbb{Z}$.

3   Gevrey solutions of $\mathcal{M}_A(\beta)$

The definition of the irregularity (or the irregularity complex) of a left coherent $\mathcal{D}_X$–module $\mathcal{M}$ has been given by Z. Mebkhout [16, (2.1.2), page 98 and (6.3.7)]. In dimension 2 this definition is the following:

Definition 3.1. (a) Let Z be a curve in X. The irregularity of $\mathcal{M}$ along Z (denoted by $\operatorname{Irr}_Z(\mathcal{M})$) is the solution complex of $\mathcal{M}$ with values in $\mathcal{Q}_Z$, i.e.

$$\operatorname{Irr}_Z(\mathcal{M}) := \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{Q}_Z).$$

(b) If Y is a smooth curve in X and for each $1 \leq s \leq \infty$, the irregularity of order s of $\mathcal{M}$ with respect to Y is the complex

$$\operatorname{Irr}_Y^{(s)}(\mathcal{M}) := \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{Q}_Y(s)).$$

Since $\mathcal{O}_{\widehat{X|Y}}(\infty) = \mathcal{O}_{\widehat{X|Y}}$ we have $\operatorname{Irr}_Y^{(\infty)}(\mathcal{M}) = \operatorname{Irr}_Y(\mathcal{M})$. By definition, the irregularity of $\mathcal{M}$ along Z (resp. $\operatorname{Irr}_Y^{(s)}(\mathcal{M})$) is a complex in the derived category $D^b(\mathbb{C}_X)$ and its support is contained in Z (resp. in Y).

In Sections 4 and 5 we will describe the cohomology of the irregularity complex $\operatorname{Irr}_Y(\mathcal{M}_A(\beta))$, and moreover we will compute a basis of the vector spaces

$$\mathcal{H}^i(\operatorname{Irr}_Y^{(s)}(\mathcal{M}_A(\beta)))_p = \mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p$$

for $p \in Y$, $i \in \mathbb{N}$ and $1 \leq s \leq \infty$.

Lemma 3.2. A free resolution of $\mathcal{M}_A(\beta)$ is given by

$$0 \longrightarrow \mathcal{D} \xrightarrow{\psi_1} \mathcal{D}^2 \xrightarrow{\psi_0} \mathcal{D} \xrightarrow{\pi} \mathcal{M}_A(\beta) \longrightarrow 0 \tag{3.1}$$

where ψ_0 is defined by the column matrix $(P, E)^t$, ψ_1 is defined by the row matrix $(E + ab, -P)$ and π is the canonical projection.

Remark 3.3. *For any left $\mathcal{D}_X$–module $\mathcal{F}$ the solution complex $\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_A(\beta), \mathcal{F})$ is represented by*

$$0 \longrightarrow \mathcal{F} \xrightarrow{\psi_0^*} \mathcal{F} \oplus \mathcal{F} \xrightarrow{\psi_1^*} \mathcal{F} \longrightarrow 0$$

where $\psi_0^*(f) = (P(f), E(f))$ and $\psi_1^*(f_1, f_2) = (E + ab)(f_1) - P(f_2)$ for f, f_1, f_2 local sections in $\mathcal{F}$.

4 Description of $\mathrm{Irr}_Y(\mathcal{M}_A(\beta))_{(0,0)}$

Theorem 4.1. *With the previous notations we have $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))_{(0,0)} = 0$ for all $\beta \in \mathbb{C}$ and $1 \leq s \leq \infty$. In other words*

$$\mathcal{E}xt^i(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_{(0,0)} = 0$$

for all $\beta \in \mathbb{C}$, $1 \leq s \leq \infty$ and $i \in \mathbb{N}$. Here the $\mathcal{E}xt$ groups are taken over the sheaf of rings $\mathcal{D}_X$.

Previous result is related to [17, Th. 1].
Let us denote by $V_A(\beta, s)$ and $W_A(\beta, s)$ the vector spaces

$$\left\{ \sum_{\alpha \in \mathbb{N}^2} a_\alpha x^\alpha \in \mathcal{O}_{\widehat{X|Y}}(s)_{(0,0)} : a_\alpha = 0 \text{ if } A\alpha = \beta \right\}, \quad \left\{ \sum_{\alpha \in \mathbb{N}^2} a_\alpha x^\alpha \in \mathcal{O}_{\widehat{X|Y}}(s)_{(0,0)} : a_\alpha = 0 \text{ if } A\alpha \neq \beta \right\}$$

respectively. Notice that $V_A(\beta, s) = \mathcal{O}_{\widehat{X|Y}}(s)_{(0,0)}$ if and only if $\beta \notin a\mathbb{N} + b\mathbb{N}$.

Lemma 4.2. *i) The $\mathbb{C}$–linear map $E_A(\beta) : V_A(\beta, s) \to V_A(\beta, s)$ is an automorphism for all $1 \leq s \leq \infty$ and $\beta \in \mathbb{C}$. In particular, if $\beta \notin a\mathbb{N} + b\mathbb{N}$ then $E_A(\beta)$ is an automorphism of $\mathcal{O}_{\widehat{X|Y}}(s)_{(0,0)}$ for all $1 \leq s \leq \infty$.*
ii) The $\mathbb{C}$–linear map $P : W_A(\beta, s) \to W_A(\beta - ab, s)$ is surjective for all $1 \leq s \leq \infty$ and $\beta \in \mathbb{C}$.

Proof. Part *ii)* is obvious. Let's prove part *i)*. We have $E_A(\beta) = ax_1\partial_1 + bx_2\partial_2 - \beta$ then for $f = \sum_{\alpha \in \mathbb{N}^2} f_\alpha x^\alpha \in \mathbb{C}[[x_1, x_2]]$ we have

$$E_A(\beta)(f) = \sum_{\alpha \in \mathbb{N}^2} f_\alpha (A\alpha - \beta) x^\alpha.$$

This implies that $E_A(\beta)$ is an automorphism of $V_A(\beta, \infty)$. It is also clear that $E_A(\beta)$ is an automorphism of $V_A(\beta, 1)$. For any $1 < s < \infty$ we have $\rho_s E_A(\beta) = E_A(\beta)\rho_s$ and then $E_A(\beta)$ is an automorphism of $V_A(\beta, s)$ (see Section 1 for the definition of ρ_s). $\qquad\square$

Corollary 4.3. *$E_A(\beta)$ is an automorphism of the vector space $\mathcal{Q}_Y(s)_{(0,0)}$ for $1 \leq s \leq \infty$ and $\beta \in \mathbb{C}$.*

Proof. [**Theorem 4.1**] Let us simply write $E := E_A(\beta)$. The complex $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))_{(0,0)}$ is represented by the germ at $(0,0)$ of the following complex

$$0 \longrightarrow \mathcal{Q}_Y(s) \xrightarrow{\psi_0^*} \mathcal{Q}_Y(s) \oplus \mathcal{Q}_Y(s) \xrightarrow{\psi_1^*} \mathcal{Q}_Y(s) \longrightarrow 0$$

where $\psi_0^*(f) = (P(f), E(f))$ and $\psi_1^*(f_1, f_2) = (E + ab)(f_1) - P(f_2)$ for f, f_1, f_2 germs in $\mathcal{Q}_Y(s)$ (see Remark 3.3). In particular, we only need to prove the statement for $i = 0, 1, 2$.

For $i = 0, 2$ the statement follows from Corollary 4.3 Let us see the case $i = 1$. Let us consider $(\overline{f}, \overline{g}) \in \mathrm{Ker}(\psi_1^*)_{(0,0)}$ (i.e. $(E + ab)(\overline{f}) = P(\overline{g})$). We want to prove that there exists $\overline{h} \in \mathcal{Q}_Y(s)_{(0,0)}$ such that $P(\overline{h}) = \overline{f}$ and $E(\overline{h}) = \overline{g}$, where $(\overline{})$ means modulo $\mathcal{O}_{X|Y,(0,0)} = \mathbb{C}\{x\}$.

From Corollary 4.3 we have that there exists a unique $\overline{h} \in \mathcal{Q}_Y(s)_{(0,0)}$ such that $E(\overline{h}) = \overline{g}$. Since $PE = (E + ab)P$ and $(E + ab)(\overline{f}) = P(\overline{g})$ we have:

$$(E + ab)(\overline{f}) = P(\overline{g}) = P(E(\overline{h})) = (E + ab)(P(\overline{h})).$$

Since for all $\beta \in \mathbb{C}$, $E + ab = E_A(\beta - ab)$ is an automorphism of $\mathcal{Q}_Y(s)_{(0,0)}$ (see Corollary 4.3) we also have $\overline{f} = \overline{P(h)}$. So $(\overline{f}, \overline{g}) = (P(\overline{h}), E(\overline{h})) \in \mathrm{Im}(\psi_0^*)_{(0,0)}$. $\qquad\square$

Remark 4.4. *From Theorem 4.1 and the long exact sequence of cohomology associated with the exact sequence (1.1) we have*

$$\mathcal{E}xt^i(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y})_{(0,0)} \simeq \mathcal{E}xt^i(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s))_{(0,0)}$$

for $1 \leq s \leq \infty$, $i \in \mathbb{N}$ and $\beta \in \mathbb{C}$. In fact we have the following two propositions.

Proposition 4.5. *With the previous notations we have $\mathcal{E}xt^i(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s))_{(0,0)} = 0$ for all $\beta \notin a\mathbb{N} + b\mathbb{N}$, $1 \leq s \leq \infty$ and $i \in \mathbb{N}$.*

Proof. The proof is similar to the one of Theorem 4.1 because $E = E_A(\beta)$ is an automorphism of $\mathcal{O}_{\widehat{X|Y}}(s))_{(0,0)}$. □

Proposition 4.6. *With the previous notations we have*

$$\dim_{\mathbb{C}}(\mathcal{E}xt^i(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s))_{(0,0)}) = \begin{cases} 1 & if \quad i = 0, 1 \\ 0 & if \quad i \geq 2 \end{cases}$$

for all $\beta \in a\mathbb{N} + b\mathbb{N}$ and $1 \leq s \leq \infty$. Moreover, $\mathcal{E}xt^i(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s))_{(0,0)}$ is generated by a polynomial ϕ_{v^q} when $i = 0$ and by the class of $(0, \phi_{v^q})$ when $i = 1$ (see the proof of Lemma 5.1 for the definition of ϕ_{v^q}).

Proof. By Remark 3.3 it is enough to consider $i = 0, 1, 2$. Let's treat first the case $i = 2$. Let $h \in \mathcal{O}_{\widehat{X|Y}}(s)_{(0,0)}$ and write $h = h_1 + h_2$ with $h_1 \in V_A(\beta - ab, s)$ and $h_2 \in W_A(\beta - ab, s)$. From Lemma 4.2 there exist $f \in V_A(\beta - ab, s)$ and $g \in W_A(\beta, s)$ such that $(E + ab)(f) = h_1$ and $P(g) = -h_2$. Then $(E + ab)(f) - P(g) = h_1 + h_2 = h$.

Let's see now that the $\mathcal{E}xt^0$ has dimension 1. Assume that $h \in \mathcal{O}_{\widehat{X|Y}}(s)_{(0,0)}$ satisfies $P(h) = E(h) = 0$ and let's write $h = h_1 + h_2$ with $h_1 \in V_A(\beta, s)$ and $h_2 \in W_A(\beta, s)$. We have $E(h_2) = 0$ and then $E(h) = E(h_1) = 0$ implies $h_1 = 0$ because of Lemma 4.2. Now, from $P(h) = P(h_2) = 0$ we get $h_2 = \lambda \phi_{v^q}$ for some $\lambda \in \mathbb{C}$ (see Proposition 5.4).

Finally, let's prove that the $\mathcal{E}xt^1$ has dimension 1. Let's consider $(f, g) \in \mathcal{O}_{\widehat{X|Y}}(s)_{(0,0)}$ such that $(E + ab)(f) = P(g)$. Let's write $f = f_1 + f_2$, $g = g_1 + g_2$ with $f_1 \in V_A(\beta - ab, s)$, $f_2 \in W_A(\beta - ab, s)$, $g_1 \in V_A(\beta, s)$ and $g_2 \in W_A(\beta, s)$. As $(E + ab)(f_2) = 0$ we have $(E + ab)(f) = (E + ab)(f_1) = P(g_1) + P(g_2)$. This implies $P(g_2) = 0$ since $(E + ab)(f_1)$ and $P(g_1)$ belong to $V_A(\beta - ab, s)$. By Lemma 4.2 there exists $h_1 \in V_A(\beta, s)$ such that $E(h_1) = g_1$. We also have $(E + ab)(f_1 - P(h_1)) = (E + ab)(f_1) - PE(h_1) = 0$ and again by Lemma 4.2 we have $(f_1, g_1) = (P(h_1), E(h_1))$.

By Lemma 4.2 there exists $h_2 \in W_A(\beta, a)$ such that $P(h_2) = f_2$. So, $(f_2, g_2) - (P(h_2), E(h_2)) = (0, g_2) = \lambda(0, \phi_{v^q})$ for some $\lambda \in \mathbb{C}$ since $P(g_2) = 0$ (see Proposition 5.4). □

5 Description of $\mathrm{Irr}_Y(\mathcal{M}_A(\beta))_p$ for $p \in Y$, $p \neq (0,0)$

We will compute a basis of the vector space $\mathcal{E}xt^i(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p$ for $1 \leq s \leq \infty$, $i \in \mathbb{N}$, $p \in Y$, $p \neq (0,0)$. In this section we are writing $p = (\epsilon, 0) \in Y$ with $\epsilon \in \mathbb{C}^*$.

We are going to use Γ–series following ([6], [8, Section 1]) and in the way they are handled in [19, Section 3.4].

We will consider the family $v^k = (\frac{\beta - kb}{a}, k) \in \mathbb{C}^2$ for $k = 0, \ldots, a - 1$. They satisfy $Av^k = \beta$ and the corresponding Γ–series are

$$\phi_{v^k} = x^{v^k} \sum_{m \geq 0} \Gamma[v^k; u(m)] x_1^{-bm} x_2^{am} \in x^{v^k} \mathbb{C}[[x_1^{-1}, x_2]]$$

where $u(m) = (-bm, am)$ for $m \in \mathbb{Z}$.

Although ϕ_{v^k} does not define in general any holomorphic germ at $(0,0)$ it is clear that it defines a germ $\phi_{v^k,p}$ in $\mathcal{O}_{\widehat{X|Y},p}$ for $k = 0, 1, \ldots, a - 1$. Let us write $x_1 = t_1 + \epsilon$ and remind that $\epsilon \in \mathbb{C}^*$. We have

$$\phi_{v^k,p} = (t_1 + \epsilon)^{\frac{\beta - bk}{a}} x_2^k \sum_{m \geq 0} \Gamma[v^k; u(m)](t_1 + \epsilon)^{-bm} x_2^{am}.$$

Lemma 5.1. *1. If $\beta \in a\mathbb{N} + b\mathbb{N}$ then there exists a unique $0 \leq q \leq a - 1$ such that ϕ_{v^q} is a polynomial. Moreover, the Gevrey index of $\phi_{v^k,p} \in \mathcal{O}_{\widehat{X|Y},p}$ is $\frac{b}{a}$ for $0 \leq k \leq a - 1$ and $k \neq q$.*

2. If $\beta \notin a\mathbb{N} + b\mathbb{N}$ then the Gevrey index of $\phi_{v^k,p} \in \mathcal{O}_{\widehat{X|Y},p}$ is $\frac{b}{a}$ for $0 \leq k \leq a - 1$.

Proof. The notion of Gevrey index is given in Definition 1.1. Let assume first that $\beta \in a\mathbb{N} + b\mathbb{N}$. Then there exists a unique $0 \leq q \leq a - 1$ such that $\beta = qb + a\mathbb{N}$. Then for $m \in \mathbb{N}$ big enough $\frac{\beta - qb}{a} - bm$ is a negative integer and the coefficient $\Gamma[v^q; u(m)]$ is zero. So ϕ_{v^q} is a polynomial in $\mathbb{C}[x_1, x_2]$ (and then $\phi_{v^q,p}(t_1, x_2)$ is a polynomial in $\mathbb{C}[t_1, x_2]$) since for $\frac{\beta - qb}{a} - bm \geq 0$ the expression

$$x^{v^q} x_1^{-bm} x_2^{am}$$

is a monomial in $\mathbb{C}[x_1, x_2]$.

Let us consider an integer number k with $0 \leq k \leq a - 1$. Assume $\frac{\beta - bk}{a} \notin \mathbb{N}$. Then the formal power series $\phi_{v^k,p}(t_1, x_2)$ is not a polynomial. We will see that its Gevrey index is b/a. It is enough to prove that the Gevrey index of

$$\psi(t_1, x_2) := \sum_{m \geq 0} \Gamma[v^k; u(m)](t_1 + \epsilon)^{-bm} x_2^{am} = \sum_{m \geq 0} \Gamma[v^k; u(m)] \left(\frac{x_2^a}{(t_1 + \epsilon)^b} \right)^m$$

is b/a, i.e. the series

$$\rho_s(\psi(t_1, x_2)) = \sum_{m \geq 0} \frac{\Gamma[v^k; u(m)]}{(am)!^{s-1}} \left(\frac{x_2^a}{(t_1 + \epsilon)^b} \right)^m$$

is convergent for $s = b/a$ and divergent for $s < b/a$.

Considering $\rho_s(\psi(t_1, x_2))$ as a power series in $(x_2^a/(t_1 + \epsilon)^b)$ and writing

$$c_m := \frac{\Gamma[v^k; u(m)]}{(am)!^{s-1}}$$

we have that

$$\lim_{m \to \infty} \left| \frac{c_{m+1}}{c_m} \right| = \lim_{m \to \infty} \frac{(bm)^b}{(am)^{as}}$$

and then by using the d'Alembert's ratio test it follows that the power series $\rho_s(\psi(t_1, x_2))$ is convergent for $b \leq as$ and divergent for $b > as$. $\qquad \square$

Proposition 5.2. *We have $\dim_{\mathbb{C}} \left(\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}})_p \right) = a$ for all $\beta \in \mathbb{C}$, $p \in Y \setminus \{(0,0)\}$.*

Proof. Recall that $p = (\epsilon, 0)$ with $\epsilon \in \mathbb{C}^*$. The operators defining $\mathcal{M}_A(\beta)_p$ are (using coordinates (t_1, x_2)) $P = \partial_1^b - \partial_2^a$ and $E_p(\beta) := at_1\partial_1 + bx_2\partial_2 + a\epsilon\partial_1 - \beta$. We will simply write $E_p = E_p(\beta)$.

First of all, we will prove the inequality

$$\dim_{\mathbb{C}} \left(\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}})_p \right) \leq a.$$

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Assume that $f \in \mathbb{C}[[t_1, x_2]]$, $f \neq 0$, satisfies $E_p(f) = P(f) = 0$. Then choosing $\omega \in \mathbb{R}^2_{>0}$ such that $a\omega_2 > b\omega_1$, we have $\mathrm{in}_{(-\omega,\omega)}(E_p) = a\epsilon\partial_1$ and $\mathrm{in}_{(-\omega,\omega)}(P) = \partial_2^a$, where $\mathrm{in}_{(-\omega,\omega)}(-)$ stands for the initial part with respect to the weights $weight(x_i) = -w_i$, $weight(\partial_i) = w_i$.

Then (see [19, Th. 2.5.5]) $\partial_1(\mathrm{in}_\omega(f)) = \partial_2^a(\mathrm{in}_\omega(f)) = 0$. So, $\mathrm{in}_\omega(f) = \lambda_l x_2^l$ for some $0 \leq l \leq a-1$ and some $\lambda_l \in \mathbb{C}$. This implies the inequality.

Now, remind that

$$\phi_{v^k,p} = (t_1 + \epsilon)^{\frac{\beta - bk}{a}} x_2^k \sum_{m \geq 0} \Gamma[v^k; u(m)](t_1 + \epsilon)^{-bm} x_2^{am}$$

and that the support of such a formal series in $\mathbb{C}[[t_1, x_2]]$ is contained in $\mathbb{N} \times (k + a\mathbb{N})$ for $k = 0, 1, \ldots, a-1$. Then the family $\{\phi_{v^k,p} \mid k = 0, \ldots, a-1\}$ is $\mathbb{C}$-linearly independent and they all satisfy the equations defining $\mathcal{M}_A(\beta)_p$. $\square$

Proposition 5.3. *If $\beta \notin a\mathbb{N} + b\mathbb{N}$ then*

$$\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s))_p = \begin{cases} \sum_{k=0}^{a-1} \mathbb{C}\phi_{v^k,p} & \text{if } s \geq \frac{b}{a} \\ 0 & \text{if } s < \frac{b}{a} \end{cases}$$

for all $p = (\epsilon, 0) \in \mathbb{C}^* \times \{0\}$.

Proof. From the proof of Proposition 5.2 and Lemma 5.1 it follows that any linear combination $\sum_{k=0}^{a-1} \lambda_k \phi_{v^k,p}$ with $\lambda_k \in \mathbb{C}$ has Gevrey index equal to b/a if $\beta \notin a\mathbb{N} + b\mathbb{N}$. $\square$

Proposition 5.4. *If $\beta \in a\mathbb{N} + b\mathbb{N}$ then*

$$\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s))_p = \begin{cases} \sum_{k=0}^{a-1} \mathbb{C}\phi_{v^k,p} & \text{if } s \geq \frac{b}{a} \\ \mathbb{C}\phi_{v^q} & \text{if } s < \frac{b}{a} \end{cases}$$

for all $p = (\epsilon, 0) \in \mathbb{C}^* \times \{0\}$ where q is the unique $k \in \{0, 1, \ldots, a-1\}$ such that $\beta \in kb + a\mathbb{N}$.

Proof. The proof is analogous to the one of Proposition 5.3 and follows from Lemma 5.1. $\square$

Lemma 5.5. *The germ of $E := E_A(\beta)$ at any point $p = (\epsilon, 0) \in \mathbb{C}^* \times \{0\}$ induces a surjective endomorphism on $\mathcal{O}_{\widehat{X|Y}}(s)_p$ for all $\beta \in \mathbb{C}$, $1 \leq s \leq \infty$.*

Proof. We will prove that $E_p : \mathcal{O}_{\widehat{X|Y}}(s)_{(0,0)} \longrightarrow \mathcal{O}_{\widehat{X|Y}}(s)_{(0,0)}$ is surjective (using coordinates (t_1, x_2)). It is enough to prove that $F := \partial_1 + bx_2 u(t_1)\partial_2 - \beta u(t_1)$ yields a surjective endomorphism on $\mathcal{O}_{\widehat{X|Y}}(s)_{(0,0)}$, where $u(t_1) = (a(t_1 + \epsilon))^{-1} \in \mathbb{C}\{t_1\}$. For $s = 1$, the surjectivity of F follows from Cauchy-Kovalevskaya theorem. To finish the proof it is enough to notice that $\rho_s \circ F = F \circ \rho_s$ for $1 \leq s < \infty$. For $s = \infty$ the result is obvious. $\square$

Corollary 5.6. *We have:*

   i) $\mathcal{E}xt^2(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s))_p = 0$ for all $p \in Y$, $p \neq (0,0)$, $\beta \in \mathbb{C}$ and $1 \leq s \leq \infty$.

   ii) $\mathcal{E}xt^2(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p = 0$ for all $p \in Y$, $p \neq (0,0)$, $\beta \in \mathbb{C}$ and $1 \leq s \leq \infty$.

Proof. i) We first consider the germ at p of the solution complex of $\mathcal{M}_A(\beta)$ as described in Remark 3.3 for $\mathcal{F} = \mathcal{O}_{\widehat{X|Y}}(s)$. Then we apply that $E + ab$ is surjective on $\mathcal{O}_{\widehat{X|Y}}(s)_p$ (Lemma 5.5).
ii) It follows from *i)* and the long exact sequence in cohomology associated with (1.1). $\square$

5.1 Computation of $\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p$ for $p \in Y$, $p \neq (0,0)$

Lemma 5.7. *Assume that $f \in \mathbb{C}[[t_1, x_2]]$ satisfies $E_p(f) = 0$. Then $f = \sum_{k=0}^{a-1} f^{(k)}$ where*

$$f^{(k)} = \sum_{m \geq 0} f_{k+am}(t_1 + \epsilon)^{\frac{\beta - bk}{a} - bm} x_2^{k+am}$$

with $f_{k+am} \in \mathbb{C}$.

Proof. Let us sketch the proof. We know that $\text{in}_{(-\omega,\omega)}(E_p)(\text{in}_\omega(f)) = 0$ [19, Th. 2.5.5] for all $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2_{\geq 0}$.

If $\omega_1 > 0$ then $\text{in}_{(-\omega,\omega)}(E_p) = a\epsilon \partial_1$ and so, $\text{in}_\omega(f) \in \mathbb{C}[[x_2]]$ for all ω with $\omega_1 > 0$. On the other hand, if $w_1 = 0$ then $\text{in}_{(-\omega,\omega)}(E_p) = E_p$ and in particular $E_p(\text{in}_{(0,1)}(f)) = 0$ and $\text{in}_\omega(\text{in}_{(0,1)}(f)) \in \mathbb{C}[x_2]$, for all $\omega \in \mathbb{R}^2_{>0}$.

There exists a unique (k, m) with $k \in \{0, \ldots, a-1\}$ and $m \in \mathbb{N}$ such that $\text{in}_{(0,1)}(f) = x_2^{am+k} h(t_1)$ for some $h(t_1) \in \mathbb{C}[[t_1]]$ with $h(0) \neq 0$.

There exists $f_{am+k} \in \mathbb{C}^*$ such that t_1 divides

$$\text{in}_{(0,1)}(f) - f_{am+k}(t_1 + \epsilon)^{\frac{\beta - bk}{a} - bm} x_2^{k+am} \in \mathbb{C}[[t_1]] x_2^{am+k}.$$

But we have

$$E_p(\text{in}_{(0,1)}(f) - f_{am+k}(t_1 + \epsilon)^{\frac{\beta - bk}{a} - bm} x_2^{k+am}) = 0.$$

This implies that $\text{in}_{(0,1)}(f) = f_{am+k}(t_1 + \epsilon)^{\frac{\beta - bk}{a} - bm} x_2^{k+am}$.

We finish by induction by applying the same argument to $f - \text{in}_{(0,1)}(f)$ since $E_p(f - \text{in}_{(0,1)}(f)) = 0$. $\square$

Let's recall that $Y = (x_2 = 0) \subset X = \mathbb{C}^2$ and $v^k = (\frac{\beta - bk}{a}, k)$ for $k = 0, \ldots, a-1$.

Remark 5.8. *As in the proof of Lemma 5.1 if $\beta \in a\mathbb{N} + b\mathbb{N}$ then there exists a unique $0 \leq q \leq a-1$ such that $\beta \in qb + a\mathbb{N}$. Let us write $m_0 = \frac{\beta - qb}{a}$.*

The series ϕ_{v^q} is in fact a polynomial in $\mathbb{C}[x_1, x_2]$ since for $m_0 - bm \geq 0$ the expression $x^{v^q} x_1^{-bm} x_2^{am}$ is a monomial in $\mathbb{C}[x_1, x_2]$.

Let us write m' the smallest integer number satisfying $bm' \geq m_0 + 1$ and

$$\widetilde{v^q} := v^q + u(m') = (m_0 - bm', q + am').$$

Let us notice that $A\widetilde{v^q} = \beta$ and that $\widetilde{v^q}$ does not have minimal negative support (see [19, p. 132-133]) and then the Γ–series $\phi_{\widetilde{v^q}}$ is not a solution of $H_A(\beta)$. We have

$$\phi_{\widetilde{v^q}} = x^{\widetilde{v^q}} \sum_{m \in \mathbb{N};\, bm \geq m_0 + 1} \Gamma[\widetilde{v^q}; u(m)] x_1^{-bm} x_2^{am}.$$

It is easy to prove that $H_A(\beta)_p(\phi_{\widetilde{v^q},p}) \subset \mathcal{O}_{X,p}$ for all $p = (\epsilon, 0) \in X$ with $\epsilon \neq 0$, and that $\phi_{\widetilde{v^q},p}$ is a Gevrey series of index b/a.

Theorem 5.9. *For all $p \in Y \setminus \{(0,0)\}$ and $\beta \in \mathbb{C}$ we have*

$$\dim_{\mathbb{C}}(\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p) = \begin{cases} a & \text{if } \quad s \geq b/a \\ 0 & \text{if } \quad s < b/a \end{cases}$$

Moreover, we also have

i) If $\beta \notin a\mathbb{N} + b\mathbb{N}$ *then:*

$$\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p = \sum_{k=0}^{a-1} \mathbb{C}\overline{\phi_{v^k,p}}$$

for all $s \geq b/a$

ii) If $\beta \in a\mathbb{N} + b\mathbb{N}$ *then for all* $s \geq b/a$ *we have :*

$$\mathcal{E}xt^0(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p = \sum_{k=0,k\neq q}^{a-1} \mathbb{C}\overline{\phi_{v^k,p}} + \mathbb{C}\overline{\phi_{\widetilde{v^q},p}}$$

with $\phi_{\widetilde{v^q}}$ *as in Remark 5.8.*

Here $\overline{\phi}$ *stands for the class modulo* $\mathcal{O}_{X|Y,p}$ *of* $\phi \in \mathcal{O}_{\widehat{X|Y}}(s)_p$.

Proof. It follows from Propositions 5.3 and 5.4, from the proofs of Lemma 5.1 and Proposition 5.2 (by using the long exact sequence in cohomology) and Theorem 5.10 below. $\qquad\square$

5.2 Computation of $\mathcal{E}xt^1(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p$ for $p \in Y$, $p \neq (0,0)$

Theorem 5.10. *For all* $\beta \in \mathbb{C}$ *we have*

$$\mathcal{E}xt^1(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s))_p = 0$$

for all $s \geq b/a$ *and for all* $p \in Y$, $p \neq (0,0)$.

Proof. We will use the germ at p of the solution complex of $\mathcal{M}_A(\beta)$ with values in $\mathcal{F} = \mathcal{O}_{\widehat{X|Y}}(s)$ (see Remark 3.3):

$$0 \to \mathcal{O}_{\widehat{X|Y}}(s) \xrightarrow{\psi_0^*} \mathcal{O}_{\widehat{X|Y}}(s) \oplus \mathcal{O}_{\widehat{X|Y}}(s) \xrightarrow{\psi_1^*} \mathcal{O}_{\widehat{X|Y}}(s) \to 0$$

Let us consider $(f,g) \in (\mathcal{O}_{\widehat{X|Y}}(s)_p)^2$ in the germ at p of $\mathrm{Ker}(\psi_1^*)$, i.e. $(E_p + ab)(f) = P(g)$. We want to prove that there exists $h \in \mathcal{O}_{\widehat{X|Y}}(s)_p$ such that $P(h) = f$ and $E_p(h) = g$.

From Lemma 5.5, there exists $\widehat{h} \in \mathcal{O}_{\widehat{X|Y}}(s)_p$ such that $E_p(\widehat{h}) = g$. Then:

$$(f,g) = (P(\widehat{h}), E_p(\widehat{h})) + (\widehat{f}, 0)$$

where $\widehat{f} = f - P(\widehat{h}) \in \mathcal{O}_{\widehat{X|Y}}(s)_p$ and $(\widehat{f}, 0) \in \mathrm{Ker}(\psi_1^*)$. In order to finish the proof it is enough to prove that there exists $h \in \mathcal{O}_{\widehat{X|Y}}(s)_p$ such that $P(h) = \widehat{f}$ and $E_p(h) = 0$.

Since $h, \widehat{f} \in \mathbb{C}[[t_1, x_2]]$, $(E_p + ab)(\widehat{f}) = 0$ and $E_p(h)$ must be 0, it follows from Lemma 5.7 that

$$h = \sum_{k=0}^{a-1} \sum_{m \geq 0} h_{k+am}(t_1 + \epsilon)^{\frac{\beta - bk}{a} - bm} x_2^{k+am},$$

$$\widehat{f} = \sum_{k=0}^{a-1} \sum_{m \geq 0} f_{k+am}(t_1 + \epsilon)^{\frac{\beta - bk}{a} - b(m+1)} x_2^{k+am}$$

with $h_{k+am}, f_{k+am} \in \mathbb{C}$.

The equation $P(h) = \widehat{f}$ is equivalent to the recurrence relation:

$$h_{k+a(m+1)} = \frac{1}{(k + a(m+1))_a} \left(\left(\frac{\beta - bk}{a} - bm \right)_b h_{k+am} - f_{k+am} \right) \qquad (5.1)$$

for $k = 0, \ldots, a - 1$ and $m \in \mathbb{N}$. The solution to this recurrence relation proves that there exists $h \in \mathbb{C}[[t_1, x_2]]$ such that $P(h) = \widehat{f}$ and $E_p(h) = 0$.

We need to prove now that $h \in \mathcal{O}_{\widehat{X|Y}}(s)_p$.

Dividing (5.1) by $((k + a(m+1))!)^{s-1}$ we get:

$$\frac{h_{k+a(m+1)}}{(k + a(m+1))!^{s-1}} =$$

$$\frac{1}{((k + a(m+1))_a)^s} \left(\left(\frac{\beta - bk}{a} - bm \right)_b \frac{h_{k+am}}{(k + am)!^{s-1}} - \frac{f_{k+am}}{(k + am)!^{s-1}} \right)$$

So it is enough to prove that there exists $C, D > 0$ such that

$$\left| \frac{h_{k+am}}{(k + am)!^{s-1}} \right| \leq CD^m \tag{5.2}$$

for all $0 \leq k \leq a - 1$ and $m \geq 0$. We will argue by induction on m.

Since $\rho_s(\widehat{f})$ is convergent, there exists $\widetilde{C}, \widetilde{D} > 0$ such that

$$\frac{|f_{k+am}|}{(k + am)!^{s-1}} \leq \widetilde{C}\widetilde{D}^m$$

for all $m \geq 0$ and $k = 0, \ldots, a - 1$.

Since $s \geq b/a$, we have

$$\lim_{m \to \infty} \frac{|(\frac{\beta - bk}{a} - bm)_b|}{((k + a(m+1))_a)^s} \leq (b/a)^b$$

and then there exists an upper bound $C_1 > 0$ of the set

$$\left\{ \frac{|(\frac{\beta - bk}{a} - bm)_b|}{((k + a(m+1))_a)^s} : m \in \mathbb{N} \right\}.$$

Let us consider

$$C = \max\{\widetilde{C}, \frac{|h_k|}{k!^{s-1}} ; k = 0, \ldots, a - 1\}$$

and

$$D = \max\{\widetilde{D}, C_1 + 1\}.$$

So, the case $m = 0$ of (5.2) follows from the definition of C. Assume $|\frac{h_{k+am}}{(k+am)!^{s-1}}| \leq CD^m$. We will prove inequality (5.2) for $m + 1$. From the recurrence relation we deduce:

$$\left| \frac{h_{k+a(m+1)}}{(k + a(m+1))!^{s-1}} \right| \leq C_1 \left| \frac{h_{k+am}}{(k + am)!^{s-1}} \right| + \widetilde{C}\widetilde{D}^m$$

and using the induction hypothesis and the definition of C, D we get:

$$\left| \frac{h_{k+a(m+1)}}{(k + a(m+1))!^{s-1}} \right| \leq (C_1 + 1)CD^m \leq CD^{m+1}.$$

In particular $\rho_s(h)$ converges and $h \in \mathcal{O}_{\widehat{X|Y}}(s)_p$. $\qquad \square$

Lemma 5.11. *Assume that $h \in \mathcal{O}_{\widehat{X|Y},p}$, $p \in Y$, $p \neq (0, 0)$, satisfies $E(h) = 0$ and $P(h) \in \mathcal{O}_{\widehat{X|Y}}(s)_p$ with $s < b/a$. Then:*

i) If $\beta \notin a\mathbb{N} + b\mathbb{N}$ there exists $g \in \mathcal{O}_{\widehat{X|Y}}(s)_p$ with $P(h) = P(g)$ and $E(g) = 0$.

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ii) If $\beta \in a\mathbb{N} + b\mathbb{N}$ there exists $g \in \mathcal{O}_{\widehat{X|Y}}(s)_p$ with $P(h) = P(g + \lambda_q \phi_{\widehat{v^q},p})$ and $E(g) = 0$.

Proof. Since $E(h) = 0$ then $(E + ab)(\widehat{f}) = 0$ for $\widehat{f} := P(h)$. Reasoning as in the proof of Theorem 5.10 we have the recurrence relation (5.1) for the coefficients of h and $\widehat{f}$. Let us prove first that for all $k = 0, \ldots, a-1$ such that $\frac{\beta - bk}{a} \notin \mathbb{N}$ there exists $\lambda_k \in \mathbb{C}$ with $h^{(k)} - \lambda_k \phi_{v^k,p} \in \mathcal{O}_{\widehat{X|Y}}(s)_p$.

Since $h_k x_1^{\frac{\beta - bk}{a}} x_2^k$ is holomorphic in a neighborhood of p and $E(h_k x_1^{\frac{\beta - bk}{a}} x_2^k) = 0$ we can assume without loss of generality that $h_k = 0$ obtaining:

$$h_{k+a(m+1)} = -\frac{(\frac{\beta - bk}{a})_{b(m+1)}}{(k + a(m+1))!} \sum_{r=0}^{m} \frac{(k+ar)!}{(\frac{\beta - bk}{a})_{b(r+1)}} f_{k+ar}. \tag{5.3}$$

Recall that the coefficient of $x_1^{\frac{\beta - bk}{a} - bm} x_2^{k+am}$ in $\phi_{v^k,p}$ is

$$\Gamma[v^k; u(m)] = \frac{(\frac{\beta - bk}{a})_{bm} k!}{(k+am)!}.$$

Therefore for all $\lambda_k \in \mathbb{C}$ we get:

$$h_{k+a(m+1)} - \lambda_k \Gamma[v^k; u(m+1)] = \frac{(\frac{\beta - bk}{a})_{b(m+1)}}{(k + a(m+1))!}\left(-k!\lambda_k - \sum_{r=0}^{m} \frac{(k+ar)! f_{k+ar}}{(\frac{\beta - bk}{a})_{b(r+1)}}\right).$$

Since $as < b$ we can choose

$$\lambda_k = -\sum_{r \geq 0} \frac{(k+ar)! f_{k+ar}}{k!(\frac{\beta - bk}{a})_{b(r+1)}} \in \mathbb{C}$$

and, because $f^{(k)} \in \mathcal{O}_{\widehat{X|Y}}(s)_p$, there exist real numbers $C > 0, D > 0$ such that $|f_{k+ar}| \leq CD^r(k + ar)!^{s-1}$ for all $r \geq 0$. Then:

$$h_{k+a(m+1)} - \lambda_k \Gamma[v^k; u(m+1)] = \frac{(\frac{\beta - bk}{a})_{b(m+1)}}{(k + a(m+1))!} \sum_{r \geq m+1} \frac{(k+ar)! f_{k+ar}}{(\frac{\beta - bk}{a})_{b(r+1)}}.$$

Equivalently,

$$h_{k+a(m+1)} - \lambda_k \Gamma[v^k; u(m+1)] = \sum_{r \geq 0} \frac{(k+a(r+m+1))_{ar} f_{k+a(r+m+1)}}{(\frac{\beta - bk}{a} - (m+1)b)_{b(r+1)}}.$$

The series

$$g_m(z) = \sum_{r \geq 0} \frac{(k+a(r+m+1))_{ar}^s}{|(\frac{\beta - bk}{a} - (m+1)b)_{b(r+1)}|} z^r$$

is an entire function in the variable z for all $m \geq 0$. To prove that it is enough to apply the d'Alembert's ratio test using $b > sa$:

$$\lim_{r \to \infty} \frac{(k+a(r+m+1))^s(k+a(r+m+1)-1)^s \cdots (k+a(r+m)+1)^s}{|\frac{\beta - bk}{a} - b(r+m+1)| \cdots |\frac{\beta - bk}{a} - (r+m+2)b+1|} =$$

$$\lim_{r \to \infty} \frac{(ar)^{as}}{(br)^b} = 0.$$

In particular, $0 < g_m(D) < \infty$ and

$$|h_{k+a(m+1)} - \lambda_k \Gamma[v^k; u(m+1)]| \leq Cg_m(D)D^{m+1}(k+am)!^{s-1}.$$

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It can be proved (by using elementary properties of the Pochhammer symbol and standard estimates) that there exists $m_2, \widehat{C} \in \mathbb{N}$ such that $\forall m \geq m_2$, $g_{m+1}(D) \leq \widehat{C}g_m(D)$. This implies that $|g_{m+1}(D)| \leq \widehat{C}^{m+1-m_2}g_{m_2}(D)$, $\forall m \geq m_2$. Then, taking $\widetilde{C} = \widehat{C}^{-m_2}g_{m_2}(D) > 0$, we obtain:

$$|h_{k+a(m+1)} - \lambda_k\Gamma[v^k; u(m+1)]| \leq \widetilde{C}(\widehat{C}D)^{m+1}(k+am)!^{s-1}$$

for all $m \geq 0$. Hence $h^{(k)} - \lambda_k\phi_{v^k,p} \in \mathcal{O}_{\widehat{X|Y}}(s)_p$. If $\beta \notin a\mathbb{N} + b\mathbb{N}$ then we have that $g := h - \sum_{k=0}^{a-1}\lambda_k\phi_{v^k,p} \in \mathcal{O}_{X|Y}(s)_p$ satisfies statement $i)$.

$ii)$ If $\beta \in a\mathbb{N} + b\mathbb{N}$ there exists a unique $q \in \{0, 1, \ldots, a-1\}$ verifying $\frac{\beta-bq}{a} = m_0 \in \mathbb{N}$. For $k \neq q$ we have as before that $h^{(k)} - \lambda_k\phi_{v^k,p} \in \mathcal{O}_{\widehat{X|Y}}(s)_p$. For $k = q$ we can assume without loss of generality that $h_{q+am} = 0$ for $m = 0, 1, \ldots, [m_0/b]$ (here $[-]$ denotes the integer part) obtaining an expression for $h_{q+a(m+1)}$ similar to Equality (5.3) for $m \geq [m_0/b]$. Then by using $\phi_{\widetilde{v^q}}$ instead of ϕ_{v^q} we get, alike in $i)$, that $h - \lambda_q\phi_{\widetilde{v^q}} \in \mathcal{O}_{\widehat{X|Y}}(s)_p$. Hence $g := h - \sum_{k=0, k\neq q}^{a-1}\lambda_k\phi_{v^k,p} - \lambda_q\phi_{\widetilde{v^q}} \in \mathcal{O}_{\widehat{X|Y}}(s)_p$ satisfies $ii)$. $\quad\square$

Theorem 5.12. *We have*

$$\dim_{\mathbb{C}}(\mathcal{E}xt^1(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s))_p) = \begin{cases} 1 & \text{for} \quad \beta \in a\mathbb{N} + b\mathbb{N} \\ 0 & \text{for} \quad \beta \notin a\mathbb{N} + b\mathbb{N} \end{cases}$$

for all $p \in Y$, $p \neq (0,0)$ and $1 \leq s < \frac{b}{a}$. Moreover, if $\beta \in a\mathbb{N} + b\mathbb{N}$ then $\mathcal{E}xt^1(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s))_p$ is generated by the class of $(P(\phi_{\widetilde{v^q},p}), 0)$.

Proof. By definition

$$\mathcal{E}xt^1(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s))_p = \frac{\{(f, g) \in (\mathcal{O}_{\widehat{X|Y}}(s)_p)^2 : \ (E + ab)(f) = P(g)\}}{\{(P(h), E(h)) : \ h \in \mathcal{O}_{\widehat{X|Y}}(s)_p\}}$$

As in the proof of Theorem 5.10 we can assume $g = 0$ and then $(E + ab)(f) = 0$. This implies that $f = \sum_{k=0}^{a-1}f^{(k)}$ (see Lemma 5.7) with

$$f^{(k)} = \sum_{m \geq 0} f_{k+am}x_1^{\frac{\beta-bk}{a}-b(m+1)}x_2^{k+am}$$

We can then consider $h \in \mathcal{O}_{\widehat{X|Y},p}$ as in (5.3) such that $P(h) = f$ and $E(h) = 0$ and apply Lemma 5.11. Furthermore, it is easy to prove that $P(\phi_{\widetilde{v^q}})$ is a Laurent monomial term with pole along $\{x_1 = 0\}$ and hence holomorphic at any point $p \in Y \setminus \{(0,0)\}$. This finishes the proof. $\quad\square$

Remark 5.13. *Notice that the generator $(P(\phi_{\widetilde{v^q}}), 0)$ does not define a germ at the origin although the dimension of $\mathcal{E}xt^1(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s))_{(0,0)}$ is one (see Proposition 4.6). Nevertheless, it can be checked that the class of $(0, \phi_{v^q})$ is a generator of $\mathcal{E}xt^1(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s))_p$ at any point of $p \in Y$.*

Proposition 5.14. *For all $\beta \in \mathbb{C}$ we have*

$$\mathcal{E}xt^1(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s)) = 0$$

for all $1 \leq s \leq \infty$.

Proof. Since $\mathcal{E}xt^1(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p = 0$ for $p = (0,0)$ (see Section 4) it is enough to prove the equality for all $p \in Y \setminus \{(0,0)\}$.

From Corollary 5.6 (for $s = 1$), Theorem 5.10 and the long exact sequence in cohomology we get the equality for $s \geq b/a$. Using again Corollary 5.6 (for $s = 1$), Theorem 5.12, Theorem 5.9 (only necessary in the case $\beta \in a\mathbb{N} + b\mathbb{N}$) and the long exact sequence in cohomology we get the equality for $1 \leq s < b/a$. $\quad\square$

Remark 5.15. *We can also prove, with similar methods to the ones presented in this paper, that the irregularity complex* $\mathrm{Irr}_Z(\mathcal{M}_A(\beta))$ *is zero for any* $\beta \in \mathbb{C}$ *and for* $A = (-a\ b)$ *with* a, b *strictly positive integer numbers and* $\gcd(a, b) = 1$. *Here* Z *is either* $x_1 = 0$ *or* $x_2 = 0$. *Let us notice that the characteristic variety of* $\mathcal{M}_A(\beta)$ *is defined by the ideal* $(\xi_1\xi_2, -ax_1\xi_1 + bx_2\xi_2)$ *and then its singular support is the union of the two coordinates axes in* $\mathbb{C}^2$.

Conclusions

1) In Sections 4 and 5 we have proved that the irregularity complex $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ is zero for $1 \leq s < b/a$ and is concentrated in degree 0 for $b/a \leq s \leq \infty$ (see Theorems 4.1 and 5.9 and Proposition 5.14). Moreover, the description of a basis of $\mathcal{E}xt^0_{\mathcal{D}_X}(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))_p$ for $p \in Y$, $p \neq (0,0)$, and $b/a \leq s \leq \infty$ (see Theorem 5.9) proves that the cohomology of the complex $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ is *constructible* on Y, with respect to the stratification given by $\{\{(0,0)\}, Y \setminus \{(0,0)\}\}$. This can be also deduced from a Theorem of M. Kashiwara [11]. From the form of the basis it is also easy to see that the eigenvalues of the corresponding monodromy are simply $\exp(\frac{2\pi i(\beta - bk)}{a})$ for $k = 0, \ldots, a-1$. Notice that for $\beta \in \mathbb{Z}$ one eigenvalue (the one corresponding to the unique $k = 0, \ldots, a-1$ such that $\frac{\beta - bk}{a} \in \mathbb{Z}$) is just 1. See Subsection 5.1 for notations.

2) From the previous results we can also prove that the complex $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ is a *perverse sheaf* on Y for any $1 \leq s \leq \infty$. This is a very particular case of a general result of Z. Mebkhout [16, Th. 6.3.3]. To this end, as $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ is concentrated in degree 0, it is enough to prove the *co-support* condition, which is equivalent (see [2]) to prove that the hypercohomology $\mathcal{H}^0_{\{p\}}(\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta)))$ with support on $\{p\}$ is zero, for $p \in Y$. This is obvious because $\mathcal{E}xt^0_{\mathcal{D}_X}(\mathcal{M}_A(\beta), \mathcal{Q}_Y(s))$ has no sections supported on points.

3) We have also proved (see Theorem 5.9 and Proposition 5.14) that the *Gevrey filtration* $\mathrm{Irr}_Y^{(s)}(\mathcal{M}_A(\beta))$ has a unique gap for $s = b/a$. So the only *analytic slope* of $\mathcal{M}_A(\beta)$ with respect to Y is b/a [16, Déf. 6.3.7]. On the other hand it is also known (see [9, Th. 3.3] and [3]) that the only *algebraic slope* of $\mathcal{M}_A(\beta)$ is also b/a. This fact is a very particular case of the slope comparison theorem of Y. Laurent and Z. Mebkhout [12, Th. 2.4.2].

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