

# On the singular braid monoid of an orientable surface

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## Abstract

In this paper we show that the singular braid monoid of an orientable surface can be embedded in a group. The proof is purely topological, making no use of the monoid presentation.

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## 1 Introduction

The aim of this paper is proving, in an easy and fully topological way, that the singular braid monoid of an orientable surface can be embedded into a group.

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The singular braid monoids appear naturally when studying braid groups of surfaces. They were introduced in [1] and [4] and they arise in some situations connected with the Vassiliev knot invariants ([2]).

A particular case of our theorem (that of the open disk) was proved by Fenn, Keyman and Rourke in [6], where it was pointed out that all the topological difficulties of the proof relied on the so-called diamond lemma. Their proof, although purely topological, was both quite involved and not adaptable (at least in an easy way) for general orientable surfaces. On the other hand our approach results in a simpler proof, even for the open disk case.

An algebraic generalization of this result can be found in [3], which in fact may include the surface case if applied together with the results of [7] concerning the monoid presentation. However, we think that a topological and more down-to-earth proof of this result may shed some light and contribute towards a better understanding of the subject.

A final comment is in order : a different proof of this result, also based in the monoid presentation, has been announced to us by Bellingeri in a private communication.

## 2 The singular braid monoid and the singular braid group

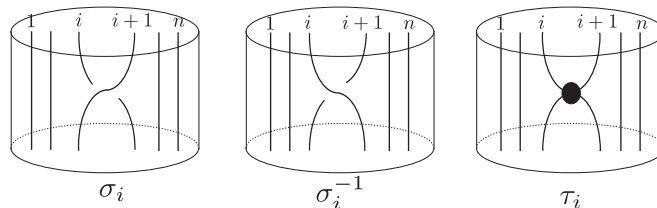
**Definition.**— Let  $U$  be any orientable surface. If we fix beforehand  $n$  distinct points  $P_1, \dots, P_n \in U$ , a geometric braid is a set of  $n$  differentiable disjoint paths  $b_1, \dots, b_n$  inside the cylinder  $U \times [0, 1]$ , such that  $b_i$  (also called the  $i$ -th string) goes, monotonically in  $t \in [0, 1]$ , from  $(P_i, 0)$  to  $(P_j, 1)$ .

A singular geometric braid is just the same as a geometric braid, except for the fact that we allow a finite number of transversal intersections between two strings. The intersection points will be called singular points.

If we consider the braids modulo isotopies of  $U \times [0, 1]$  leaving fixed  $U \times \{0, 1\}$ , we get the singular braid monoid (of  $U$ ), noted  $SB_n$ , with the operation defined by concatenation.

We will note by  $\alpha, \beta, \gamma, \dots$  the geometric braids and by  $\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \dots$  their corresponding classes in  $SB_n$ .

When  $U$  is the open disk  $D$ ,  $SB_n$  is a very well-known monoid ([5]), generated by the braids  $\widehat{\sigma}_i, \widehat{\sigma}_i^{-1}, \widehat{\tau}_i$ , where  $\sigma_i, \sigma_i^{-1}, \tau_i$  are the geometric braids shown below. We will also speak of  $\widehat{\sigma}_i, \widehat{\sigma}_i^{-1}, \widehat{\tau}_i$  in the general case, by means of the natural embedding of the singular braid monoid of the disk into  $SB_n$ .



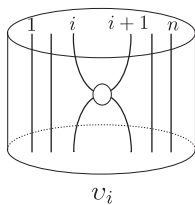
In what follows, we will draw braids only in the cylinder  $D \times [0, 1]$  (for the sake of simplicity), although our arguments will work for  $U \times [0, 1]$ .

We will introduce now the analogous setup to that of Fenn, Keyman and Rourke in [6] for the disk case.

**Definition.**— We will call  $M$  the monoid where the elements are geometric singular braids (modulo isotopies) in which every singular point is assigned a colour, black or white.

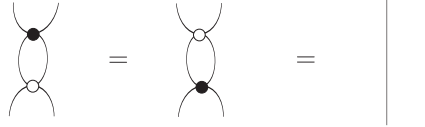
We will then note by  $\alpha, \beta, \gamma, \dots$  the geometric braids (with coloured singular points) and by  $\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \dots$  their corresponding classes in  $M$ .

**Remark.**— There is a surjection from  $M$  to  $SB_n$  which consists on forgetting the colours of the singular points. On the other hand,  $SB_n$  can be embedded in  $M$  by assigning the black colour to every singular point. The image of  $\widehat{\tau}_i$  under this injection will be noted also by  $\widehat{\tau}_i$ , while we will note by  $v_i$  the braid obtained by assigning the white colour to the singular point of  $\tau_i$ . We will call  $\widehat{v}_i$  the opposite of  $\widehat{\tau}_i$ .



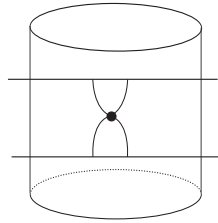
Without explicit mention we will consider  $SB_n$  as a submonoid of  $M$ , using the above injection.

**Definition.**— If we add to  $M$  the relations  $\widehat{\tau}_i \widehat{v}_i = \widehat{v}_i \widehat{\tau}_i = 1$ , we obtain a group, called the singular braid group (of  $U$ ), noted  $SG_n$ .



The relations  $\widehat{\tau}_i \widehat{v}_i = \widehat{v}_i \widehat{\tau}_i = 1$  in  $SG_n$ .

From now on, if a singular braid has the form, say  $\widehat{\alpha} = \alpha_1 \widehat{\tau}_i \alpha_2$ , and we do not care about who  $\alpha_1$  and  $\alpha_2$  are, we will draw it as follows:



### 3 The embedding theorem

**Theorem.**— The natural map  $SB_n \rightarrow SG_n$  is one-to-one.

Before proving the result we will introduce a few notations. For  $\alpha, \beta$  geometric singular braids, we will note  $\alpha \nearrow \beta$  if there exist  $\alpha_1, \beta_1$  such that  $\widehat{\alpha} = \widehat{\alpha}_1, \widehat{\beta} = \widehat{\beta}_1$  and we can obtain  $\beta_1$  from  $\alpha_1$  by adding a pair of consecutive opposite singular points. In the same way, we will note  $\alpha \searrow \beta$  if there exist  $\alpha_1, \beta_1$  such that  $\widehat{\alpha} = \widehat{\alpha}_1, \widehat{\beta} = \widehat{\beta}_1$  and we can obtain  $\beta_1$  from  $\alpha_1$  by erasing a pair of consecutive opposite singular points.

**Remark.**— Note that  $\widehat{\alpha}$  and  $\widehat{\beta}$  define the same element on  $SG_n$  if and only if there exist  $\alpha = \alpha_0, \alpha_1, \dots, \alpha_k = \beta$  such that either  $\alpha_i \nearrow \alpha_{i+1}$  or  $\alpha_i \searrow \alpha_{i+1}$  for all  $i = 0, \dots, k - 1$ .

The following result is the key part of the proof of the theorem, as noted in [6], where it is proved for the case  $U = D$ . The proof will be given in the following section.

**Diamond lemma.**— Let  $\alpha, \beta, \gamma$  be geometric singular braids such that  $\alpha \nearrow \beta \searrow \gamma$ . Then, either  $\widehat{\alpha} = \widehat{\gamma}$ , or there exists  $\eta$  such that  $\alpha \searrow \eta \nearrow \gamma$ .

**Definition.**— We will say  $\widehat{\alpha} \in M$  is irreducible if there is no  $\beta$  with  $\alpha \searrow \beta$ .

**Corollary.**— If  $\widehat{\alpha}, \widehat{\beta} \in M$  are irreducible and define the same element in  $SG_n$ , then  $\widehat{\alpha} = \widehat{\beta}$  in  $M$ .

The proof is immediate from the diamond lemma (see [6]) and, besides, as the elements of  $SB_n$  are irreducible (they only have black singular points), this proves that the map from  $SB_n$  to  $SG_n$  is an embedding. Hence the theorem is proved, up to the diamond lemma.

## 4 The diamond lemma

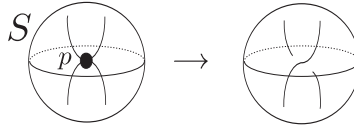
Given a geometric singular braid  $\beta$ , with a singular point  $p$  on it, we will denote by  $\beta(p^+)$  and  $\beta(p^-)$  the braid obtained from  $\beta$  by replacing  $p$  by a positive and negative crossing, respectively. That is, if  $\beta = \beta_1 \tau_i \beta_2$ , (where  $p$  is the singular point in  $\tau_i$ ) we will have  $\beta(p^+) = \beta_1 \sigma_i \beta_2$  and  $\beta(p^-) = \beta_1 \sigma_i^{-1} \beta_2$ . In a similar way, if we have  $p_1, \dots, p_m$  singular points on  $\beta$ , we will write  $\beta(p_1^{s_1}, \dots, p_m^{s_m})$  the braid obtained by replacing each  $p_i$  by a crossing with sign  $s_i \in \{+, -\}$ .

If we have a singular braid  $\beta$ , with a singular point  $p$  on it, we will write  $\beta(p^\bullet)$  and  $\beta(p^\circ)$  the braid obtained from  $\beta$  by replacing  $p$  (no matter which colour it has) by a black point and a white point, respectively.

**Lemma 1.**— Let  $\beta$  and  $\beta'$  be geometric singular braids,  $p$  a singular point on  $\beta$ . Assume  $\widehat{\beta} = \widehat{\beta}'$  and let us call also  $p$  the point corresponding to  $p$  in  $\beta'$ . Then:

1.  $\widehat{\beta(p^+)} = \widehat{\beta'(p^+)}$ .
2.  $\widehat{\beta(p^-)} = \widehat{\beta'(p^-)}$ .

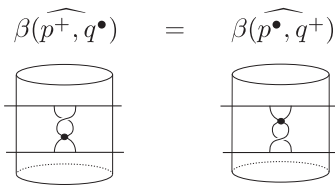
**Proof.**— As both cases are analogous, we will do the first case only. As  $\widehat{\beta} = \widehat{\beta}'$ , there exists an isotopy of the cylinder,  $H_t$ , with  $H_0(\beta) = \beta$ ,  $H_1(\beta) = \beta'$ . Take a sphere  $S$  centered at  $p$  with radius small enough such that the only strings intersecting  $S$  are those forming the singular point. We can assume that  $\beta(p^+)$  and  $\beta$  coincide outside  $S$ .



We can suppose that  $H_1(S)$  is also a sphere centered at  $p$  on  $\beta'$ . Hence, if we apply  $H_1$  to  $\beta(p^+)$ , we obtain the braid  $\beta'$ , except for the fact that the point  $p$  has been replaced by a positive crossing (since  $U$  is orientable). That is, we get  $\beta'(p^+)$ . So  $H_1(\beta(p^+)) = \beta'(p^+)$ .

**Lemma 2.**— Let  $\widehat{\beta} \in M$ , having two consecutive singular points,  $p$  and  $q$ . Then  $\beta(\widehat{p^s}, \widehat{q^c}) = \beta(\widehat{p^c}, \widehat{q^s})$ , with  $s \in \{+, -\}$ ,  $c \in \{\bullet, \circ\}$ .

**Proof.**— This result is a straightforward consequence of the well known relation  $\sigma_i \tau_i = \tau_i \sigma_i$  in the singular braid monoid of the disk.



We can now proceed to prove the diamond lemma. As  $\alpha \nearrow \beta$  we can consider, with no loss of generality, that  $\alpha = \beta(p^+, q^-)$  for some pair  $p, q$  of opposite consecutive points in  $\beta$ . On the other side, as  $\beta \searrow \gamma$ , let us call  $\beta'$  the braid verifying  $\widehat{\beta} = \widehat{\beta}'$  and, as above,  $\gamma = \beta'(r^+, s^-)$  for a pair  $r, s$  of opposite consecutive points in  $\beta'$ .

Let us call  $H$  the cylinder isotopy taking  $\beta$  into  $\beta'$ . Then, using our previous notation, we can write  $\alpha = \beta(p^+, q^-) \nearrow \beta \xrightarrow{H} \beta' \searrow \beta'(r^+, s^-) = \gamma$ .

The technique used for the proof is heavily related to the proof of lemma 1. In fact, we will make an extensive use of the fact that  $H$  brings together  $r$  and  $s$ , while possibly moving apart  $p$  and  $q$ , and  $H^{-1}$  acts the other way around. This fact, together with the using of the small spheres as in lemma 1, will give us the result.

We will assume that  $p$  is above  $q$  and  $r$  is above  $s$  from now on. Then we need to distinguish three cases:

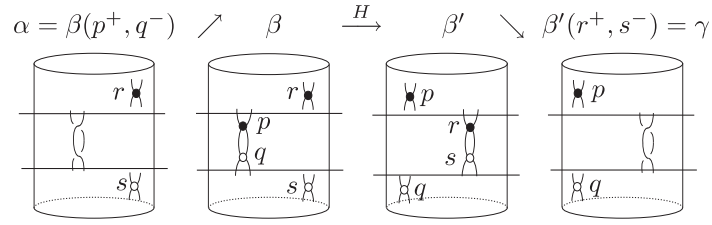
- (a)  $p = r, q = s$ .

(b)  $p, q, r, s$  are all distinct.

(c)  $p = s$  or  $q = r$ .

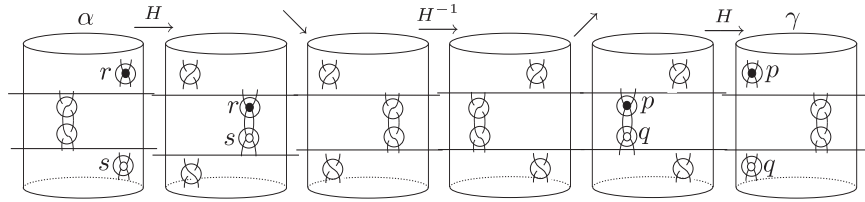
**Case (a)** This is the easiest situation. As  $\widehat{\beta} = \widehat{\beta}'$ , from lemma 1 we have  $\widehat{\beta(p^+)} = \widehat{\beta'(p^+)}$ . Hence  $\widehat{\alpha} = \widehat{\beta(p^+, q^-)} = \widehat{\beta'(p^+, q^-)} = \widehat{\gamma}$ .

**Case (b)** In this case  $\beta$  has two pairs of opposite singular points,  $(p, q)$  and  $(r, s)$ , of which  $(p, q)$  are consecutive. Hence  $\beta$  can be assumed to have the form  $\beta_1 \tau_i v_i \beta_2$ , and the analogous thing happens with  $\beta'$  and  $(r, s)$ .



Then we can write

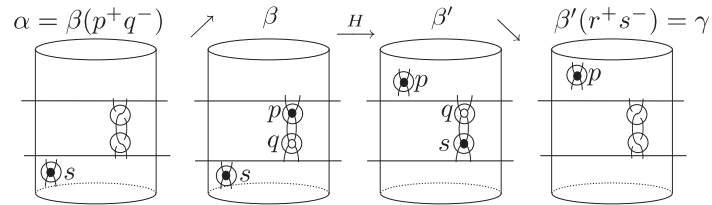
$$\begin{aligned} \alpha &= \beta(p^+, q^-) \xrightarrow{H} \beta'(p^+, q^-) \searrow \beta'(p^+, q^-, r^+, s^-) \xrightarrow{H^{-1}} \\ &\xrightarrow{H^{-1}} \beta(p^+, q^-, r^+, s^-) \nearrow \beta(r^+, s^-) \xrightarrow{H} \beta'(r^+, s^-) = \gamma. \end{aligned}$$



This proves the existence of  $\eta = \beta'(p^+, q^-, r^+, s^-)$  with  $\alpha \searrow \eta \nearrow \gamma$ .

**Case (c)** We will do the subcase  $q = r$ , the other one being symmetric.

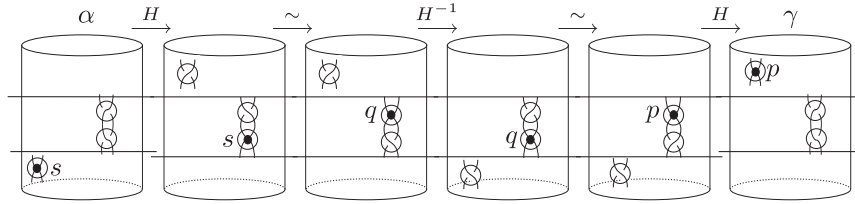
We have



Then, from lemma 2,

$$\begin{aligned} \alpha = \beta(p^+, q^-) &\xrightarrow{H} \beta'(p^+, q^-) = \beta'(p^+, q^-, s^\bullet) \xrightarrow{\sim} \beta'(p^+, q^\bullet, s^-) \xrightarrow{H^{-1}} \\ &\xrightarrow{H^{-1}} \beta(p^+, q^\bullet, s^-) \xrightarrow{\sim} \beta(p^\bullet, q^+, s^-) \xrightarrow{H} \beta'(p^\bullet, q^+, s^-) = \gamma, \end{aligned}$$

where  $\xrightarrow{\sim}$  stands for the (non-specified) isotopies which come from applying the lemma 2. This proves  $\hat{\alpha} = \hat{\gamma}$  and concludes the proof of the diamond lemma.



## References

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