

# ON THE UBIQUITY OF TRIVIAL TORSION ON ELLIPTIC CURVES

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ABSTRACT. The purpose of this paper is to give a *down-to-earth* proof of the well-known fact that a randomly chosen elliptic curve over the rationals is most likely to have trivial torsion.

## 1. INTRODUCTION

Let us consider an elliptic curve  $E$ , defined over the rationals and written in short Weierstrass form

$$(1) \quad E : Y^2 = X^3 + AX + B, \quad A, B \in \mathbb{Z}.$$

We will use the standard notations for:

- $\Delta = -16(4A^3 + 27B^2) \neq 0$ , the discriminant of  $E$ ;
- $E(\mathbb{Q})$ , the finitely generated abelian group of rational points on  $E$ , and
- $\mathcal{O}$ , the identity element of  $E(\mathbb{Q})$ .

Given  $P \in E(\mathbb{Q})$ , we will also write as customary  $[m]P$  for the point resulting after adding  $m$  times  $P$ .

The problem of computing the torsion of  $E(\mathbb{Q})$  has been solved in a lot of very efficient ways [2, 3, 6], and most computer packages (say **Maple-Apecs**, **PARI/GP**, **Magma** or **Sage**) calculate the torsion of curves with huge coefficients in very few seconds. The major result which made this possible (along with others, like the Nagell–Lutz Theorem ([18],[15]) or the embedding theorem for good reduction primes (see, for example, [21, VIII.7] or [12, Chap. 5])) was Mazur’s Theorem [16, 17] who listed the fifteen possible torsion groups.

In the above papers, it is proved that the possible structures of the torsion group of  $E(\mathbb{Q})$  are

$$\mathbb{Z}/n\mathbb{Z} \text{ for } n = 2, \dots, 10, 12, \quad \text{or} \quad \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2n\mathbb{Z} \text{ for } n = 1, \dots, 4.$$

Besides, the fifteen of them actually happen as torsion subgroups of elliptic curves. Notice that thanks to the above theorem, the possible prime orders for a torsion point defined over  $\mathbb{Q}$  are 2, 3, 5 or 7.

Let  $p$  be a prime number and let  $E[p]$  be the group of points of order  $p$  on  $E(\overline{\mathbb{Q}})$ , where  $\overline{\mathbb{Q}}$  denotes an algebraic closure of  $\mathbb{Q}$ . The action of the

absolute Galois group  $G_{\mathbb{Q}} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  on  $E[p]$  defines a mod  $p$  Galois representation

$$\rho_{E,p} : G_{\mathbb{Q}} \rightarrow \text{Aut}(E[p]) \cong \text{GL}_2(\mathbb{F}_p).$$

Let  $\mathbb{Q}(E[p])$  be the number field generated by the coordinates of the points of  $E[p]$ . Therefore, the Galois extension  $\mathbb{Q}(E[p])/\mathbb{Q}$  has Galois group

$$\text{Gal}(\mathbb{Q}(E[p])/\mathbb{Q}) \cong \rho_{E,p}(G_{\mathbb{Q}}).$$

The prime  $p$  is called exceptional for  $E$  if  $\rho_{E,p}$  is not surjective. If  $E$  has complex multiplication then any odd prime number is exceptional. On the other hand, if  $E$  does not have complex multiplication then Serre [20] proved that  $E$  has only finitely many exceptional primes.

Duke [4] proved that *almost all* elliptic curves over  $\mathbb{Q}$  have no exceptional primes. More precisely, given an elliptic curve  $E$  in a short Weierstrass form as in (1), the height of the elliptic curve is defined as

$$H(E) = \max(|A|^3, |B|^2).$$

Let  $M$  be a positive integer, and let  $\mathcal{C}_H(M)$  be the set of elliptic curves  $E$  with  $H(E) \leq M^6$ . For any prime  $p$  denote by  $\mathcal{E}_p(M)$  the set of elliptic curves  $E \in \mathcal{C}_H(M)$  such that  $p$  is an exceptional prime for  $E$ , and by  $\mathcal{E}(M)$  the union of  $\mathcal{E}_p(M)$  for all primes. Actually in both sets the elliptic curves were considered up to  $\mathbb{Q}$ -isomorphisms. Duke then proved that

$$\lim_{M \rightarrow \infty} \frac{|\mathcal{E}(M)|}{|\mathcal{C}_H(M)|} = 0.$$

His proof is based on a version of the Chebotarev density theorem, and uses a two-dimensional large sieve inequality together with results of Deuring, Hurwitz and Masser-Wüstholz.

Duke also conjectured the following fact, later proved by Grant [10]

$$|\mathcal{E}(M)| \sim c\sqrt{M}.$$

Being a bit more precise, Grant showed that, in order to efficiently estimate  $|\mathcal{E}(M)|$ , only  $\mathcal{E}_2(M)$  and  $\mathcal{E}_3(M)$  had to be actually taken into account.

Now recall that there is a tight relationship between exceptional primes and torsion orders, because if there is a point of order  $p$ , then  $p$  is an exceptional prime [20]. Our aim is then giving a down-to-earth proof of the fact that *almost all* elliptic curves over  $\mathbb{Q}$  have trivial torsion, motivated by Duke's paper.

We will use in order to achieve this the characterization of torsion structures given in [7, 8], Mazur's Theorem [16, 17]; and a theorem by Schmidt [19] on Thue inequalities. Note that we have used a different height notion, more naive in some sense, but nevertheless better suited for our purposes.

Let us change a bit the notation and let us call

$$E_{(A,B)} : Y^2 = X^3 + AX + B$$

and, provided  $\Delta \neq 0$ , we will denote by  $E_{(A,B)}(\mathbb{Q})[m]$  the group of points  $P \in E_{(A,B)}(\mathbb{Q})$  such that  $[m]P = \mathcal{O}$ . Let us write as well

$$\begin{aligned} \mathcal{C}(M) &= \{(A, B) \in \mathbb{Z}^2 \mid \Delta = -16(4A^3 + 27B^2) \neq 0, \quad |A|, |B| \leq M\}. \\ \mathcal{T}_p(M) &= \{(A, B) \in \mathcal{C}(M) \mid E_{(A,B)}(\mathbb{Q})[p] \neq \{\mathcal{O}\}\}. \\ \mathcal{T}(M) &= \bigcup_{p \text{ prime}} \mathcal{T}_p(M) \end{aligned}$$

Our version of Duke's result is then as follows.

**Theorem 1.** *With the notations above,*

$$\lim_{M \rightarrow \infty} \frac{|\mathcal{T}(M)|}{|\mathcal{C}(M)|} = 0.$$

The proof will lead to extremely coarse bounds for  $|\mathcal{T}_p(M)|$  which will be proved unsatisfactory in view of experimental data, which we will display subsequently.

## 2. PROOF OF THEOREM 1.

Recall that the possible prime orders of a torsion point defined over  $\mathbb{Q}$  are 2, 3, 5 or 7.

We will make extensive use of the parametrizations of curves with a point of prescribed order given in [7, 8, 14]. These results have recently been proved useful in showing new properties of the torsion subgroup (see, for instance [1, 9, 13]).

First, note that, for a given  $A$  with  $|A| \leq M$  there are, at most, two possible choices for  $B$  such that  $\Delta = 0$  (and hence, the corresponding curve  $E_{(A,B)}$  is not an elliptic curve). Therefore

$$|\mathcal{C}(M)| \geq (2M + 1)^2 - 2(2M + 1) = 4M^2 - 1.$$

Let us recall from [7] that a curve  $E_{(A,B)}$  with a point of order 2 must verify that there exist  $z_1, z_2 \in \mathbb{Z}$  such that

$$A = z_1 - z_2^2, \quad B = z_1 z_2.$$

Therefore  $z_1 | B$  and for a chosen  $z_1$ , both  $z_2$  and  $A$  are determined. Hence, there is at most one pair in  $\mathcal{T}_2(M)$  for every divisor of  $B$ .

We need now an estimate for the average order of the function  $d(x)$ , the number of positive divisors of  $x$ . The simplest estimation is, probably, the one that can be found in [11],

$$d(1) + d(2) + \dots + d(x) \sim x \log(x).$$

Therefore, as  $M$  tends to infinity,

$$|\mathcal{T}_2(M)| \leq \sum_{x=1}^M 2d(x) + \sum_{x=1}^M 2d(x) + 2M,$$

taking into account that we need to consider both positive and negative divisors, the cases where  $x \in \{-M, \dots, -1\}$  and the  $2M$  curves with  $B = 0$ . Hence  $|\mathcal{T}_2(M)| \sim c_2 M \log(M)$ , where we can, in fact, take  $c_2 = 4$ .

As for points of order 3 we can find in [7] a similar characterization (a bit more complicated this time) based on the existence of  $z_1, z_2 \in \mathbb{Z}$  such that

$$A = 27z_1^4 + 6z_1z_2, \quad B = z_2^2 - 27z_1^6.$$

Analogously  $z_1|A$  and, once we fix such a divisor,  $z_2$  is necessarily given by

$$z_2 = \frac{A - 27z_1^4}{6z_1},$$

which implies that again there is at most one pair in  $\mathcal{T}_3(M)$  for every divisor of  $A$ . Hence, as  $M$  tends to infinity

$$|\mathcal{T}_3(M)| \leq c_3 M \log(M),$$

and again  $c_3 = 4$  suits us.

Points of order 5 and 7 need a similar, yet slightly different argument. From [8] we know that if there is a point of order 5 in  $E_{(A,B)}(\mathbb{Q})$ , then there must exist  $p, q \in \mathbb{Z}$  verifying:

$$\begin{aligned} A &= -27(q^4 - 12q^3p + 14q^2p^2 + 12p^3q + p^4), \\ B &= 54(p^2 + q^2)(q^4 - 18q^3p + 74q^2p^2 + 18p^3q + p^4). \end{aligned}$$

The first equation is an irreducible Thue equation, hence we can apply the following result by Schmidt:

**Theorem (Schmidt [19]).**— Let  $F(x, y)$  be an irreducible binary form of degree  $r > 3$ , with integral coefficients. Suppose that not more than  $s + 1$  coefficients are nonzero. Then the number of solutions of the inequality  $|F(x, y)| \leq h$  is, at most,

$$(rs)^{1/2} h^{2/r} \left(1 + \log^{1/r}(h)\right).$$

As for our interests are concerned, this gives a bound for the number of possible  $(p, q)$  such that

$$|-27(q^4 - 12q^3p + 14q^2p^2 + 12p^3q + p^4)| \leq M.$$

Hence, as every such solution determines at most one pair in  $\mathcal{T}_5(M)$ ,

$$|\mathcal{T}_5(M)| \leq 4\sqrt{M} \left(1 + \log^{1/4}(M)\right).$$

A similar result can be applied for points of order 7. The equations which must have a solution are now

$$\begin{aligned} A &= -27k^4(p^2 - pq + q^2)(q^6 + 5q^5p - 10q^4p^2 - 15q^3p^3 + \\ &\quad 30q^2p^4 - 11qp^5 + p^6), \\ B &= 54k^6(p^{12} - 18p^{11}q + 117p^{10}q^2 - 354p^9q^3 + 570p^8q^4 - 486p^7q^5 \\ &\quad + 273p^6q^6 - 222p^5q^7 + 174p^4q^8 - 46p^3q^9 - 15p^2q^{10} + 6pq^{11} + q^{12}). \end{aligned}$$

either for  $k = 1$  or for  $k = 1/3$ . Hence, using the polynomial defining  $B$  and with a similar argument as above

$$|\mathcal{T}_7(M)| \leq 24\sqrt[6]{M} \left(1 + \log^{1/12}(M)\right).$$

Therefore, for all  $p$  there is an absolute constant  $c_p \in \mathbb{Z}_+$  such that

$$\lim_{M \rightarrow \infty} \frac{|\mathcal{T}_p(M)|}{|C(M)|} \leq \lim_{M \rightarrow \infty} \frac{c_p M \log(M)}{4M^2 - 1} = 0.$$

This proves the theorem.

**Remark.**— It must be noted here that our arguments are counting pairs  $(A, B)$ . So, in fact, isomorphic curves may appear as separated cases. Both Duke and Grant estimated isomorphism classes (over  $\mathbb{Q}$ ) rather than curves.

But this can also be achieved by the arguments above with a little extra work. We will show now that these instances of isomorphic curves are actually negligible as for counting is concerned.

First note that if two curves  $E_{(A,B)}$  and  $E_{(A',B')}$  are isomorphic over  $\mathbb{Q}$ , there must be some  $u \in \mathbb{Q}$  such that  $A = u^4 A'$  and  $B = u^6 B'$ . Hence, there exists some prime  $l$  such that, say,  $l^4 | A$  and  $l^6 | B$  (the case  $l^4 | A'$  and  $l^6 | B'$  is analogous). Let us write, for a fixed prime  $l$

$$P_n(M, l) = \{x \in \mathbb{Z}_+ \mid 1 \leq x \leq M, l^n | M\},$$

and by  $P_n(M)$  the union of  $P_n(M, l)$ , where  $l$  run the set of prime divisors of  $M$ .

Then it is clear that

$$\begin{aligned} |P_n(M^n)| &\leq \sum_{l \leq M} |P_n(M^n, l)| = \sum_{l \leq M} \left[ \frac{M^n}{l^n} \right] = \sum_{l \leq M} \left( \frac{M^n}{l^n} + O(1) \right) = \\ &= M^n \sum_{l \leq M} \left( \frac{1}{l^n} \right) + O(M) = M^n \sum_{l \text{ prime}} \frac{1}{l^n} + O(M) = M^n \mathcal{P}(n) + O(M), \end{aligned}$$

where  $\mathcal{P}$  is the prime zeta function (see [5], for instance). So, changing  $M^n$  for  $M$  we get

$$\begin{aligned} |P_4(M)| &\leq P(4)M + O\left(\sqrt[4]{M}\right) \simeq 0.0769931M + O\left(\sqrt[4]{M}\right), \\ |P_6(M)| &\leq P(6)M + O\left(\sqrt[6]{M}\right) \simeq 0.0170701M + O\left(\sqrt[6]{M}\right). \end{aligned}$$

Hence, if we are interested in curves up to  $\mathbb{Q}$ -isomorphism, our bounds for  $|\mathcal{T}_p(M)|$  are still correct, while we should change

$$|C(M)| \geq 4M^2 - 1$$

by

$$|C(M)| \geq (4 - P(4)P(6))M^2 + O\left(\sqrt[6]{M}\right)$$

which obviously makes no difference in the result.

*Remark 1.* While all of our boundings for  $|\mathcal{T}_p(M)|$  are of the form  $c_p M \log(M)$ , computational data show that the actual number of curves on  $\mathcal{T}_p(M)$  depends heavily on  $p$ , as one might predict after the estimation given by Grant [10] for  $\mathcal{E}_p(M)$ , the set of elliptic curves  $E \in \mathcal{C}_H(M)$  such that  $p$  is an exceptional prime for  $E$ . In fact, a hands-on `Magma` program gave us the following output

$M$	$ \mathcal{T}_2(M) $	$ \mathcal{T}_3(M) $	$ \mathcal{T}_5(M) $	$ \mathcal{T}_7(M) $
$10^4$	204, 220	507	1	1
$10^5$	2, 484, 196	1, 935	3	1
$10^6$	29, 430, 050	5, 873	11	4
$10^7$	340, 334, 782	18, 387	24	5

These actual figures are quite smaller than the bounds obtained.

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