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ON THE FUNCTORIALITY OF COHOMOLOGY OF CATEGORIES

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ABSTRACT. In this paper we show that the Baues-Wirsching complex used to define cohomology of categories is a 2-functor from a certain 2-category of natural systems of abelian groups to the 2-category of chain complexes, chain homomorphism and relative homotopy classes of chain homotopies. As a consequence we derive (co)localization theorems for this cohomology.

1. INTRODUCTION

Baues-Wirsching cohomology of a small category $\mathbf{C}$ with coefficients in a natural system D on $\mathbf{C}$ was defined in [2] as the cohomology of a certain cochain complex $F^*(\mathbf{C}, D)$. This cohomology generalizes some other cohomologies previously known, as for example

- the Hochschild-Mitchell cohomology of $\mathbf{C}$ with coefficients in a functor $D\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Ab}$ ([10]),
- the cohomology of the classifying space $B\mathbf{C}$ with local coefficients D,
- the Mac Lane cohomology of a ring ([8], [7]).

The Baues-Wirsching complex $F^*(\mathbf{C}, D)$ as well as its cohomology $H^*(\mathbf{C}, D)$ are known to be functors on a certain category $\mathbf{Nat}$ of pairs $(\mathbf{C}, D)$. It is also known that equivalences of categories induce homotopy equivalences in Baues-Wirsching complexes and isomorphisms in cohomology groups, however this does not follow immediately from the fact that F^* and H^* are functors in $\mathbf{Nat}$. More precisely, it is not known the behaviour of $F^*(\mathbf{C}, D)$ and $H^*(\mathbf{C}, D)$ with respect to natural transformations between functors in the first variable. The goal of this paper is to shed some light on that issue.

We define a new 2-category $\mathbf{Nat}_{\mathcal{F}}$ of pairs $(\mathbf{C}, D)$ containing $\mathbf{Nat}$ and prove that F^* is in fact a 2-functor in $\mathbf{Nat}_{\mathcal{F}}$ (Theorem 5.1). The 2-morphisms in the category of cochain complexes will be homotopy classes of homotopies relative to the boundary of the cylinder. The category obtained from $\mathbf{Nat}_{\mathcal{F}}$ by taking sets of connected components on morphism categories turns out to be a quotient of $\mathbf{Nat}$ (Proposition 5.2) and the cohomology functor H^* factors through this quotient category (Corollary 5.3).

As an application of these results we obtain localization and colocalization theorems for Baues-Wirsching cohomology (Theorems 6.5 and 6.12). We also give some examples of how these (co)localization theorems can be used to carry out computations in cohomology of categories.

Pirashvili and Waldhausen defined in [12] the homology of a small category $\mathbf{C}$ with coefficients in a functor $D\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Ab}$ by using a complex $F_*(\mathbf{C}, D)$ which is similar, and in some sense dual, to the Baues-Wirsching complex. They also proved that this homology extends Mac Lane homology of rings ([8]), which is isomorphic to topological Hochschild homology in the sense of [3]. We claim that the definition of the Pirashvili-Waldhausen homology $H_*(\mathbf{C}, D)$ can be extended to natural systems D as coefficient objects and the functorial properties of F_* and H_* are similar to those described here for F^* and H^*, in particular (co)localization theorems should hold.

2. Notation and conventions

Compelled by the necessarily intricate notation of this paper, we have decided to include at the beginning a section to fix the meaning of some symbols. Additional notation will also appear in the development of the paper, but it will not contradict in any case that introduced here.
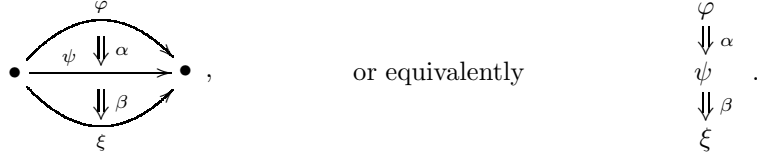
symbols	meaning
$\mathbf{C}, \mathbf{D}, \mathbf{E}$	small categories.
X, Y	objects in those categories.
f, g, h, k, σ	morphisms in those categories.
$\varphi, \psi, \xi, \zeta$	functors between those categories.
$\alpha, \beta, \gamma, \varepsilon, t, s$	natural transformations.
$\to$	arrow for morphisms in ordinary categories, 1-morphisms in 2-categories, functors, and 2-functors.
$\Rightarrow$	arrow for 2-morphisms and natural transformations.
1	identity morphism or identity 2-morphism, a subscript will clarify the meaning in ambiguous cases.
$\mathbf{Ab}$	the category of *objects:* abelian groups, *morphisms:* homomorphisms.
$\mathbf{Cat}$	the category of *objects:* small categories, *morphisms:* functors.
$\mathbf{Cat}_2$	the standard 2-category of *objects:* small categories, *1-morphisms:* functors, *2-morphisms:* natural transformations.
D, E, G	natural systems, see Definition 4.1.
c	cochain in cohomology of categories.
A^*, B^*, C^*	cochain complexes of abelian groups.
d	the differential in all cochain complexes.
p, q, h, r	graded morphisms between graded abelian groups.

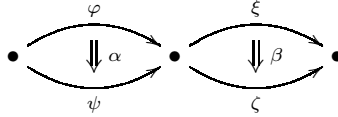These symbols can be altered by adding superscripts or subscripts.

In 2-categories the word "morphism" will be synonym of "1-morphism". All categories can be regarded as 2-categories with only the trivial 2-morphisms.

The symbol • stands for an unspecified object in an arbitrary category. It can appear several times in a diagram, however in general it will stand for a different object each time.

The composite of morphisms, say $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$, or functors will be indicated by juxtaposition gf, as well as the vertical composition of 2-morphisms or natural transformations $\beta\alpha\colon \varphi \Rightarrow \xi$ as in the diagram

$$
\bullet \underset{\xi}{\overset{\varphi}{\rightrightarrows}} \bullet \quad , \qquad \text{or equivalently} \qquad \begin{matrix} \varphi \\ \Downarrow \alpha \\ \psi \\ \Downarrow \beta \\ \xi \end{matrix} \quad .
$$

We will use the symbol $*$ for the horizontal composition of 2-morphisms or natural transformations $\beta * \alpha\colon \xi\varphi \Rightarrow \zeta\psi$ as in the following diagram

$$
\bullet \underset{\psi}{\overset{\varphi}{\rightrightarrows}} \bullet \underset{\zeta}{\overset{\xi}{\rightrightarrows}} \bullet
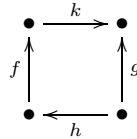$$

3. Factorization categories

Factorization categories are the source of the coefficient objects for the Baues-Wirsching cohomology of small categories, see Definition 4.1. A thorough study of their properties is essential to study in depth the functoriality of the Baues-Wirsching complex in Section 5.

Definition 3.1. The *factorization category* $\mathcal{F}\mathbf{C}$ of a small category $\mathbf{C}$ has

    objects: morphisms in $\mathbf{C}$,

    morphisms: $(h,k)\colon f \to g$ are pairs of morphisms in $\mathbf{C}$ such that $kfh = g$, that is commutative diagrams in $\mathbf{C}$

$$
\begin{array}{ccc} \bullet & \xrightarrow{k} & \bullet \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle g} \\ \bullet & \xleftarrow{h} & \bullet \end{array}
$$

    and composition is defined by $(h',k')(h,k) = (hh', k'k)$.

One can easily check that factorization categories define a functor

$$\mathcal{F}\colon \mathbf{Cat} \longrightarrow \mathbf{Cat}.$$

This functor is defined on morphisms as follows: a functor $\varphi\colon \mathbf{C} \to \mathbf{D}$ is sent to another one

$$\mathcal{F}(\varphi)\colon \mathcal{F}\mathbf{C} \longrightarrow \mathcal{F}\mathbf{D},$$

which is given

    on objects: by $\mathcal{F}(\varphi)(f) = \varphi(f)$,

    on morphisms: by $\mathcal{F}(\varphi)(h,k) = (\varphi(h), \varphi(k))$.

Notice that the functor $\mathcal{F}$ preserves products, therefore we can consider the 2-category $\mathbf{Cat}_{\mathcal{F}}$ obtained from $\mathbf{Cat}_2$ by applying the functor $\mathcal{F}$ to morphism categories. Let us make explicit the structure of $\mathbf{Cat}_{\mathcal{F}}$:

(3.A)   *objects:* are small categories;

1-*morphisms:* $\alpha\colon \mathbf{C} \to \mathbf{D}$ are actually natural transformations $\alpha\colon \varphi \Rightarrow \psi$ between functors $\varphi, \psi\colon \mathbf{C} \to \mathbf{D}$, and composition $\beta\alpha$ in $\mathbf{Cat}_{\mathcal{F}}$ is horizontal composition $\beta * \alpha$ of natural transformations;

2-*morphisms:* $(\varepsilon, \gamma)\colon \alpha \Rightarrow \beta$ are natural transformations such that $\gamma\alpha\varepsilon = \beta$, that is commutative diagrams of natural transformations

$$
\begin{array}{ccc}
\psi & \overset{\gamma}{\Longrightarrow} & \zeta \\
\alpha \Big\Uparrow & & \Big\Uparrow \beta \\
\varphi & \underset{\varepsilon}{\Longleftarrow} & \xi
\end{array}
$$

vertical composition of 2-morphisms is given by $(\varepsilon', \gamma')(\varepsilon, \gamma) = (\varepsilon\varepsilon', \gamma'\gamma)$, and the horizontal composition of (ε, γ) and (ε', γ') as in the following diagram in $\mathbf{Cat}_{\mathcal{F}}$

$$
\mathbf{C} \underset{\beta}{\overset{\alpha}{\Rrightarrow}} (\varepsilon,\gamma) \; \mathbf{D} \underset{\beta'}{\overset{\alpha'}{\Rrightarrow}} (\varepsilon',\gamma') \; \mathbf{E}
$$

is $(\varepsilon', \gamma') * (\varepsilon, \gamma) = (\varepsilon' * \varepsilon, \gamma' * \gamma)$.

There is a unique 2-functor

$$\mathbf{Cat} \longrightarrow \mathbf{Cat}_2$$

which is the identity on objects and 1-morphisms. Moreover, there is also a unique 2-functor

$$\mathbf{Cat} \longrightarrow \mathbf{Cat}_{\mathcal{F}}$$

which is the identity on objects and sends a functor $\varphi\colon \mathbf{C} \to \mathbf{D}$ to the identity natural transformation $1_{\varphi}\colon \varphi \Rightarrow \varphi$ regarded as a morphism $1_{\varphi}\colon \mathbf{C} \to \mathbf{D}$ in $\mathbf{Cat}_{\mathcal{F}}$.

Proposition 3.2. *There is defined a 2-functor $\mathcal{F}\colon \mathbf{Cat}_{\mathcal{F}} \to \mathbf{Cat}_2$ fitting into a commutative diagram*

$$
\begin{array}{ccc}
\mathbf{Cat} & \overset{\mathcal{F}}{\longrightarrow} & \mathbf{Cat} \\
\downarrow & & \downarrow \\
\mathbf{Cat}_{\mathcal{F}} & \underset{\mathcal{F}}{\longrightarrow} & \mathbf{Cat}_2
\end{array}
$$

where the vertical arrows are the 2-functors previously defined.

Proof. The new 2-functor $\mathcal{F}$ is defined in the following way, we use the notation in Definition 3.1 and (3.A):

on objects: $\mathcal{F}\mathbf{C}$ is the factorization category,

on 1-morphisms: the functor $\mathcal{F}(\alpha)\colon \mathcal{F}\mathbf{C} \to \mathcal{F}\mathbf{D}$ is defined

on objects: given an object f in $\mathcal{F}\mathbf{C}$, which is a morphism $f\colon X \to Y$ in $\mathbf{C}$, $\mathcal{F}(\alpha)(f) = \alpha_Y\varphi(f) = \psi(f)\alpha_X$;

on morphisms: $\mathcal{F}(\alpha)(h, k) = (\varphi(h), \psi(k))$.

on 2-morphisms: $\mathcal{F}(\varepsilon, \gamma)\colon \mathcal{F}(\alpha) \Rightarrow \mathcal{F}(\beta)$ is the natural transformation which evaluated on f as above is the morphism $\mathcal{F}(\varepsilon, \gamma)_f = (\varepsilon_X, \gamma_Y)$ in $\mathcal{F}\mathbf{D}$.

It is a straightforward exercise to check that this definition is consistent and $\mathcal{F}$ is indeed a 2-functor. Moreover, the diagram in the statement commutes because all 2-functors are the identity on objects, $\mathcal{F}(1_\varphi)(f) = \varphi(f) = \mathcal{F}(\varphi)(f)$ and $\mathcal{F}(1_\varphi)(h, k) = (\varphi(h), \varphi(k)) = \mathcal{F}(\varphi)(h, k)$. $\qquad \square$

4. Baues-Wirsching cohomology of categories

Definition 4.1. Recall from [2] that a *natural system* on $\mathbf{C}$ is a functor $D\colon \mathcal{F}\mathbf{C} \to \mathbf{Ab}$. The *Baues-Wirsching complex* $F^*(\mathbf{C}, D)$ of a small category $\mathbf{C}$ with coefficients in a natural system D on $\mathbf{C}$ is a cochain complex of abelian groups concentrated in non-negative dimensions. In dimension n this complex is given by the following product indexed by all sequences of morphisms of length $n-1$ in $\mathbf{C}$

$$F^n(\mathbf{C}, D) = \prod_{\bullet \xleftarrow{\sigma_1} \cdots \xleftarrow{\sigma_n} \bullet} D(\sigma_1 \cdots \sigma_n).$$

In this formula we assume that a sequence of length 0 is an object X in $\mathbf{C}$ which we also identify with the identity morphism 1_X. The coordinate of $c \in F^n(\mathbf{C}, D)$ in $\bullet \xleftarrow{\sigma_1} \cdots \xleftarrow{\sigma_n} \bullet$ will be denoted by $c(\sigma_1, \ldots, \sigma_n)$. The differential d is defined as

$$
\begin{aligned}
d(c)(\sigma_1, \ldots, \sigma_{n+1}) \;=\; & D(1, \sigma_1) c(\sigma_2, \ldots, \sigma_{n+1}) \\
& + \sum_{i=1}^{n} (-1)^i c(\sigma_1, \ldots, \sigma_i \sigma_{i+1}, \ldots, \sigma_{n+1}) \\
& + (-1)^{n+1} D(\sigma_{n+1}, 1) c(\sigma_1, \ldots, \sigma_n).
\end{aligned}
$$

over an n-cochain c for $n \geq 1$, and $d(c)(\sigma) = D(1, \sigma)c(X) - D(\sigma, 1)c(Y)$ for $n = 0$ and $\sigma\colon X \to Y$.

The cohomology of $\mathbf{C}$ with coefficients in D is the cohomology of the complex $F^*(\mathbf{C}, D)$, it is denoted by $H^*(\mathbf{C}, D)$.

Baues and Wirsching noticed that H^* and F^* are functors in the category $\mathbf{Nat}$ defined as follows:

objects: are pairs $(\mathbf{C}, D)$ where D is a natural system on $\mathbf{C}$,

morphisms: $(\varphi, t)\colon (\mathbf{C}, D) \to (\mathbf{D}, E)$ are pairs given by a functor $\varphi\colon \mathbf{D} \to \mathbf{C}$ and a natural transformation $t\colon D\mathcal{F}(\varphi) \Rightarrow E$, and composition is given by the formula $(\psi, s)(\varphi, t) = (\varphi\psi, s(t * 1_{\mathcal{F}(\psi)}))$.

Let **Cochain** be the category of cochain complexes of abelian groups and cochain homomorphisms. As a functor

$$F^*\colon \mathbf{Nat} \longrightarrow \mathbf{Cochain}$$

is defined as follows

on objects: $F^*(\mathbf{C}, D)$ is the Baues-Wirsching complex;

on morphisms: $F^n(\varphi, t)(c)(\sigma_1, \ldots, \sigma_n) = t_\sigma(c(\varphi(\sigma_1), \ldots, \varphi(\sigma_n)))$, where $\sigma = \sigma_1 \cdots \sigma_n$;

and

$$H^n = H^n F^*\colon \mathbf{Nat} \longrightarrow \mathbf{Ab}, \;\; n \in \mathbb{Z}.$$

5. The Baues-Wirsching complex as a 2-functor

This section is the core of the paper. Its main goal is to extend F^* to a 2-functor from an adequate 2-category $\mathbf{Nat}_{\mathcal{F}}$ with the same objects as $\mathbf{Nat}$ to the following 2-category $\mathbf{Cochain}_2$:

objects: cochain complexes of abelian groups;

1-morphisms: cochain homomorphisms, that is graded homomorphisms $p\colon A^* \to B^*$ of degree 0 such that $dp = pd$;

2-morphisms: $[h]\colon p \Rightarrow q$ are relative homotopy classes of homotopies between p and q, that is $[h]$ is represented by a degree -1 homomorphism $h\colon A^* \to B^*$ such that $dh + hd = -p + q$ and $[h] = [h']$ if there exists $r\colon A^* \to B^*$ of degree -2 such that $dr - rd = -h + h'$; vertical composition of 2-morphisms is given by $[h'][h] = [h' + h]$, and the horizontal composition of $[h]$ and $[h']$ in the following diagram

$$A^* \quad \underset{q}{\overset{p}{\rightrightarrows}} \Downarrow h \quad B^* \quad \underset{q'}{\overset{p'}{\rightrightarrows}} \Downarrow h' \quad C^*$$

is $[h'] * [h] = [h'p + q'h] = [p'h + h'q]$; one can use the degree -2 homomorphism $h'h\colon A^* \to C^*$ to check the last equality.

Notice that morphism categories in $\mathbf{Cochain}_2$ are in fact groupoids. Moreover, there is a unique 2-functor

(5.A) $\imath\colon \mathbf{Cochain} \longrightarrow \mathbf{Cochain}_2$

which is the identity on objects and 1-morphisms.

Let us define the 2-category $\mathbf{Nat}_{\mathcal{F}}$:

(5.B)   *objects:* are pairs $(\mathbf{C}, D)$ where D is a natural system on $\mathbf{C}$;

1-morphisms: $(\alpha, t)\colon (\mathbf{C}, D) \to (\mathbf{D}, E)$ are pairs given by a natural transformation $\alpha\colon \varphi \Rightarrow \psi$ between functors $\varphi, \psi\colon \mathbf{D} \to \mathbf{C}$, or equivalently a morphism $\alpha\colon \mathbf{D} \to \mathbf{C}$ in $\mathbf{Cat}_{\mathcal{F}}$, see (3.A), and a natural transformation $t\colon D\mathcal{F}(\alpha) \Rightarrow E$, where $\mathcal{F}$ is the functor defined in Proposition 3.2, and composition is defined as $(\beta, s)(\alpha, t) = (\alpha * \beta, s(t * 1_{\mathcal{F}(\beta)}))$;

2-morphisms: $(\varepsilon, \gamma)\colon (\alpha, t) \Rightarrow (\beta, s)$ are 2-morphisms $(\varepsilon, \gamma)\colon \alpha \Rightarrow \beta$ in $\mathbf{Cat}_{\mathcal{F}}$ such that $t = s(1_D * \mathcal{F}(\varepsilon, \gamma))$, that is the following diagram of natural transformations commutes

(5.C)

$$\begin{array}{ccc} & D\mathcal{F}(\alpha) & \\ {\scriptstyle 1_D * \mathcal{F}(\varepsilon,\gamma)} \Big\Downarrow & & \overset{t}{\searrow} \\ & & E \\ & \underset{s}{\nearrow} & \\ & D\mathcal{F}(\beta) & \end{array}$$

vertical and horizontal compositions of 2-morphisms in $\mathbf{Nat}_{\mathcal{F}}$ are defined as in $\mathbf{Cat}_{\mathcal{F}}$, that is $(\varepsilon', \gamma')(\varepsilon, \gamma) = (\varepsilon\varepsilon', \gamma'\gamma)$ and given a diagram in $\mathbf{Nat}_{\mathcal{F}}$

the horizontal composition $(\varepsilon', \gamma') * (\varepsilon, \gamma) = (\varepsilon * \varepsilon', \gamma * \gamma')$ in $\mathbf{Nat}_{\mathcal{F}}$ coincides with the horizontal composition of the following diagram in $\mathbf{Cat}_{\mathcal{F}}$

It is tedious but straightforward to check that $\mathbf{Nat}_{\mathcal{F}}$ is indeed a well-defined 2-category. Moreover, there is a unique 2-functor

$$\jmath \colon \mathbf{Nat} \longrightarrow \mathbf{Nat}_{\mathcal{F}}$$

which is the identity on objects and sends a morphism (φ, t) to $(1_{\varphi}, t)$. This makes sense because of the commutativity of the diagram in Proposition 3.2.

Theorem 5.1. *There is defined a 2-functor $F^* \colon \mathbf{Nat}_{\mathcal{F}} \to \mathbf{Cochain}_2$ fitting into a commutative diagram*

$$
\begin{array}{ccc}
\mathbf{Nat} & \xrightarrow{\;F^*\;} & \mathbf{Cochain} \\
{\scriptstyle \jmath}\big\downarrow & & \big\downarrow{\scriptstyle \imath} \\
\mathbf{Nat}_{\mathcal{F}} & \xrightarrow[\;F^*\;]{} & \mathbf{Cochain}_2
\end{array}
$$

Proof. The new 2-functor F^* is defined as follows, we use the notation in (3.B) and (5.B):

> *on objects:* $F^*(\mathbf{C}, D)$ is the Baues-Wirsching complex;

> *on 1-morphisms:*
>
> $$F^*(\alpha, t)(c)(\sigma_1, \ldots, \sigma_n) = t_{\sigma}(1_D * \mathcal{F}(1_{\varphi}, \alpha))_{\sigma} c(\varphi(\sigma_1), \ldots, \varphi(\sigma_n)),$$
>
> here $\sigma = \sigma_1 \cdots \sigma_n$ and the formula makes sense because
>
> $$c(\varphi(\sigma_1), \ldots, \varphi(\sigma_n)) \in D(\varphi(\sigma)) = (D\mathcal{F}(1_{\varphi}))(\sigma);$$

> *on 2-morphisms:* $F^*(\varepsilon, \gamma) = [h_{(\varepsilon, \gamma)}] \colon F^*(\alpha, t) \to F^*(\beta, s)$ where for an $(n+1)$-dimensional cochain c if $n > 0$ $h_{(\varepsilon, \gamma)}(c)$ is defined as

$h_{(\varepsilon, \gamma)}(c)(\sigma_1, \ldots, \sigma_n) =$
$\quad s_{\sigma}(1_D * \mathcal{F}(1_{\xi}, \gamma\alpha))_{\sigma} \sum_{i=0}^{n} (-1)^i c(\varphi(\sigma_1), \ldots, \varphi(\sigma_i), \varepsilon_{X_i}, \xi(\sigma_{i+1}), \ldots, \xi(\sigma_n)),$

> where X_i is the source of σ_i and/or the target of σ_{i+1}, notice that
>
> $$c(\varphi(\sigma_1), \ldots, \varphi(\sigma_i), \varepsilon_{X_i}, \xi(\sigma_{i+1}), \ldots, \xi(\sigma_n)) \in (D\mathcal{F}(\varepsilon))(\sigma);$$
>
> and if $n = 0$
>
> $$h_{(\varepsilon, \gamma)}(c)(X) = s_{1_X}(1_D * \mathcal{F}(1_{\xi}, \gamma\alpha))_{1_X} c(\varepsilon_X).$$

A tedious but straightforward computation shows that indeed

$$dh_{(\varepsilon,\gamma)} + h_{(\varepsilon,\gamma)}d = -F^*(\alpha, t) + F^*(\beta, s).$$

For this, essentially, one only needs to use the naturality property of natural transformations and the commutativity of (5.C).

It is easy to see that F^* preserves composition of 1-morphisms.

In order to check that F^* preserves vertical composition of 2-morphisms we consider a diagram in $\mathbf{Nat}_{\mathcal{F}}$

where

is a commutative diagram of natural transformations between functors

$$\varphi, \varphi', \psi, \psi', \xi, \zeta \colon \mathbf{D} \to \mathbf{C}.$$

We define a degree -2 homomorphism

$$r_{(\varepsilon',\gamma');(\varepsilon,\gamma)} \colon F^*(\mathbf{C}, D) \longrightarrow F^*(\mathbf{D}, E)$$

in the following way, if c is an $(n+2)$-cochain with $n > 0$ then

$$r_{(\varepsilon',\gamma');(\varepsilon,\gamma)}(c)(\sigma_1, \ldots, \sigma_n) = s_\sigma(1_D * \mathcal{F}(1_\xi, \gamma'\gamma\alpha))_\sigma \sum_{i=0}^n \sum_{j=i}^n (-1)^{i+j} c(\varphi(\sigma_1), \ldots,$$
$$\varphi(\sigma_i), \varepsilon_{X_i}, \varphi'(\sigma_{i+1}), \ldots, \varphi'(\sigma_j), \varepsilon'_{X_j}, \xi(\sigma_{j+1}), \ldots, \xi(\sigma_n))$$

and for $n = 0$

$$r_{(\varepsilon',\gamma');(\varepsilon,\gamma)}(c)(X) = s_{1_X}(1_D * \mathcal{F}(1_\xi, \gamma'\gamma\alpha))_{1_X} c(\varepsilon_X, \varepsilon'_X).$$

It is hard but straightforward to check that

$$dr_{(\varepsilon',\gamma');(\varepsilon,\gamma)} - r_{(\varepsilon',\gamma');(\varepsilon,\gamma)}d = -h_{(\varepsilon,\gamma)} - h_{(\varepsilon',\gamma')} + h_{(\varepsilon\varepsilon',\gamma'\gamma)},$$

therefore $F^*(\varepsilon', \gamma')F^*(\varepsilon, \gamma) = F^*(\varepsilon\varepsilon', \gamma'\gamma)$.

Let us see that F^* preserves horizontal composition of 2-morphisms. Consider a diagram in $\mathbf{Nat}_{\mathcal{F}}$

Here

$$\begin{array}{ccc} \psi \xrightarrow{\ \gamma\ } \zeta & & \psi' \xrightarrow{\ \gamma'\ } \zeta' \\ {\scriptstyle\alpha}\Uparrow \quad \Uparrow{\scriptstyle\beta} & \text{and} & {\scriptstyle\alpha'}\Uparrow \quad \Uparrow{\scriptstyle\beta'} \\ \varphi \xleftarrow{\ \varepsilon\ } \xi & & \varphi' \xleftarrow{\ \varepsilon'\ } \xi' \end{array}$$

are commutative diagrams of natural transformations between functors

$$\varphi, \psi, \xi, \zeta \colon \mathbf{D} \to \mathbf{C} \quad \text{and} \quad \varphi', \psi', \xi', \zeta' \colon \mathbf{E} \to \mathbf{D}.$$

We define a degree -2 homomorphism

$$r'_{(\varepsilon',\gamma');(\varepsilon,\gamma)} \colon F^*(\mathbf{C}, D) \longrightarrow F^*(\mathbf{E}, G)$$

over an $(n+2)$-cochain c with $n > 0$ as

$$r'_{(\varepsilon',\gamma');(\varepsilon,\gamma)}(c)(\sigma_1, \ldots, \sigma_n) =$$
$$s'_\sigma(s * 1_{\mathcal{F}(\beta')})_\sigma (1_D * \mathcal{F}(1_{\xi\xi'}, (\gamma * \gamma')(\alpha * \alpha')))_\sigma \sum_{i=0}^{n} \sum_{j=i}^{n} (-1)^{i+j} c(\varphi\varphi'(\sigma_1), \ldots,$$
$$\varphi\varphi'(\sigma_i), \varphi(\varepsilon'_{X_i}), \varphi\xi'(\sigma_{i+1}), \ldots, \varphi\xi'(\sigma_j), \varepsilon_{\xi'(X_j)}, \xi\xi'(\sigma_{j+1}), \ldots, \xi\xi'(\sigma_n))$$

and for $n = 0$

$$r'_{(\varepsilon',\gamma');(\varepsilon,\gamma)}(c)(X) = s'_{1_X}(s*1_{\mathcal{F}(\beta')})_{1_X}(1_D*\mathcal{F}(1_{\xi\xi'}, (\gamma*\gamma')(\alpha*\alpha')))_{1_X} c(\varphi(\varepsilon'_X), \varepsilon_{\xi'(X)}).$$

After a laborious computation one can check that

$$dr'_{(\varepsilon',\gamma');(\varepsilon,\gamma)} - r'_{(\varepsilon',\gamma');(\varepsilon,\gamma)}d = -h_{(\varepsilon',\gamma')}F^*(\alpha,t) - F^*(\beta',s')h_{(\varepsilon,\gamma)} + h_{(\varepsilon*\varepsilon',\gamma*\gamma')},$$

hence $F^*(\varepsilon',\gamma') * F^*(\varepsilon,\gamma) = F^*(\varepsilon*\varepsilon', \gamma*\gamma')$.

The commutativity of the diagram in the statement follows easily from the commutativity of the diagram in Proposition 3.2. □

The set $\pi_0\mathbf{C}$ of connected components of a small category $\mathbf{C}$ is formed by equivalence classes $\{X\}$ of objects in $\mathbf{C}$. Two objects X, Y are equivalent $\{X\} = \{Y\}$ if there exists a sequence of (non-composable) morphisms in $\mathbf{C}$ connecting them

$$X \to \bullet \leftarrow \cdots \to \bullet \leftarrow Y.$$

This defines a product-preserving functor from small categories to sets

$$\pi_0 \colon \mathbf{Cat} \longrightarrow \mathbf{Set}$$

with $\pi_0(\varphi)\{X\} = \{\varphi(X)\}$. Moreover, one can obtain an ordinary category $\mathbf{M}^0$ from a 2-category $\mathbf{M}$ by taking π_0 on morphism categories and also an ordinary functor $\rho^0 \colon \mathbf{M}^0 \to \mathbf{N}^0$ from a 2-functor $\rho \colon \mathbf{M} \to \mathbf{N}$. If $\mathbf{M}$ is a category regarded as a 2-category with only the trivial 2-morphisms then $\mathbf{M}^0 = \mathbf{M}$.

The homotopy category of cochain complexes $\mathbf{Cochain}/\simeq$ coincides with $\mathbf{Cochain}_2^0$ and the 2-functor $\imath \colon \mathbf{Cochain} \to \mathbf{Cochain}_2$ in (5.A) induces the natural projection $\imath^0 \colon \mathbf{Cochain} \to \mathbf{Cochain}/\simeq$ onto the quotient category.

By Theorem 5.1 there is a commutative diagram of functors

(5.D)
$$\begin{array}{ccc} \mathbf{Nat} & \xrightarrow{\ F^*\ } & \mathbf{Cochain} \\ {\scriptstyle\jmath^0}\downarrow & & \downarrow{\scriptstyle\imath^0} \\ \mathbf{Nat}_{\mathcal{F}}^0 & \xrightarrow[\ (F^*)^0\ ]{} & \mathbf{Cochain}/\simeq \end{array}$$

Proposition 5.2. *The functor $\jmath^0 \colon \mathbf{Nat} \to \mathbf{Nat}_{\mathcal{F}}^0$ is full.*

Proof. Let $(\alpha, t) \colon (\mathbf{C}, D) \to (\mathbf{D}, E)$ be a morphism in $\mathbf{Nat}_{\mathcal{F}}$, as in (5.B). We consider the new morphism $(1_{\varphi}, t(1_D * \mathcal{F}(1_{\varphi}, \alpha))) \colon (\mathbf{C}, D) \to (\mathbf{D}, E)$. Notice that $(1_{\varphi}, t(1_D * \mathcal{F}(1_{\varphi}, \alpha))) = \jmath(\varphi, t(1_D * \mathcal{F}(1_{\varphi}, \alpha)))$. There is a 2-morphism in $\mathbf{Nat}_{\mathcal{F}}$

$$(1_{\varphi}, \alpha) \colon (1_{\varphi}, t(1_D * \mathcal{F}(1_{\varphi}, \alpha))) \Rightarrow (\alpha, t),$$

therefore

$$\{(\alpha, t)\} = \{\jmath(\varphi, t(1_D * \mathcal{F}(1_{\varphi}, \alpha)))\} = \jmath^0(\varphi, t(1_D * \mathcal{F}(1_{\varphi}, \alpha))).$$

$\square$

Proposition 5.2 shows that $\mathbf{Nat}_{\mathcal{F}}^0$ is a quotient category of $\mathbf{Nat}$, because $\jmath^0$ is the identity on objects, and diagram (5.D) yields a factorization of the Baues-Wirsching cohomology functors through $\mathbf{Nat}_{\mathcal{F}}^0$.

Corollary 5.3. *The functors* $H^n \colon \mathbf{Nat} \to \mathbf{Ab}$ ($n \in \mathbb{Z}$) *factor through the quotient category* $\mathbf{Nat}_{\mathcal{F}}^0$.

6. (Co)localization and cohomology

In this section we prove two theorems, the localization and colocalization theorems, which allow to easy computations in Baues-Wirsching cohomology of small categories when the coefficient natural system satisfy some (co)locality conditions that will be made precise later.

Here we adopt the following definition for a localization.

Definition 6.1. A *localization* is an adjoint pair (φ left adjoint to ψ)

$$\mathbf{C} \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathbf{D}$$

such that $\varphi\psi = 1_{\mathbf{D}}$ and the counit is the identity natural transformation.

Remark 6.2.   (1) A localization can also be defined as an idempotent functor $\xi \colon \mathbf{C} \to \mathbf{C}$ together with a natural transformation $\alpha \colon 1_{\mathbf{C}} \Rightarrow \xi$ such that $1_{\xi} * \alpha = 1_{\xi} = \alpha * 1_{\xi}$. This and the former definition are equivalent. On one hand, in this situation we can take $\mathbf{D}$ to be the image of ξ, $\psi \colon \mathbf{D} \to \mathbf{C}$ the (faithful) inclusion functor and $\varphi \colon \mathbf{C} \to \mathbf{D}$ the unique (full) functor such that $\xi = \psi\varphi$, then we obtain a localization in the sense of Definition 6.1 being α the unit of the adjunction. On the other hand, if we are as in the previous definition the functor $\xi = \psi\varphi$ is idempotent and the unit of the adjunction $\alpha \colon 1 \Rightarrow \xi$ satisfies $1_{\xi} * \alpha = 1_{\xi} = \alpha * 1_{\xi}$.

(2) A related concept is that of a category of fractions. Given a set Σ of morphisms in $\mathbf{C}$ the *category of fractions* $\mathbf{C}[\Sigma^{-1}]$ is a category together with a functor $\mathbf{C} \to \mathbf{C}[\Sigma^{-1}]$ which is universal among all functors which send morphisms in Σ to isomorphisms. Let $\mathbf{C}_{\Sigma}$ be the full subcategory of $\mathbf{C}$ formed by those objects X such that $\mathbf{C}(f, X)$ is a bijection for any $f \in \Sigma$ and $\psi \colon \mathbf{C}_{\Sigma} \to \mathbf{C}$ the full inclusion. If there exists a functor $\varphi \colon \mathbf{C} \to \mathbf{C}_{\Sigma}$ as in Definition 6.1 then the composite $\mathbf{C}_{\Sigma} \to \mathbf{C} \to \mathbf{C}[\Sigma^{-1}]$ is known to be an equivalence of categories. On the other hand if we are in the situation of Definition 6.1 and Σ is the set of morphisms f such that $\varphi(f)$ is an isomorphism then ψ induces an equivalence between $\mathbf{D}$ and $\mathbf{C}_{\Sigma}$.

Definition 6.3. In the situation of Definition 6.1, a natural system D on $\mathbf{C}$ is said to be $\mathbf{D}$-*local* if given $f\colon X \to Y$ in $\mathbf{C}$ such that $\varphi(f)$ is an isomorphism then $D(1_X, f)\colon D(1_X) \to D(f)$ is also an isomorphism.

The following proposition gives alternative characterizations of $\mathbf{D}$-local natural systems.

Proposition 6.4. *Given a natural system D on $\mathbf{C}$ the following conditions are equivalent:*

(1) *D is $\mathbf{D}$-local,*
(2) *D is naturally isomorphic to $E\mathcal{F}(\alpha)$ for some natural system E on $\mathbf{C}$,*
(3) *D is naturally isomorphic to $D\mathcal{F}(\alpha)$.*

Here $\alpha\colon 1_{\mathbf{C}} \Rightarrow \psi\varphi$ is the unit of the adjunction.

Proof. Let $f\colon X \to Y$ be a morphism in $\mathbf{C}$ such that $\varphi(f)$ is an isomorphism, then $E\mathcal{F}(\alpha)(1_X, f) = E(1_X, \psi\varphi(f))$ is an isomorphism because $(1_X, \psi\varphi(f))$ is an isomorphism in $\mathcal{F}\mathbf{C}$ with inverse $(1_X, \psi(\varphi(f)^{-1}))$.

Suppose now that D is $\mathbf{D}$-local. The natural transformation

$$1_D * \mathcal{F}(1_{1_{\mathbf{C}}}, \alpha)\colon D = D\mathcal{F}(1_{1_{\mathbf{C}}}) \longrightarrow D\mathcal{F}(\alpha)$$

is a natural isomorphism since given an object f in $\mathcal{F}\mathbf{C}$, which is a morphism $f\colon X \to Y$ in $\mathbf{C}$, we have that $(1_D * \mathcal{F}(1_{1_{\mathbf{C}}}, \alpha))_f = D(1_X, \alpha_Y)$ and $\varphi(\alpha_Y) = 1_{\varphi(Y)}$. For this last equality we use that φ is left-adjoint to ψ, α is the unit, and $\varphi\psi = 1$. $\square$

Theorem 6.5. *Under the conditions of Definition 6.1, if D is a $\mathbf{D}$-local natural system on $\mathbf{C}$ then ψ induces isomorphisms $(n \in \mathbb{Z})$*

$$H^n(\mathbf{C}, D) \xrightarrow{\simeq} H^n(\mathbf{D}, D\mathcal{F}(\psi)).$$

Proof. The morphism in the statement is $H^n(\psi, 1_{D\mathcal{F}(\psi)})$. If $\alpha\colon 1_{\mathbf{C}} \Rightarrow \psi\varphi$ is the unit we have that $\alpha * \alpha = \alpha$, see Remark 6.2 (1) and the beginning of the proof of the following lemma, hence by Proposition 6.4 we can suppose without loss of generality that $D = D\mathcal{F}(\alpha)$. Consider the natural transformation

$$1_D * \mathcal{F}(\alpha, 1_{\psi\varphi})\colon D\mathcal{F}(\psi)\mathcal{F}(\varphi) = D\mathcal{F}(\psi\varphi) = D\mathcal{F}(1_{\psi\varphi}) \Rightarrow D\mathcal{F}(\alpha) = D.$$

We claim that $H^n(1_\varphi, 1_D * \mathcal{F}(\alpha, 1_{\psi\varphi}))$ is the inverse of

$$H^n(\psi, 1_{D\mathcal{F}(\psi)}) = H^n(1_\psi, 1_{D\mathcal{F}(1_\psi)}),$$

for this equality we use the commutativity of the diagram in Proposition 5.1.

On one hand we have the following equalities in $\mathbf{Nat}_\mathcal{F}$

$$
\begin{aligned}
(1_\psi, 1_{D\mathcal{F}(1_\psi)})(1_\varphi, 1_D * \mathcal{F}(\alpha, 1_{\psi\varphi})) &= (1_\varphi * 1_\psi, 1_{D\mathcal{F}(1_\psi)}(1_D * \mathcal{F}(\alpha, 1_{\psi\varphi}) * 1_{\mathcal{F}(1_\psi)})) \\
&= (1_{\varphi\psi}, 1_D * \mathcal{F}(\alpha, 1_{\psi\varphi}) * \mathcal{F}(1_\psi, 1_\psi)) \\
&= (1_{\mathbf{D}}, 1_D * \mathcal{F}(\alpha * 1_\psi, 1_{\psi\varphi} * 1_\psi)) \\
&= (1_{\mathbf{D}}, 1_D * \mathcal{F}(1_\psi, 1_{\psi\varphi\psi})) \\
&= (1_{\mathbf{D}}, 1_D * \mathcal{F}(1_\psi, 1_\psi)) \\
&= (1_{\mathbf{D}}, 1_D * 1_{\mathcal{F}(1_\psi)}) \\
&= (1_{\mathbf{D}}, 1_{D\mathcal{F}(1_\psi)}).
\end{aligned}
$$

Here we use the equality $\alpha * 1_\psi = 1_\psi$ which can be checked by using the equality $\varphi\psi = 1_{\mathbf{D}}$ and elementary properties of adjoint functors.

On the other hand

$$(1_\varphi, 1_D * \mathcal{F}(\alpha, 1_{\psi\varphi}))(1_\psi, 1_{D\mathcal{F}(1_\psi)}) = (1_\psi * 1_\varphi, (1_D * \mathcal{F}(\alpha, 1_{\psi\varphi}))(1_{D\mathcal{F}(1_\psi)} * 1_{\mathcal{F}(1_\varphi)}))$$
$$= (1_{\psi\varphi}, 1_D * \mathcal{F}(\alpha, 1_{\psi\varphi})).$$

Now the theorem follows from Corollary 5.3, Remark 6.2 (1) and the following lemma. $\qquad\square$

Lemma 6.6. *Let $\xi\colon \mathbf{C} \to \mathbf{C}$ and $\alpha\colon 1_{\mathbf{C}} \Rightarrow \xi$ be a localization in the sense of Remark 6.2 (1) and D any natural system on $\mathbf{C}$, then there is a diagram in $\mathbf{Nat}_{\mathcal{F}}$ as follows*

Proof. We have that $\alpha * \alpha = (\alpha * 1_\xi)(1_{1_{\mathbf{C}}} * \alpha) = 1_\xi \alpha = \alpha$. By using the functor $\mathcal{F}$ defined in Proposition 3.2 we see that $\mathcal{F}(\alpha)\mathcal{F}(1_\xi) = \mathcal{F}(\alpha * 1_\xi) = \mathcal{F}(1_\xi)$, $\mathcal{F}(\alpha, 1_\xi)\colon \mathcal{F}(1_\xi) \to \mathcal{F}(\alpha)$, and $\mathcal{F}(\alpha)\mathcal{F}(\alpha) = \mathcal{F}(\alpha * \alpha) = \mathcal{F}(\alpha)$, therefore the 1-morphisms in the diagram are well defined. Moreover, the 2-morphisms are also well-defined because $1_{D\mathcal{F}(\alpha)} = 1_D * 1_{\mathcal{F}(\alpha)}$, $1_{\mathcal{F}(\alpha)} = \mathcal{F}(1_{1_{\mathbf{C}}}, 1_\xi)$ and

$$\mathcal{F}(1_{1_{\mathbf{C}}}, 1_\xi) * \mathcal{F}(1_{1_{\mathbf{C}}}, \alpha) = \mathcal{F}(1_{1_{\mathbf{C}}} * 1_{1_{\mathbf{C}}}, 1_\xi * \alpha) = \mathcal{F}(1_{1_{\mathbf{C}}}, 1_\xi),$$

$$\mathcal{F}(1_{1_{\mathbf{C}}}, 1_\xi) * \mathcal{F}(\alpha, 1_\xi) = \mathcal{F}(1_{1_{\mathbf{C}}} * \alpha, 1_\xi * 1_\xi) = \mathcal{F}(\alpha, 1_\xi).$$

$\qquad\square$

Example 6.7. Here we exhibit some examples of the usefulness of Theorem 6.5 as a computational tool in cohomology of categories. We recall from [2] that any functor $\mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Ab}$ yields a natural system on $\mathbf{C}$ through the natural functor $\mathcal{F}\mathbf{C} \to \mathbf{C}^{\mathrm{op}} \times \mathbf{C}$ which sends an object f, which is a morphism $f\colon X \to Y$ in $\mathbf{C}$, to the pair (X, Y).

(1) In his study of the 2-dimensional cohomology groups of the category $\mathbf{Ab}_0$ of finitely generated abelian groups Hartl [6] computed

(6.A) $$H^2(\mathbf{Ab}_0, \mathrm{Ext}^1_{\mathbb{Z}}(-, - \otimes \mathbb{Z}/2)) \simeq \mathbb{Z}/2.$$

The generator is the characteristic class associated to the stable homotopy category of compact Moore spaces, see [2] 3.2. Let $\psi\colon \mathbf{vect}(\mathbb{Z}/2) \to \mathbf{Ab}_0$ be the inclusion of the full subcategory of finite-dimensional $\mathbb{Z}/2$-vector spaces, whose left-adjoint is $\varphi = - \otimes \mathbb{Z}/2$. Clearly $\mathrm{Ext}^1_{\mathbb{Z}}(-, - \otimes \mathbb{Z}/2)$ is $\mathbf{vect}(\mathbb{Z}/2)$-local. Moreover, there is a natural isomorphism

$$\mathrm{Ext}^1_{\mathbb{Z}}(-, - \otimes \mathbb{Z}/2)\mathcal{F}(\psi) \simeq \mathrm{Hom}_{\mathbb{Z}/2}$$

induced by the change of rings spectral sequence associated to the natural projection $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2$, see [9] 10.2 (c), therefore by Theorem 6.5

$$H^n(\mathbf{Ab}_0, \mathrm{Ext}^1_{\mathbb{Z}}(-, - \otimes \mathbb{Z}/2)) \simeq H^n(\mathbf{vect}(\mathbb{Z}/2), \mathrm{Hom}_{\mathbb{Z}/2}), \quad n \in \mathbb{Z}.$$

Moreover, by Theorem A and Corollary 3.11 in [7] this is the Mac Lane cohomology of $\mathbb{Z}/2$,

$$H^n(\mathbf{vect}(\mathbb{Z}/2), \mathrm{Hom}_{\mathbb{Z}/2}) \simeq H^n_{ML}(\mathbb{Z}/2, \mathbb{Z}/2) \simeq \begin{cases} \mathbb{Z}/2, & n \text{ even;} \\ \\ 0, & n \text{ odd;} \end{cases}$$

which was computed in [4], therefore we recover the isomorphism (6.A) from this complete calculation of $H^*(\mathbf{Ab}_0, \mathrm{Ext}^1_{\mathbb{Z}}(-, - \otimes \mathbb{Z}/2))$.

(2) More generally by Theorems A and B in [7] for any prime ℓ there is a spectral sequence converging to $H^{n+m}_{ML}(\mathbb{Z}, \mathbb{Z}/\ell)$ with

$$E_2^{nm} = H^n(\mathbf{Ab}_0, \mathrm{Ext}^m_{\mathbb{Z}}(-, - \otimes \mathbb{Z}/\ell)).$$

Let $\psi \colon \mathbf{vect}(\mathbb{Z}/\ell) \to \mathbf{Ab}_0$ be the full inclusion of the category of finite-dimensional $\mathbb{Z}/\ell$-vector spaces, whose left-adjoint is $\varphi = - \otimes \mathbb{Z}/\ell$. It is obvious that $\mathrm{Ext}^m_{\mathbb{Z}}(-, - \otimes \mathbb{Z}/\ell)$ is $\mathbf{vect}(\mathbb{Z}/\ell)$-local for all $m \geq 0$, moreover

$$\mathrm{Ext}^m_{\mathbb{Z}}(-, - \otimes \mathbb{Z}/\ell)\mathcal{F}(\psi) \simeq \begin{cases} \mathrm{Hom}_{\mathbb{Z}/\ell}, & \text{for } m = 0, 1; \\ \\ 0, & \text{otherwise;} \end{cases}$$

hence for $m = 0, 1$

$$H^n(\mathbf{Ab}_0, \mathrm{Ext}^m_{\mathbb{Z}}(-, - \otimes \mathbb{Z}/\ell)) \simeq H^n(\mathbf{vect}(\mathbb{Z}/\ell), \mathrm{Hom}_{\mathbb{Z}/\ell}) \simeq H^n_{ML}(\mathbb{Z}/\ell, \mathbb{Z}/\ell)$$

and zero for $m \neq 0, 1$. Therefore we recover from the previous spectral sequence a long exact sequence ($n \in \mathbb{Z}$)

$$\cdots \to H^{n-2}_{ML}(\mathbb{Z}/\ell, \mathbb{Z}/\ell) \to H^n_{ML}(\mathbb{Z}/\ell, \mathbb{Z}/\ell) \to H^n_{ML}(\mathbb{Z}, \mathbb{Z}/\ell) \to H^{n-1}_{ML}(\mathbb{Z}/\ell, \mathbb{Z}/\ell) \to \cdots$$

which was used in [4] to compute the Mac Lane cohomology $H^*_{ML}(\mathbb{Z}, \mathbb{Z}/\ell)$.

Definition 6.8. A *colocalization* is an adjoint pair (φ right adjoint to ψ)

$$\mathbf{C} \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathbf{D}$$

such that $\varphi\psi = 1_{\mathbf{D}}$ and the unit is the identity natural transformation.

Dually to Remark 6.2 we have the following observations.

Remark 6.9.   (1) A colocalization can also be regarded as an idempotent functor $\xi \colon \mathbf{C} \to \mathbf{C}$ together with a natural transformation $\alpha \colon \xi \Rightarrow 1_{\mathbf{C}}$ such that $1_\xi * \alpha = 1_\xi = \alpha * 1_\xi$. This is equivalent to the previous definition. Definition 6.8 yields this structure by taking $\xi = \psi\varphi$ and α the counit. Moreover, in the situation of this example we obtain a colocalization as in Definition 6.8 by taking $\mathbf{D}$ as the image of ξ, $\psi \colon \mathbf{D} \to \mathbf{C}$ the inclusion and $\varphi \colon \mathbf{C} \to \mathbf{D}$ the unique functor such that $\xi = \psi\varphi$.

(2) Given a set Σ of morphisms in $\mathbf{C}$ we define $_\Sigma\mathbf{C}$ to be the full subcategory of $\mathbf{C}$ formed by those objects X such that $\mathbf{C}(X, f)$ is a bijection for any $f \in \Sigma$. Let $\psi \colon {_\Sigma}\mathbf{C} \to \mathbf{C}$ be the full inclusion. If there exists a functor $\varphi \colon \mathbf{C} \to {_\Sigma}\mathbf{C}$ as in Definition 6.8 then the composite $_\Sigma\mathbf{C} \to \mathbf{C} \to \mathbf{C}[\Sigma^{-1}]$ is an equivalence of categories. On the other hand if we are in the situation of Definition 6.8 and Σ is the set of morphisms f such that $\varphi(f)$ is an isomorphism then ψ induces an equivalence between $\mathbf{D}$ and $_\Sigma\mathbf{C}$.

Definition 6.10. Under the conditions of Definition 6.8, a natural system D on $\mathbf{C}$ is said to be $\mathbf{D}$-*colocal* if given $f\colon X \to Y$ in $\mathbf{C}$ such that $\varphi(f)$ is an isomorphism then $D(f, 1_Y)\colon D(1_Y) \to D(f)$ is also an isomorphism.

Proposition 6.11. *Given a natural system D on $\mathbf{C}$ the following conditions are equivalent:*

    (1) *D is $\mathbf{D}$-colocal,*
    (2) *D is naturally isomorphic to $E\mathcal{F}(\alpha)$ for some natural system E on $\mathbf{C}$,*
    (3) *D is naturally isomorphic to $D\mathcal{F}(\alpha)$.*

Here $\alpha\colon \psi\varphi \Rightarrow 1_\mathbf{C}$ is the counit of the adjunction.

Proof. Given a morphism $f\colon X \to Y$ in $\mathbf{C}$ such that $\varphi(f)$ is an isomorphism, then $E\mathcal{F}(\alpha)(f, 1_Y) = E(\psi\varphi(f), 1_Y)$ is an isomorphism because $(\psi\varphi(f), 1_Y)$ is an isomorphism in $\mathcal{F}\mathbf{C}$ with inverse $(\psi(\varphi(f)^{-1}), 1_Y)$.

Conversely if D is $\mathbf{D}$-colocal one can readily check that the natural transformation

$$1_D * \mathcal{F}(\alpha, 1_{1_\mathbf{C}})\colon D = D\mathcal{F}(1_{1_\mathbf{C}}) \longrightarrow D\mathcal{F}(\alpha)$$

is a natural isomorphism, compare the proof of Proposition 6.4. $\qquad\square$

Theorem 6.12. *Under the conditions of Definition 6.8, if D is a $\mathbf{D}$-colocal natural system on $\mathbf{C}$ then ψ induces isomorphisms ($n \in \mathbb{Z}$)*

$$H^n(\mathbf{C}, D) \xrightarrow{\simeq} H^n(\mathbf{D}, D\mathcal{F}(\psi)).$$

The proof of this theorem is very similar to that of Theorem 6.5, for this reason we leave it to the reader. In this case one has to use Proposition 6.11, Remark 6.9 (1) and the following lemma instead of Proposition 6.4, Remark 6.2 (1) and Lemma 6.6.

Lemma 6.13. *Let $\xi\colon \mathbf{C} \to \mathbf{C}$ and $\alpha\colon \xi \Rightarrow 1_\mathbf{C}$ be a colocalization in the sense of Remark 6.9 (1) and D any natural system on $\mathbf{C}$, then there is a diagram in $\mathbf{Nat}_\mathcal{F}$ as follows*

The proof of this lemma is a mere verification as in Lemma 6.6.

Example 6.14. Here we combine the localization and colocalization theorems (Theorems 6.5 and 6.12) to reduce the computation of certain cohomology groups of a triangulated category, see [5] Chapter IV for basic facts.

More precisely, let $\mathbf{D}$ be a small triangulated category, whose translation functor is denoted by Σ, equipped with a t-structure $(\mathbf{D}^{\leq 0}, \mathbf{D}^{\geq 0})$ and $\mathbf{A} = \mathbf{D}^{\leq 0} \cap \mathbf{D}^{\geq 0}$ its core. We write $t_{\leq 0}\colon \mathbf{D} \to \mathbf{D}^{\leq 0}$ and $t_{\geq 0}\colon \mathbf{D} \to \mathbf{D}^{\geq 0}$ for the right and left adjoints of the corresponding full inclusions. Notice that the restriction of $t_{\geq 0}$ to $\mathbf{D}^{\leq 0}$, which is usually denoted by $H^0\colon \mathbf{D}^{\leq 0} \to \mathbf{A}$, is also a left-adjoint to the full inclusion.

We consider the natural system on $\mathbf{D}$ given by the functor

$$(6.\mathrm{B}) \qquad\qquad \mathbf{D}(t_{\leq 0}, \Sigma^n t_{\geq 0})\colon \mathbf{D}^{\mathrm{op}} \times \mathbf{D} \longrightarrow \mathbf{Ab}.$$

The restriction to $\mathbf{A}^{\mathrm{op}} \times \mathbf{A}$ will be denoted by $\mathrm{Ext}_{\mathbf{D}}^n$. Moreover, it coincides with the ordinary extension functor $\mathrm{Ext}_{\mathbf{A}}^n$ in the abelian category $\mathbf{A}$ when $\mathbf{D} = \mathcal{D}^b(\mathbf{A})$ is the bounded derived category of $\mathbf{A}$ with the canonical t-structure.

By using elementary properties of triangulated categories and t-structures one can check that the natural system defined by (6.B) is $\mathbf{D}^{\leq 0}$-colocal. Moreover, its restriction to $\mathbf{D}^{\leq 0}$, that is $\mathbf{D}^{\leq 0}(-, \Sigma^n H^0)$, is $\mathbf{A}$-local, therefore

$$H^*(\mathbf{D}, \mathbf{D}(t_{\leq 0}, \Sigma^n t_{\geq 0})) \simeq H^*(\mathbf{D}^{\leq 0}, \mathbf{D}^{\leq 0}(-, \Sigma^n H^0)) \simeq H^*(\mathbf{A}, \mathrm{Ext}_{\mathbf{D}}^n).$$

This and similar examples are relevant for the approach to homotopy theory given by the tower of categories in [1]. An explicit application can be found in [11].

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