EXTRACTA MATHEMATICAE Vol. **20**, Núm. 1, 1 – 12 (2005)

About a family of Naturally Graded no p-filiform Lie algebras †

L.M. Camacho, J.R. Gómez, A.J. González

Departamento de Matemática Aplicada I, Universidad de Sevilla
Avda. Reina Mercedes s/n, 41012 *Sevilla, Spain*
e-mail: lcamacho@us.es, jrgomez@us.es, agonzalez@us.es

(Presented by Santos González)

AMS *Subject Class.* (2000): 17B45, 17B70 *Received December* 20, 2004

1. Introduction

The knowledge of naturally graded Lie algebras of a particular Lie algebras class gives a valuable information about the structure of the rest of algebras of that class. In 1970, Vergne [9] obtained the classification in finite arbitrary dimension, n, for the case of filiform (nilindex $n-1$). In [8, 7] Goze and Khakimdjanov gave the geometric description of the characteristically nilpotent filiform Lie algebras using the naturally graded filiform Lie algebras. In [6] Gómez and Jiménez-Merchán, obtained the classification in finite arbitrary dimension for the case 2-filiform (nilindex $n-2$). There are two subcases for the nilindex $n-3$: 3-filiform Lie algebras and the Lie algebras with characteristic sequence $(n-3,2,1)$. In [4, 5], Cabezas, Gómez and Pastor gave the classification of naturally graded p-filiform Lie algebras.

Consistently, for nilindex $n-3$, only rest to study the case of characteristic sequence $(n-3,2,1)$. In this work we offer the classification in arbitrary finite dimension of the family of naturally graded Lie algebras $\mathfrak{g}$ with the above characteristic sequence such that the dimension of the derived ideal is minimum, that is, with $\dim[\mathfrak{g},\mathfrak{g}] = n-3$.

The two first acceptable dimensions are 5 and 6, but the general situation occurs only for $n \geq 7$.

† This paper has been partially supported by the PAICYT, of Junta de Andalucía (Spain), and by the Ministerio de Ciencia y Tecnología (Spain), ref. BFM 2000-1047

2. Preliminaries

The *descending central sequence* of a Lie algebra $\mathfrak{g}$ is defined by $(\mathcal{C}^i(\mathfrak{g}))$, $i \in \mathbb{N} \cup \{0\}$, where $\mathcal{C}^0(\mathfrak{g}) = \mathfrak{g}$ and $\mathcal{C}^i(\mathfrak{g}) = [\mathfrak{g}, \mathcal{C}^{i-1}(\mathfrak{g})]$.

A Lie algebra $\mathfrak{g}$ is called *nilpotent* if there exists $k \in \mathbb{N}$ such that $\mathcal{C}^k(\mathfrak{g}) = \{0\}$. The smallest integer verifying this equation is called the *nilindex* of $\mathfrak{g}$.

A Lie algebra $\mathfrak{g}$, with $\dim(\mathfrak{g}) = n$, is called *filiform* (or 1-*filiform*) if it verifies $\dim(\mathcal{C}^i(\mathfrak{g})) = n-i-1$ for $1 \leq i \leq n-1$. These algebras have maximal nilindex $n-1$. The Lie algebras with a nilindex $n-2$ are called *quasifiliform* (or 2-*filiform*) and those whose nilindex is 1 are called *abelian*.

Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension n.

For all $X \in \mathfrak{g} - [\mathfrak{g}, \mathfrak{g}]$, $c(X) = (c_1(X), c_2(X), \ldots, 1)$ is the sequence, in decreasing order, of the dimensions of the characteristic subspaces of the *nilpotent operator* $\mathrm{ad}(X)$, where the adjoint operator of an element $X \in \mathfrak{g}$, $\mathrm{ad}(X)$, is defined by

$$\begin{array}{rcl} \mathrm{ad}(X) : \mathfrak{g} & \rightarrow & \mathfrak{g} \\ Y & \mapsto & [X, Y]\,. \end{array}$$

The finite sequence $c(\mathfrak{g}) = \sup\{c(X) : X \in \mathfrak{g} - [\mathfrak{g}, \mathfrak{g}]\}$ is called the *characteristic sequence* or *Goze invariant* of the nilpotent Lie algebra $\mathfrak{g}$. The filiform, quasifiliform and abelian Lie algebras of dimension n have as their Goze invariant $(n-1, 1)$, $(n-2, 1, 1)$ and $(1, 1, \ldots, 1)$, respectively. The Lie algebras with characteristic sequence $(n-p, 1, \ldots, 1)$ are known as *p-filiform* Lie algebras [3]. We know the classification of p-filiform for the integer values of p between $n-5$ and $n-2$ ([2, 1]). Remark that, for nilindex $n-3$, there are two families with Goze invariant $(n-3, 1, 1, 1)$ and $(n-3, 2, 1)$ respectively.

Note that a complex Lie algebra $\mathfrak{g}$ is naturally filtered by the descending central sequence. This result leads to associate any Lie algebra $\mathfrak{g}$ with a graded Lie algebra, $\mathrm{gr}\,\mathfrak{g}$ with equal nilindex:

$$\mathrm{gr}\,\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i\,, \qquad \mathfrak{g}_i = \mathcal{C}^{i-1}(\mathfrak{g})/\mathcal{C}^i(\mathfrak{g})\,.$$

By nilpotency, the above graduation is finite, that is $\mathrm{gr}\,\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k$ with $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, for $i+j \leq k$. A Lie algebra $\mathfrak{g}$ is said to be naturally graded if $\mathrm{gr}\,\mathfrak{g}$ is isomorphic to $\mathfrak{g}$, what will be denoted henceforth by $\mathrm{gr}\,\mathfrak{g}=\mathfrak{g}$.

Let $\{X_0, X_1, \ldots, X_{n-3}, Y_1, Y_2\}$ be an adapted basis of $\mathfrak{g}$. We study the case where the dimension of the derived ideal is minimum, consistently $\dim[\mathfrak{g}, \mathfrak{g}] = n-3$. Thus, Y_1 is not in $[\mathfrak{g}, \mathfrak{g}]$ and, consequently, $Y_1 \in \mathfrak{g}_1$. In general, if we denote as r to the position of the vector Y_1 into the subspaces of the natural graduation, we observe that the value of r is $r = 1$. We remark that the position of Y_2 is previously determined because we have that $[X_0, Y_1] = Y_2$ and that implies $Y_2 \in \mathfrak{g}_{r+1}$ with $1 \leq r \leq n-4$. Then, in this case $Y_2 \in \mathfrak{g}_2$.

From now, Jacobi identity for the vectors X, Y, Z will be denoted as $\mathrm{Jac}(X, Y, Z)$ and the laws of the algebras, $\mathfrak{g}$, of dimension n such that $\dim[\mathfrak{g}, \mathfrak{g}]$ is minimum will be denoted as μ_n.

3. Structure theorem

In this section, we will obtain a first approximation to the structure of naturally graded Lie algebras with Goze invariant $(n-3, 2, 1)$.

Let $\mathfrak{g}$ be a naturally graded Lie algebra of Goze's invariant $(n-3, 2, 1)$ and let $\{X_0,\ X_1, \ldots, X_{n-3},\ Y_1,\ Y_2\}$ be an adapted basis of $\mathfrak{g}$, that is:

$$\begin{aligned} &[X_0, X_i] = X_{i+1} \quad (1 \leq i \leq n-4)\,, \\ &[X_0, X_{n-3}] = 0\,, \\ &[X_0, Y_1] = Y_2\,, \\ &[X_0, Y_2] = 0\,, \end{aligned}$$

where $X_0 \in \mathfrak{g} - [\mathfrak{g}, \mathfrak{g}]$. That implies

$$\begin{aligned} &\mathcal{C}^1(\mathfrak{g}) \supset \langle X_2, X_3, \ldots, X_{n-3}, Y_2 \rangle\,, \\ &\mathcal{C}^i(\mathfrak{g}) \supset \langle X_{i+1}, X_{i+2}, \ldots, X_{n-3} \rangle \quad (2 \leq i \leq n-4)\,. \end{aligned}$$

Lemma 3.1. *Let $\mathfrak{g}$ be a Lie algebra of dimension n and Goze's invariant $(n-3, 2, 1)$ and let $\{X_0, X_1, \ldots, X_{n-3}, Y_1, Y_2\}$ be an adapted basis of $\mathfrak{g}$. Then,*

$$X_1 \notin \mathcal{C}^1(\mathfrak{g})\,, \qquad X_{n-3} \in \mathcal{Z}(\mathfrak{g})\,, \qquad Y_1 \notin \mathcal{C}^{n-4}(\mathfrak{g})\,, \qquad Y_2 \notin \mathcal{C}^{n-3}(\mathfrak{g})\,.$$

Proof. Obviously, $X_{n-3} \in \mathcal{Z}(\mathfrak{g})$, $Y_1 \notin \mathcal{C}^{n-4}(\mathfrak{g})$ and $Y_2 \notin \mathcal{C}^{n-3}(\mathfrak{g})$ because, otherwise, $\mathfrak{g}$ could not be of characteristic sequence $(n-3, 2, 1)$. It is easy to prove that $X_1 \notin [\mathfrak{g}, \mathfrak{g}]$ supposing that $X_1 \in [Y_1, Y_2]$, or $X_1 \in [X_i, Y_j]$, $1 \leq i \leq n-4$, $1 \leq j \leq 2$, or $X_1 \in [X_i, X_j]$, $1 \leq i < j \leq n-3-i$, and obtaining contradiction. ∎

Remark 3.2. We identify each vector with its class, and we call $\mu(n, r)$ the family of laws of Lie algebras with Goze invariant $(n-3, 2, 1)$ where n is the dimension and r is the position of Y_1 in the subsets of the natural gradation. We remark that the position of Y_2 is previously determined because we have that $[X_0, Y_1] = Y_2$ and that implies $Y_2 \in \mathfrak{g}_{r+1}$ with $1 \leq r \leq n-4$.

Remark 3.3. It is easy to see that $\mathfrak{g}_1 \supset \langle X_0, X_1\rangle$ and $\mathfrak{g}_i \supset \langle X_i\rangle$, $2 \le i \le n-3$.

Now, we obtain the general structure of laws of naturally graded Lie algebras of characteristic sequence $(n-3, 2, 1)$ in arbitrary dimension. At first, we prove that if $Y_1 \in \mathfrak{g}_r$, then r is odd.

LEMMA 3.4. *If r is even, the case $\mu(n, r)$ is not admissible in any dimension.*

Proof. Let $\mathfrak{g}$ be a naturally graded Lie algebra of Goze invariant $(n-3, 2, 1)$, let $\{X_0, X_1, \ldots, X_{n-3}, Y_1, Y_2\}$ be an adapted basis of $\mathfrak{g}$, and let $Y_1 \in \mathfrak{g}_r$ be with r even. It is easy to prove that $Y_1 \notin [\mathfrak{g}, \mathfrak{g}]$ so $Y_1 \in \mathfrak{g}_1$ and this is impossible because r is even. ∎

THEOREM 3.5. (STRUCTURE THEOREM) *Any complex naturally graded Lie algebra $\mathfrak{g}$ of dimension $n \ge 5$, with Goze invariant $(n-3, 2, 1)$ is isomorphic to one whose law can be expressed in an adapted basis $\{X_0, X_1, \ldots, X_{n-3}, Y_1, Y_2\}$ by:*

- *If $r = 1$*

$$\begin{cases} [X_0, X_i] = X_{i+1} & (1 \le i \le n-4), \\ [X_0, Y_1] = Y_2, & \\ [X_i, X_j] = a_{ij} X_{i+j} & (1 \le i < j \le n-3-i). \end{cases}$$

- *If $3 \le r \le \frac{n-5}{2}$, r odd*

$$\begin{cases} [X_0, X_i] = X_{i+1} & (1 \le i \le n-4), \\ [X_0, Y_1] = Y_2, & \\ [X_i, X_j] = a_{ij} X_{i+j} \qquad (i+j \notin \{r, r+1\}, & 1 \le i < j \le n-3-i), \\ [X_i, X_{r-i}] = a_{i,r-i} X_r + (-1)^{i-1} Y_1 & (1 \le i \le \frac{r-1}{2}), \\ [X_i, X_{r+1-i}] = a_{i,r+1-i} X_{r+1} + (-1)^{i-1} \frac{(r+1-2i)}{2} Y_2 & (1 \le i \le \frac{r-1}{2}), \\ [X_i, Y_1] = \varepsilon X_{r+i} & (1 \le i \le n-3-r), \end{cases}$$

with $\varepsilon \in \{0, 1\}$.

• *If* $\frac{n-4}{2} \le r \le n-4$, *r odd*

$$\begin{cases} [X_0, X_i] = X_{i+1} & (1 \le i \le n-4)\,, \\ [X_0, Y_1] = Y_2\,, & \\ [X_i, X_j] = a_{ij} X_{i+j} \qquad (i+j \notin \{r, r+1\}\,,\ 1 \le i < j \le n-3-i)\,, & \\ [X_i, X_{r-i}] = a_{i,r-i} X_r + (-1)^{i-1} Y_1 & (1 \le i \le \frac{r-1}{2})\,, \\ [X_i, X_{r+1-i}] = a_{i,r+1-i} X_{r+1} & \\ \qquad\qquad +(-1)^{i-1} \frac{(r+1-2i)}{2} Y_2 & (1 \le i \le \frac{r-1}{2})\,, \\ [X_i, Y_1] = (c_1 - (i-1)c_2) X_{r+i} & (1 \le i \le n-3-r \le \frac{n-2}{2})\,, \\ [X_i, Y_2] = c_2 X_{r+1+i} & (1 \le i \le n-4-r \le \frac{n-4}{2})\,, \\ [Y_1, Y_2] = h X_{n-3} & (h = 0 \text{ if } r \ne \frac{n-4}{2})\,, \end{cases}$$

with $c_1, c_2 \in \mathbb{C}$.

Proof. If $\mathfrak{g}$ is in the condition of theorem, then a first general expression of $\mathfrak{g}$ is given by:

$$\begin{cases} [X_0, X_i] = X_{i+1} & (1 \le i \le n-4)\,, \\ [X_0, Y_1] = Y_2\,, & \\ [X_i, X_j] = a_{ij} X_{i+j} \qquad (i+j \notin \{r, r+1\}\,,\ 1 \le i < j \le n-3-i)\,, & \\ [X_i, X_{r-i}] = a_{i,r-i} X_r + b_{i1} Y_1 & (1 \le i \le \frac{r-1}{2})\,, \\ [X_i, X_{r+1-i}] = a_{i,r+1-i} X_{r+1} + b_{i2} Y_2 & (1 \le i \le \frac{r-1}{2})\,, \\ [X_1, Y_1] = c_{11} X_{r+1} + d Y_2\,, & \\ [X_i, Y_1] = c_{i1} X_{r+i} & (2 \le i \le n-3-r)\,, \\ [X_i, Y_2] = c_{i2} X_{r+1+i} & (1 \le i \le n-4-r)\,, \\ [Y_1, Y_2] = h X_{2r+1} & (\text{si } r \le \frac{n-4}{2})\,. \end{cases}$$

Some elementary changes of basis jointly with Jacobi identity implies that:

• If $1 \le r \le \frac{n-5}{2}$ the coefficients can be expressed by

$$c_{i,1} = c_1 \quad (1 \le i \le n-3-r) \qquad \text{and} \qquad c_{i,2} = 0 \quad (1 \le i \le n-4-r)\,.$$

• If $\frac{n-4}{2} \le r \le n-4$ the coefficients can be expressed by

$$c_{i,1} = c_1 - (i-1)c_2 \quad (1 \le i \le n-3-r) \qquad \text{and} \qquad c_{i,2} = c_2 \quad (1 \le i \le n-4-r)\,.$$

By using Jacobi identity it is posible to obtain that

$$b_{i,2} = (-1)^{(i-1)} \frac{r+1-2i}{2} b_1, \qquad 1 \le i \le \frac{r-1}{2}\,.$$

Furthermore, $b_1 \neq 0$ (in other case $Y_1 \notin \mathcal{C}^1(\mathfrak{g})$ and then $Y_1 \notin \mathfrak{g}_r =< X_r, Y_1 >$ with $r \geq 3$). Next, an easy change of basis allows to suppose $b_1 = 1$. Then,

• If $3 \leq r \leq \frac{n-5}{2}$. As $b_1 \neq 0$, if $c_1 \neq 0$ an easy change of basis allows to suppose $c_1 = 1$, and consistently $c_1 \in \{0, 1\}$.

• If $r = 1$, the case must be studied separately. ∎

4. Dimensions $n = 5$ and $n = 6$.

Even if our main aim is to study the case of dimension n finite arbitrary, the low dimensional cases are special and we will study them previously. The lowest cases are for dimensions $n = 5$ and $n = 6$ and they have a special treatment.

Theorem 4.1. *Any complex naturally graded Lie algebra of dimension* 5 *with Goze invariant* $(2, 2, 1)$ *is isomorphic to one whose law can be expressed in an adapted basis* $\{X_0, X_1, X_2, Y_1, Y_2\}$ *by:*

$$\mu_5 : \begin{cases} [X_0, X_1] = X_2\,, \\ [X_0, Y_1] = Y_2\,. \end{cases}$$

Proof. The proof is trivial. ∎

Theorem 4.2. *Any complex naturally graded Lie algebra of dimension* 6 *with Goze invariant* $(3, 2, 1)$ *is isomorphic to one whose law can be expressed in an adapted basis* $\{X_0, X_1, X_2, X_3, Y_1, Y_2\}$ *by:*

$$\mu_6^1 : \begin{cases} [X_0, X_i] = X_{i+1} \quad (1 \leq i \leq 2)\,, \\ [X_0, Y_1] = Y_2\,, \end{cases} \qquad \mu_6^2 : \begin{cases} [X_0, X_i] = X_{i+1} \quad (1 \leq i \leq 2)\,, \\ [X_0, Y_1] = Y_2\,, \\ [X_1, X_2] = X_3\,. \end{cases}$$

Proof. In dimension six the graduation is

$$\langle X_0, X_1, Y_1 \rangle \oplus \langle X_2, Y_2 \rangle \oplus \langle X_3 \rangle\,,$$

and by Theorem 3.5 the laws of these algebras are the following:

$$\begin{cases} [X_0, X_1] = X_2\,, \\ [X_0, X_2] = X_3\,, \\ [X_0, Y_1] = Y_2\,, \\ [X_1, X_2] = a_{12} X_3\,. \end{cases}$$

By using a generic change of basis we prove that nullity of coefficient a_{12} is an invariant.

• If $a_{12} \neq 0$, it is easy to obtain the algebra of law μ_6^2.

• If $a_{12} = 0$, we obtain the algebra of law μ_6^1. ∎

5. Dimension $n \geq 7$.

Now, we present the classification of the naturally graded Lie algebras with Goze invariant $(n-3, 2, 1)$, dimension $n \geq 7$ and $\dim[\mathfrak{g}, \mathfrak{g}]$ minimum, that is, equal to $n-3$. The first expression of this family is given by the following lemma:

Lemma 5.1. *Let $\mathfrak{g}$ be a naturally graded Lie algebra with Goze invariant $(n-3, 2, 1)$, $\dim(\mathfrak{g}) = n \geq 7$ and $\dim[\mathfrak{g}, \mathfrak{g}] = n-3$. Then, there exists a characteristic vector X_0 and an adapted basis $\{X_0, X_1, \ldots, X_{n-3}, Y_1, Y_2\}$, which lead us to express the laws of $\mathfrak{g}$ by:*

$$\mu_n^a : \begin{cases} [X_0, X_i] = X_{i+1} & (1 \leq i \leq n-4)\,, \\ [X_0, Y_1] = Y_2\,, \\ [X_1, X_i] = aX_{i+1} & (2 \leq i \leq n-4)\,, \end{cases}$$

if n is odd, or

$$\mu_n^{a,b} : \begin{cases} [X_0, X_i] = X_{i+1} & (1 \leq i \leq n-4)\,, \\ [X_0, Y_1] = Y_2\,, \\ [X_1, X_i] = aX_{i+1} & (2 \leq i \leq n-5)\,, \\ [X_1, X_{n-4}] = (a+b)X_{n-3}\,, \\ [X_i, X_{n-3-i}] = (-1)^{i+1} b X_{n-3} & (2 \leq i \leq \frac{n-4}{2})\,, \end{cases}$$

if n is even.

Proof. By using Teorema 3.5 it follows that, in this case $(r = 1)$, there exists a characteristic vector X_0 and an adapted basis, $\{X_0, X_1, \ldots, X_{n-3}, Y_1, Y_2\}$, such that the laws of the algebra are given by

$$\mu_n : \begin{cases} [X_0, X_i] = X_{i+1} & (1 \leq i \leq n-4)\,, \\ [X_0, Y_1] = Y_2\,, \\ [X_1, X_i] = a_{ij} X_{i+j} & (2 \leq i < j \leq n-3-i)\,. \end{cases}$$

Now, we use an inductive procedure on n.

DIMENSION $n = 7$: In dimension seven the graduation is

$$\langle X_0, X_1, Y_1\rangle \oplus \langle X_2, Y_2\rangle \oplus \langle X_3\rangle \oplus \langle X_4\rangle\,,$$

and by using the Jacobi identity in the family μ_7 we obtain μ_7^a.

DIMENSION $n = 8$: Analogously, by using the Jacobi identity it is easy to obtain that μ_8 is $\mu_8^{a,b}$.

The inductive procedure is realized in function of the parity of the dimension. That is the reason why we study the cases of dimension n even and n odd separately.

DIMENSION $n > 7$, n ODD: If we suppose that the result is true for $n = k$ even, we will prove it for $n = k + 1$ odd. If k is even, we suppose that it is possible to express μ_k by

$$\mu_k^{a,b} : \begin{cases} [X_0, X_i] = X_{i+1} & (1 \le i \le k-4)\,, \\ [X_0, Y_1] = Y_2\,, & \\ [X_1, X_i] = aX_{i+1} & (2 \le i \le k-5)\,, \\ [X_1, X_{n-4}] = (a+b)X_{n-3}\,, & \\ [X_i, X_{n-3-i}] = (-1)^{i+1} b X_{n-3} & (2 \le i \le \frac{k-4}{2})\,. \end{cases}$$

Now, for $n = k + 1$, we add the brackets

$$\begin{aligned} &[X_0, X_{k-3}] = \alpha_0 X_{k-2}\,, \\ &[X_i, X_{k-2-i}] = \alpha_i X_{k-2} \quad (1 \le i \le \tfrac{k-4}{2})\,, \\ &[X_{k-3}, Y_1] = \beta_1 X_{k-2}\,, \\ &[X_{k-4}, Y_2] = \beta_2 X_{k-2}\,. \end{aligned}$$

By using Jacobi identity we prove the result.

DIMENSION $n > 8$, n EVEN: We suppose that the result is true for $n = k$ odd and we will prove it for $n = k+1$ even. If k is odd, we suppose that it is possible to express μ_k by

$$\mu_k^a : \begin{cases} [X_0, X_i] = X_{i+1} & (1 \le i \le k-4)\,, \\ [X_0, Y_1] = Y_2\,, & \\ [X_1, X_i] = aX_{i+1} & (2 \le i \le k-4)\,. \end{cases}$$

For $n = k + 1$ it is necessary to add the same brackets as in the odd case and analogously we obtain the result. ∎

6. Classification theorem

Finally, we give the theorem of classification for naturally graded Lie algebras with Goze invariant $(n-3,2,1)$, $r=1$ and $n \geq 7$.

THEOREM 6.1. *Any complex naturally graded Lie algebra of dimension n, $n \geq 7$, with Goze invariant $(n-3,2,1)$ and laws $\mu(n)$ is isomorphic to one whose law can be expressed in suitable adapted basis by*

$$\begin{array}{r} \mu^1_{(n-3,2,1)} \\ (n \geq 5) \end{array} : \begin{cases} [X_0, X_i] = X_{i+1} & (1 \leq i \leq n-4)\,, \\ [X_0, Y_1] = Y_2\,; \end{cases}$$

$$\begin{array}{r} \mu^2_{(n-3,2,1)} \\ (n \text{ even},\ n \geq 6) \end{array} : \begin{cases} [X_0, X_i] = X_{i+1} & (1 \leq i \leq n-4)\,, \\ [X_0, Y_1] = Y_2\,, \\ [X_i, X_{n-3-i}] = (-1)^{i+1} X_{n-3} & (1 \leq i \leq \frac{n-4}{2})\,; \end{cases}$$

$$\begin{array}{r} \mu^3_{(n-3,2,1)} \\ (n \geq 7) \end{array} : \begin{cases} [X_0, X_i] = X_{i+1} & (1 \leq i \leq n-4)\,, \\ [X_0, Y_1] = Y_2\,, \\ [X_1, X_i] = X_{i+1} & (2 \leq i \leq n-4)\,; \end{cases}$$

$$\begin{array}{r} \mu^4_{(n-3,2,1)} \\ (n \text{ even},\ n \geq 8) \end{array} : \begin{cases} [X_0, X_i] = X_{i+1} & (1 \leq i \leq n-4)\,, \\ [X_0, Y_1] = Y_2\,, \\ [X_1, X_i] = X_{i+1} & (2 \leq i \leq n-5)\,, \\ [X_i, X_{n-3-i}] = (-1)^{i} X_{n-3} & (2 \leq i \leq \frac{n-4}{2})\,; \end{cases}$$

$$\begin{array}{r} \mu^5_{(n-3,2,1)} \\ (n \text{ even},\ n \geq 8) \end{array} : \begin{cases} [X_0, X_i] = X_{i+1} & (1 \leq i \leq n-4)\,, \\ [X_0, Y_1] = Y_2\,, \\ [X_1, X_i] = X_{i+1} & (2 \leq i \leq n-5)\,, \\ [X_1, X_{n-4}] = 2X_{n-3}\,, \\ [X_i, X_{n-3-i}] = (-1)^{i+1} X_{n-3} & (2 \leq i \leq \frac{n-4}{2})\,. \end{cases}$$

Proof. By using the above lemma we will obtain the result. In function of the dimension of the algebra it is necessary to consider two different cases.

Let $\mathfrak{g}$ be a naturally graded Lie algebra of dimension n odd, $n \geq 7$, with Goze invariant $(n-3,2,1)$ and laws μ_n. Then, the natural graduation is given by

$$\langle X_0, X_1, Y_1 \rangle \oplus \langle X_2, Y_2 \rangle \oplus \langle X_3 \rangle \oplus \cdots \oplus \langle X_{n-3} \rangle\,.$$

• Case 1: n even, $n \geq 8$. If n is even the laws of the algebra can be expressed by

$$\mu_n^{a,b}: \begin{cases} [X_0, X_i] = X_{i+1} & (1 \leq i \leq n-4)\,, \\ [X_0, Y_1] = Y_2\,, & \\ [X_1, X_i] = aX_{i+1} & (2 \leq i \leq n-5)\,, \\ [X_1, X_{n-4}] = (a+b)X_{n-3}\,, & \\ [X_i, X_{n-3-i}] = (-1)^{i+1} b X_{n-3}, & (2 \leq i \leq \frac{n-4}{2})\,. \end{cases}$$

The general change of basis implies three generators, X_0, X_1 and Y_1:

$$X_0' = \sum_{i=0}^{n-3} P_i X_i + P_{n-2} Y_1 + P_{n-1} Y_2\,,$$
$$X_1' = \sum_{i=0}^{n-3} Q_i X_i + Q_{n-2} Y_1 + Q_{n-1} Y_2\,,$$
$$Y_1' = \sum_{i=0}^{n-3} R_i X_i + R_{n-2} Y_1 + R_{n-1} Y_2\,.$$

By using the condition of the family we obtain that

$$\begin{cases} Q_0 = 0\,, \\ R_i = 0 \qquad (0 \leq i \leq n-5)\,. \end{cases}$$

Finally, the admisible changes of basis are

$$\begin{aligned}
X_0' &= P_0 X_0 + P_1 X_1 + P_2 X_2 + \cdots + P_{n-4} X_{n-4} + P_{n-3} X_{n-3} \\
&\quad + P_{n-2} Y_1 + P_{n-1} Y_2\,, \\
X_1' &= Q_1 X_1 + Q_2 X_2 + \cdots + Q_{n-4} X_{n-4} + Q_{n-3} X_{n-3} + Q_{n-2} Y_1 + Q_{n-1} Y_2\,, \\
X_2' &= P_0 Q_1 X_2 + (P_0 Q_2 + a(P_1 Q_2 - P_2 Q_1)) X_3 + \cdots + (P_0 Q_{n-5} \\
&\quad + a(P_1 Q_{n-5} - P_{n-5} Q_1)) X_{n-4} + (P_0 Q_{n-4} + a(P_1 Q_{n-4} - P_{n-4} Q_1)) \\
&\quad + \sum_{i=1}^{\frac{n-4}{2}} (-1)^{i+1} (P_i Q_{n-3-i} - P_{n-3-i} Q_i) b X_{n-3} + (P_0 Q_{n-2} - P_{n-2} Q_0) Y_2\,, \\
X_3' &= P_0 (P_0 + aP_1) Q_1 X_3 + (P_0 + aP_1)(P_0 Q_2 + a(P_1 Q_2 - P_2 Q_1)) X_4 + \dots \\
&\quad + (P_0 + aP_1)((P_0 Q_{n-6} + a(P_1 Q_{n-6} - P_{n-6} Q_1)) X_{n-4} \\
&\quad + (P_0 + aP_1)(\dots) X_{n-3}\,, \\
&\ \vdots
\end{aligned}$$

$$\begin{aligned}
X'_{n-4} &= P_0(P_0+aP_1)^{n-6}Q_1X_{n-4} \\
&\quad + ((P_0+aP_1)^{n-7}(P_0+(a+b)P_1)((P_0Q_2+a(P_1Q_2--P_2Q_1))X_{n-3}\,, \\
X'_{n-3} &= P_0(P_0+aP_1)^{n-6}Q_1(P_0+(a+b)P_1)X_{n-3}\,, \\
Y'_1 &= R_{n-4}X_{n-4}+R_{n-3}X_{n-3}+R_{n-2}Y_1+R_{n-1}Y_2\,, \\
Y'_2 &= (P_0+(a+b)P_1)R_{n-4}X_{n-3}+P_0R_{n-2}Y_2\,,
\end{aligned}$$

with the following restrictions

$$P_0\neq 0\,,\quad Q_1\neq 0\,,\quad R_{n-2}\neq 0\,,\quad P_0+aP_1\neq 0\,,\quad P_0+(a+b)P_1\neq 0\,.$$

The nullity of a and b are invariant, because

$$a'=\frac{Q_1a}{P_0+aP_1}\qquad\text{and}\qquad b'=\frac{P_0Q_1b}{(P_0+aP_1)(P_0+(a+b)P_1)}\,.$$

Furthermore, we obtain that the nullity of $a+b$ is invariant, because

$$a'+b'=\frac{Q_1(a+b)}{P_0+(a+b)P_1}\,.$$

We consider the following cases:

- Case 2.1: $a=b=0$. Trivially, we obtain $\mu^1_{(n-3,2,1)}$.
- Case 2.2: $a\neq 0$ and $b=0$. By choosing P_0, Q_1 and P_1, we obtain $\mu^2_{(n-3,2,1)}$.
- Case 2.3: $a=0$ and $b\neq 0$. As in the above case, we obtain $\mu^3_{(n-3,2,1)}$.
- Case 2.4: $a\neq 0$, $b\neq 0$ and $a+b=0$. By choosing P_0, Q_1 and P_1, we obtain $\mu^4_{(n-3,2,1)}$.
- Case 2.5: $a\neq 0$, $b\neq 0$ and $a+b\neq 0$. It is possible to choose P_0, Q_1 and P_1 for to obtain the algebra $\mu^5_{(n-3,2,1)}$.

Furthermore, the above results prove that the algebras $\mu^1_{(n-3,2,1)}$, $\mu^2_{(n-3,2,1)}$, $\mu^3_{(n-3,2,1)}$, $\mu^4_{(n-3,2,1)}$ y $\mu^5_{(n-3,2,1)}$ are pairwise no isomorphic for n even.

• Case 2: n odd, $n\geq 7$. As follows from the above lemma we obtain that an algebra of this kind is isomorphic to one whose law can be expressed by

$$\begin{cases}
[X_0,X_i]=X_{i+1} & (1\leq i\leq n-4)\,,\\
[X_0,Y_1]=Y_2\,, & \\
[X_1,X_i]=aX_{i+1} & (2\leq i<j\leq n-3-i)\,.
\end{cases}$$

Since, the odd case is equal to even case considering $b=0$. An analogous treatment of Case 1 proves that the nullity of a is an invariant and from here, the result is obtained. ∎

References

[1] Cabezas, J.M., Camacho, L.M., Gómez, J.R., Navarro, R.M., A class of nilpotent Lie algebras, *Comm. Algebra* **28** (9) (2000), 4489 – 4499.

[2] Cabezas, J.M., Gómez, J.R., $(n-4)$-filiform Lie algebras, *Comm. Algebra* **27** (10) (1999), 4803 – 4819.

[3] Cabezas, J.M., Gómez, J.R., Jiménez-Merchán, A., Family of p-filiform Lie algebras, in "Algebra and Operator Theory", Khakimdjanov, Y., Goze, M., Ayupov, Sh., eds., Kluwer Academic Publishers, Tashkent, 1998, pp. 93 – 102.

[4] Cabezas, J.M., Gómez, J.R., Pastor, E., Structure theorem for naturally graded 3-filiform Lie algebras, in "I Colloquium on Lie Theory and Applications", University of Vigo, Vigo, 2000.

[5] Cabezas, J.M., Pastor, E., Naturally graded p-filiform Lie algebras in arbitrary finite dimension, *J. Lie Theory* **15** (2005), 379 – 391.

[6] Gómez, J.R., Jiménez-Merchán, A., Naturally graded quasifiliform Lie algebras, *J. Algebra* **256** (2002), 211 – 228.

[7] Goze, M., Hakimjanov (Khakimdjanov), Y., Sur les algèbres de Lie nilpotentes admettant un tore de dérivations, *Manuscripta Math.* **84** (1994), 115 – 224.

[8] Goze, M., Khakimdjanov, Y., "Nilpotent Lie Algebras", Kluwer Academic Publishers, 1996.

[9] Vergne, M., Cohomologie des algèbres de Lie nilpotentes. Application à l'étude de la varieté des algèbres de Lie nilpotentes, *Bull. Soc. Math. France*, **98** (1970), 81 – 116.