ITERATIVE ALGORITHMS FOR NONEXPANSIVE MAPPINGS ON HADAMARD MANIFOLDS

Chong Li[1,*], Genaro López[2] and Victoria Martín-Márquez[3]

Abstract. Two iterative algorithms for nonexpansive mappings on Hadamard manifolds, which are extensions of the well-known Halpern's and Mann's algorithms in Euclidean spaces, are proposed and proved to be convergent to a fixed point of the mapping. Some numerical examples are provided.

1. INTRODUCTION

Many practical problems can be formulated as a fixed point problem

$$(1.1) \qquad x = Tx,$$

where T is a nonlinear operator. The solutions of this equation are called fixed points of T. If T is a self-contraction defined on a complete metric space X, Banach contraction principle sets up that, for any $x \in X$, the sequence of iterates $\{T^n x\}$ converges strongly to a solution to (1.1). If the mapping T is nonexpansive we must assume additional conditions to assure the existence of fixed point of T and, even when a fixed point exists, the sequence of iterates in general does not converge to a fixed point. This fact has motivated that the study of the asymptotic behavior of nonexpansive mappings is one of the most active research areas in nonlinear analysis. Most of the research have focused on the case when T is a self-mapping defined on a convex subset K of a normed linear space and basically two types of algorithms has been considered: Halpern's and Mann's algorithms.

Let $u, x_0 \in K$ and a sequence $\{\alpha_n\} \subset (0, 1)$ be given. Halpern's algorithm presented in [12] generates a sequence via the formula

$$(1.2) \qquad x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n, \quad n \geq 0.$$

A strong effort has been dedicated to the study of the convergence of this sequence, see [12, 18, 30, 24, 31, 3, 27] and references therein, but the full picture is still not clear (cf.[32]).

Mann's algorithm generates a sequence via the following iteration

$$(1.3) \qquad x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n \geq 0.$$

In the case when a fixed point exists, while Halpern iteration converges strongly, we just get weak convergence for Mann iteration. See [9] for example. Conditions for the convergence of the algorithm can be found in [19, 13, 23, 8, 31]. Some modifications have been proposed in [20, 14] to get strong convergence.

Because of the convex structure of both algorithms, few results have been obtained out of the setting of linear spaces. Our interest here is to develop a theory of fixed point approximation for nonexpansive mappings in Hadamard manifolds (i.e. complete simply connected Riemannian manifolds of nonpositive curvature). Extensions to Riemannian manifolds of concepts and techniques which fit in Euclidean spaces are natural. Actually, in the recent years, some algorithms for solving variational inequalities and minimization problems which involves monotone operators have been extended from the Hilbert space framework to the more general setting of Riemannian manifolds (cf. [28, 5, 21, 6, 7]).

For Mann's algorithm, some results had been obtained previously in Hyperbolic metric spaces, see [15, 10, 11, 25]. This class of spaces includes Hadamard manifolds and the Hilbert ball equipped with the hyperbolic metric as special cases. For Halpern's algorithm the only precedent, as far as we know, is a result in [16] by W. A. Kirk of convergence for an implicit algorithm in the more abstract setting of geodesic spaces.

The organization of the paper is as follows. In section 2 we introduce some basic concepts, results and notations on Riemannian manifolds. In sections 3 and 4, we prove some properties of the metric defined on Hadamard manifolds and convergence results for Halpern's and Mann's algorithms. Finally, section 5 is devoted to some numerical examples to illustrate applications of the results in the present paper.

2. PRELIMINARIES

In this section we introduce some fundamental definitions, properties and notations of Riemannian manifolds, which can be found in any textbook, for example, [4, 26].

Let M be a connected m-dimensional manifold and let $p \in M$. We use T_pM to denote the tangent space of M at p. We always assume that M can be endowed with a Riemannian metric $\langle .,. \rangle$, with the corresponding norm denoted by $\|.\|$, to become a *Riemannian* manifold. Given a piecewise smooth curve $c : [a, b] \to M$ joining p to q (i.e. $c(a) = p$ and $c(b) = q$), we define the length of c by using the metric as $L(c) = \int_a^b \|c'(t)\| dt$. Then the Riemannian distance $d(p, q)$ is defined to be the minimal length over the set of all such curves joining p to q, which induces the original topology on M.

Let ∇ be the Levi-Civita connection associated to $(M, \langle , \rangle)$. Let c be a smooth curve. A smooth vector field X along c is said to be parallel if $\nabla_{c'} X = 0$. If c' itself is parallel, we say that c is a *geodesic*, and in this case $\|c'\|$ is constant. Furthermore, if $\|c'\| = 1$, c is called normalized. A geodesic joining p to q in M is said to be a *minimal* geodesic if its length equals $d(p, q)$.

A Riemannian manifold is complete if for any $p \in M$ all geodesics emanating from p are defined for all $-\infty < t < \infty$. By the Hopf-Rinow Theorem we know that if M is complete then any pair of points in M can be joined by a minimizing geodesic. Moreover, (M, d) is a complete metric space, and bounded closed subsets are compact.

Assuming that m is complete, the exponential map $\exp_p : T_pM \to M$ at $p \in M$ is defined by $\exp_p v = \gamma_v(1, p)$ for each $v \in T_pM$, where $\gamma(.) = \gamma_v(., p)$ is the geodesic starting at p with velocity v. Then $\exp_p tv = \gamma_v(t, p)$ for each real number t.

A complete simply connected Riemannian manifold of nonpositive sectional curvature is called a Hadamard manifold. Throughout the remainder of the paper, we will always assume that M is a m-dimensional Hadamard manifold. The following result is well-known and will be useful.

Proposition 2.1. [26]. *Let $p \in M$. Then $\exp_p : T_pM \to M$ is a diffeomorphism, and for any two points $p, q \in M$ there exists a unique normalized geodesic joining p to q, which is in fact a minimal geodesic.*

This proposition says that M is diffeomorphic to the Euclidean space $\mathbb{R}^m$. Then, M has the same topology and differential structure as $\mathbb{R}^m$. Moreover, Hadamard manifolds and Euclidean spaces have some similar geometrical properties. Some of them are described in the following propositions.

A geodesic triangle $\Delta(p_1, p_2, p_3)$ of a Riemannian manifold is a set consisting of three points p_1, p_2 and p_3, and three minimal geodesics γ_i joining p_i to p_{i+1}, with $i = 1, 2, 3$ (mod 3).

Proposition 2.2. [26] (Comparison theorem for triangles). *Let $\Delta(p_1, p_2, p_3)$ be a geodesic triangle. Denote, for each $i = 1, 2, 3$ (mod 3), by $\gamma_i : [0, l_i] \to M$ the geodesic joining p_i to p_{i+1}, and set $l_i := L(\gamma_i)$, $\alpha_i := \angle(\gamma_i'(0), -\gamma_{i-1}'(l_{i-1}))$. Then*

(2.1) $\alpha_1 + \alpha_2 + \alpha_3 \leq \pi,$

(2.2) $l_i^2 + l_{i+1}^2 - 2l_i l_{i+1} \cos \alpha_{i+1} \leq l_{i-1}^2.$

In terms of the distance and the exponential map, the inequality (2.2) can be rewritten as

(2.3) $d^2(p_i, p_{i+1}) + d^2(p_{i+1}, p_{i+2}) - 2\langle \exp_{p_{i+1}}^{-1} p_i, \exp_{p_{i+1}}^{-1} p_{i+2} \rangle \leq d^2(p_{i-1}, p_i),$

since

$$\langle \exp_{p_{i+1}}^{-1} p_i, \exp_{p_{i+1}}^{-1} p_{i+2} \rangle = d(p_i, p_{i+1}) d(p_{i+1}, p_{i+2}) \cos \alpha_{i+1}.$$

A subset $K \subseteq M$ is said to be convex if for any two points p and q in K, the geodesic joining p to q is contained in K, that is, if $\gamma : [a, b] \to M$ is a geodesic such that $p = \gamma(a)$ and $q = \gamma(b)$, then $\gamma((1-t)a + tb) \in K$ for all $t \in [0, 1]$. From now K will denote a nonempty, closed and convex set in M.

A real-valued function f defined on M is said to be convex if for any geodesic γ of M, the composition function $f \circ \gamma : \mathbb{R} \to \mathbb{R}$ is convex, that is,

$$(f \circ \gamma)(ta + (1-t)b) \leq t(f \circ \gamma)(a) + (1-t)(f \circ \gamma)(b)$$

for any $a, b \in \mathbb{R}$, and $0 \leq t \leq 1$.

Proposition 2.3. [26]. *Let $d : M \times M \to \mathbb{R}$ be the distance function. Then d is a convex function with respect to the product Riemannian metric, i.e. given any pair of geodesics $\gamma_1 : [0, 1] \to M$ and $\gamma_2 : [0, 1] \to M$ the following inequality holds for all $t \in [0, 1]$:*

$$d(\gamma_1(t), \gamma_2(t)) \leq (1-t)d(\gamma_1(0), \gamma_2(0)) + td(\gamma_1(1), \gamma_2(1)).$$

In particular, for each $p \in M$, the function $d(\cdot, p) : M \to \mathbb{R}$ is a convex function.

Let P_K denote the *projection* onto K defined by

$$P_K(p) = \{p_0 \in K : d(p, p_0) \leq d(p, q) \text{ for all } q \in K\}, \quad \forall p \in M.$$

Proposition 2.4. [29]. *For any point $p \in M$, $P_K(p)$ is a singleton and the following inequality holds for all $q \in K$:*

$$\langle \exp_{P_K(p)}^{-1} p, \ \exp_{P_K(p)}^{-1} q \rangle \leq 0.$$

3. HALPERN'S ALGORITHM IN HADAMARD MANIFOLDS

Let K be a closed convex subset of M and $T : K \to K$ a mapping. We say that

T is nonexpansive if for any two points $x, y \in K$ the following inequality holds:

$$d(T(x), T(y)) \leq d(x, y).$$

In order to solve the important problem of finding a fixed point of T, Kirk provided in [16] an implicit algorithm for approximating fixed points of nonexpansive mappings in more general metric spaces called geodesic spaces which contain the class of Hadamard manifolds. Applying the general result by Kirk to Hadamard manifolds, one has the following result. Let $\text{Fix}(T)$ denote the set of all fixed points of T.

Theorem 3.1. [16]. *Suppose that K is a bounded closed convex subset of M and $T : K \to K$ is nonexpansive. Let $x \in K$, and for each $t \in [0, 1)$ let x_t be the unique point such that*

$$x_t \in [x, T(x_t)] \quad and \quad d(x, x_t) = td(x, T(x_t))$$

(which exists by Banach contraction principle). Then $\lim_{t \to 1^-} x_t = \overline{x}$, the unique nearest point to x in $Fix(T)$.

In an Euclidean space $\mathbb{R}^n$, this iteration scheme turns into $x_t = (1-t)x + tT(x_t)$, which coincides with the implicit Browder's iteration:

$$x_t = tx + (1 - t)T(x_t),$$

which, as $t \to 0^+$, converges strongly to the fixed point of T which is the nearest to x in $\text{Fix}(T)$, even in any uniformly smooth space (except that the limit is a certain retraction different from the projection). See [2] and [24] for more details.

There exists a lot of research about the convergence of the explicit Halpern's iteration (1.2) in a Banach space X. Below we present an analogue of this algorithm for approximating fixed point for nonexpansive mappings on Hadamard manifolds, which coincides with the Halpern's one in the particular case of an Euclidean space. Let $x_0, \ u \in M$ and let $\{\alpha_n\} \subset (0, 1)$. Consider the iteration scheme

(3.1) $\qquad x_{n+1} = \exp_u(1 - \alpha_n) \exp_u^{-1} T(x_n), \quad \forall n \geq 0;$

or equivalently,

$$x_{n+1} = \gamma_n(1 - \alpha_n), \quad \forall n \geq 0,$$

where $\gamma_n : [0, 1] \to M$ is the geodesic joining u to $T(x_n)$ (i.e. $\gamma(0) = u$ and $\gamma(1) = T(x_n)$).

In order to prove the convergence of the sequence generated by the algorithm (3.1) we first need some lemmas.

Lemma 3.2. [31]. *Let $\{\beta_n\}$ and $\{b_n\}$ be the sequences satisfying the following conditions.*

(i) $\{\beta_n\} \subset [0,1]$, $\sum_n \beta_n = \infty$;

(ii) $\limsup_{n\to\infty} b_n \le 0$.

Let $\{a_n\}$ be a sequence of non-negative real numbers such that

$$a_{n+1} \le (1 - \beta_n)a_n + \beta_n b_n, \ n \ge 0.$$

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 3.3. [1, p. 24]. *Let $\Delta(p,q,r)$ be a geodesic triangle in M Hadamard manifold. Then, there exists $p', q', r' \in \mathbb{R}^2$ such that*

$$d(p,q) = \|p' - q'\|, \quad d(q,r) = \|q' - r'\|, \quad d(r,p) = \|r' - p'\|$$

The triangle $\Delta(p', q', r')$ is called the comparison triangle of the geodesic triangle $\Delta(p,q,r)$, which is unique up to isometry of M. The following lemma can be proved from element geometry. This is also a direct application of the Alexandrov's Lemma in $\mathbb{R}^2$ (see [1, p. 25]).

Lemma 3.4. *Consider four distinct points $p, q, r, z \in \mathbb{R}^2$. Suppose that p and q lie on opposite sides of the line through r and z. Consider the triangles $\Delta(r,p,z)$ and $\Delta(r,q,z)$. Let β be the angle of $\Delta(r,q,z)$ at the vertex q, and let θ_1 and θ_2 be the angles of $\Delta(r,p,z)$ and $\Delta(r,q,z)$ at the vertex z, respectively. Let q' be the point such that $d(r,q') = d(r,q)$ and $d(p,q') = d(p,z) + d(z,q)$. Let β' be the angles of $\Delta(r,p,q')$ at the vertex q' (see Figure 3.1). If $\theta_1 + \theta_2 \ge \pi$, then*

$$\beta \le \beta'.$$

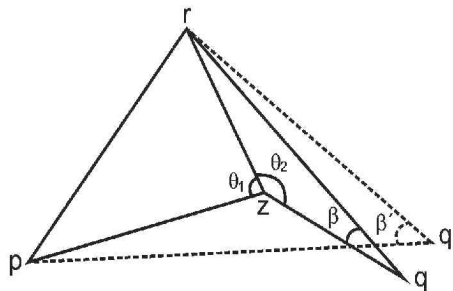Fig. 3.1

The next result shows the relation between a geodesic triangle and its comparison triangle involving angles and distances between points.

Lemma 3.5. *Let $\Delta(p, q, r)$ be a geodesic triangle in a Hadamard manifold M and $\Delta(p', q', r')$ be its comparison triangle.*

(1) *Let α, β, γ (resp. α', β', γ') be the angles of $\Delta(p, q, r)$ (resp. $\Delta(p', q', r')$) at the vertices p, q, r (resp. p', q', r'). Then, the following inequalities hold:*

(3.2) $$\alpha' \geq \alpha, \quad \beta' \geq \beta, \quad \gamma' \geq \gamma.$$

(2) *Let z be a point in the geodesic joining p to q and z' its comparison point in the interval $[p', q']$. Suppose that $d(z, p) = \|z' - p'\|$ and $d(z, q) = \|z' - q'\|$, see Figure 3.2. Then the following inequality holds:*

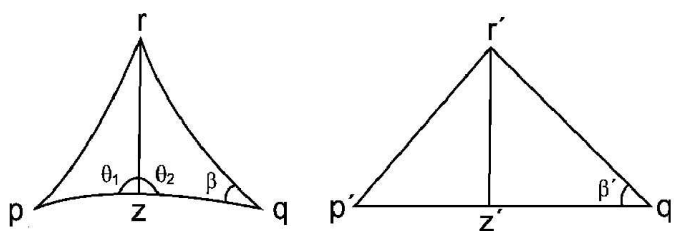(3.3) $$d(z, r) \leq \|z' - r'\|.$$

Fig. 3.2

Proof.

(1) We only prove the inequality $\beta' \geq \beta$. To do this, consider the triangle $\Delta(p', q', r')$ in $\mathbb{R}^2$. Then, by the law of cosines we have that

$$\|r' - q'\|^2 + \|q' - p'\|^2 - 2\|r' - q'\|\|q' - p'\| \cos \beta' = \|r' - p'\|^2.$$

By (2.2), one has that

$$d^2(r, q) + d^2(q, p) - 2d(r, q)d(q, p) \cos \beta \leq d^2(r, p).$$

It follows from Lemma 3.3 that

$$\cos \beta \leq \cos \beta',$$

and $\beta' > \beta$ because $\beta, \beta' \in [0, \pi]$.

(2) We fix a geodesic joining z to r. Let θ_1 and θ_2 denote respectively the angles of $\Delta(r, p, z)$ and $\Delta(r, q, z)$ at the vertex z. Let β and β' be the angles of $\Delta(r, p, q)$ and $\Delta(r', p', q')$ at the vertex q and q' respectively. See Figure 3.2.

Consider comparison triangles $\Delta(\bar{r}, \bar{p}, \bar{z})$ and $\Delta(\bar{r}, \bar{q}, \bar{z})$ for the geodesic triangles $\Delta(r, p, z)$ and $\Delta(r, q, z)$ respectively, such that they share the same edge $[\bar{r}, \bar{z}]$, and $\bar{p}$, $\bar{q}$ lie on opposite sides of the line which passes through $\bar{r}$ and $\bar{z}$. Let $\bar{\theta}_1$ and $\bar{\theta}_2$ be the angles of $\Delta(\bar{r}, \bar{p}, \bar{z})$ and $\Delta(\bar{r}, \bar{q}, \bar{z})$ at the vertex $\bar{z}$, respectively. Denote $\bar{\beta}$ the angle at the vertex $\bar{q}$. From the inequalities (3.2) we deduce that

$$\bar{\theta}_1 + \bar{\theta}_2 \geq \theta_1 + \theta_2 = \pi.$$

Thus Lemma 3.4 is applicable to getting that $\bar{\beta} \leq \beta'$. Therefore, using the law of cosines, we have $d(z, r) \leq \|z' - r'\|$. ∎

For the main theorem of this section, we consider the following hypothesis:

$(H1)$ $\lim_{n \to \infty} \alpha_n = 0$;

$(H2)$ $\sum_{n \geq 0} \alpha_n = \infty$;

$(H3)$ $\sum_{n \geq 0} |\alpha_{n+1} - \alpha_n| < \infty$;

$(H4)$ $\lim_{n \to \infty} (\alpha_n - \alpha_{n-1})/\alpha_n = 0$.

Theorem 3.6. *Let K be a closed convex subset of M and $T : K \to K$ a nonexpansive mapping with $F := Fix(T) \neq \emptyset$. Let $u, x_0 \in M$. Suppose that $\{\alpha_n\} \in (0, 1)$ satisfies (H1), (H2) and, (H3) or (H4). Then the sequence $\{x_n\}$ generated by the algorithm (3.1) converges to $P_F(u)$.*

Proof. Let $n \geq 0$ and let $\gamma_n : [0, 1] \to M$ denote the geodesic joining u to $T(x_n)$. We divide the proof into four steps.

Step 1. $\{x_n\}$ and $\{T(x_n)\}$ are bounded.

Since the boundedness of $\{T(x_n)\}$ is a direct consequence of that of $\{x_n\}$, below we only prove the boundedness of $\{x_n\}$. For this purpose, take $x \in F$ and fix n. Then, by the convexity of the distance function and the nonexpansivity of T, we have that

$$\begin{aligned} d(x_{n+1}, x) &= d(\gamma_n(1 - \alpha_n), x) \\ &\leq \alpha_n d(\gamma_n(0), x) + (1 - \alpha_n)d(\gamma_n(1), x) \\ &= \alpha_n d(u, x) + (1 - \alpha_n)d(T(x_n), x) \\ &\leq \alpha_n d(u, x) + (1 - \alpha_n)d(x_n, x). \end{aligned}$$

Then the boundness of $\{x_n\}$ is seen to hold by mathematical induction.

Step 2. $\lim_{n \to \infty} d(x_{n+1}, x_n) = 0$.

By Step 1, we can find a constant C such that

$$(3.4) \qquad d(x_n, x_{n-1}) \leq C \quad \text{and} \quad d(u, T(x_n)) \leq C, \quad \forall n \geq 0.$$

Using the convexity of the distance function, we have that, for each $n \geq 0$,

$$d(x_{n+1}, x_n) = d(\gamma_n(1 - \alpha_n), \gamma_{n-1}(1 - \alpha_{n-1}))$$

$$\leq d(\gamma_n(1 - \alpha_n), \gamma_{n-1}(1 - \alpha_n))$$

(3.5)

$$+ d(\gamma_{n-1}(1 - \alpha_n), \gamma_{n-1}(1 - \alpha_{n-1}))$$

$$\leq (1 - \alpha_n) d(T(x_n), T(x_{n-1})) + |\alpha_n - \alpha_{n-1}| d(u, T(x_{n-1})).$$

This together with (3.4) and the nonexpansivity of T implies that

$$(3.6) \qquad d(x_{n+1}, x_n) \leq (1 - \alpha_n) d(x_n, x_{n-1}) + C|\alpha_n - \alpha_{n-1}|, \quad \forall n \geq 0.$$

Thus, if (H4) holds, we apply Lemma 3.2 (with $\beta_n = \alpha_n$ and $b_n = C|\alpha_n - \alpha_{n-1}|/\alpha_n$ for each n) to conclude that $\lim_{n \to \infty} d(x_{n+1}, x_n) = 0$. As to the case when (H3) holds, let $k \leq n$. By (3.6), one gets that

$$d(x_{n+1}, x_n) \leq \prod_{i=k}^{n} (1 - \alpha_i) d(x_k, x_{k-1}) + C \sum_{i=k}^{n} |\alpha_i - \alpha_{i-1}|$$

$$\leq C \prod_{i=k}^{n} (1 - \alpha_i) + C \sum_{i=k}^{n} |\alpha_i - \alpha_{i-1}|.$$

Letting $n \to \infty$ implies that,

$$(3.7) \qquad \lim_{n \to \infty} d(x_{n+1}, x_n) \leq C \prod_{i=k}^{\infty} (1 - \alpha_i) + C \sum_{i=k}^{\infty} |\alpha_i - \alpha_{i-1}|.$$

Condition (H2) implies that $\lim_{k \to \infty} \prod_{i=k}^{\infty} (1 - \alpha_i) = 0$; while condition (H3) implies that $\lim_{k \to \infty} \sum_{i=k}^{\infty} |\alpha_i - \alpha_{i-1}| = 0$. Hence, letting $k \to \infty$ in (3.7), we get $\lim_{n \to \infty} d(x_{n+1}, x_n) = 0$ in the case when (H3) holds.

Step 3. $\limsup_{n \to \infty} \langle \exp_{P_F(u)}^{-1} u, \; \exp_{P_F(u)}^{-1} T(x_n) \rangle \leq 0$.

By Step 1, $\{ \langle \exp_{P_F(u)}^{-1} u, \; \exp_{P_F(u)}^{-1} T(x_n) \rangle \}$ is bounded; hence its upper limit exists. Thus we can find a subsequence $\{n_k\}$ of $\{n\}$ such that

$$(3.8) \quad \limsup_{n \to \infty} \langle \exp_{P_F(u)}^{-1} u, \exp_{P_F(u)}^{-1} T(x_n) \rangle = \lim_{k \to \infty} \langle \exp_{P_F(u)}^{-1} u, \exp_{P_F(u)}^{-1} T(x_{n_k}) \rangle.$$

Without loss of generality, we may assume that $x_{n_k} \to \bar{x}$ for some $\bar{x} \in M$ because $\{x_n\}$ is bounded by Step 1. By the convexity of the distance function and the definition of the algorithm,

$$d(x_{n_k+1}, T(x_{n_k})) \leq \alpha_{n_k} d(u, T(x_{n_k})).$$

Since $\{d(u, T(x_{n_k}))\}$ is bounded by Step 1, it follows that $\lim_{k\to\infty} d(x_{n_k+1}, T(x_{n_k}))=0$ as $\alpha_{n_k} \to 0$ by (H1). Noting that

$$d(x_{n_k}, T(x_{n_k})) \leq d(x_{n_k+1}, x_{n_k}) + d(x_{n_k+1}, T(x_{n_k})),$$

one sees that $\lim_{n\to\infty} d(x_{n_k}, T(x_{n_k})) = 0$. Therefore

$$d(\bar{x}, T(\bar{x})) \leq d(\bar{x}, x_{n_k}) + d(x_{n_k}, T(x_{n_k})) + d(T(x_{n_k}), T(\bar{x})) \to 0,$$

which means that $\bar{x} \in F$. Then, Proposition 2.4 implies that $\langle \exp_{P_F(u)}^{-1} u, \exp_{P_F(u)}^{-1} \bar{x}\rangle \leq 0$ and so

$$\lim_{k\to\infty} \langle \exp_{P_F(u)}^{-1} u, \exp_{P_F(u)}^{-1} T(x_{n_k})\rangle \;=\; \langle \exp_{P_F(u)}^{-1} u, \exp_{P_F(u)}^{-1} \bar{x}\rangle \leq 0.$$

Combining this with (3.8), we complete the proof of Step 3.

Step 4. $\lim_{n\to\infty} d(x_n, P_F u) = 0$.
Write

$$b_n = \alpha_n d(u, P_F u) + \langle \exp_{P_F(u)}^{-1} u, \exp_{P_F(u)}^{-1} T(x_n)\rangle, \quad \forall n \geq 0.$$

Then $\lim_{n\to\infty} b_n \leq 0$ by Step 3. Thus, by Lemma 3.2, it suffices to verify that

$$(3.9) \qquad d^2(x_{n+1}, P_F u) \leq (1 - \alpha_n) d^2(x_n, P_F u) + \alpha_n b_n, \quad \forall n \geq 0.$$

To this end, we fix $n \geq 0$ and set $p = T(x_n)$, $q = P_F u$. Consider the geodesic triangle $\Delta(u, p, q)$ and its comparison triangle $\Delta(u', p', q')$. Then

$$d(u, P_F u) = d(u, q) = \|u' - q'\| \quad \text{and} \quad d(T(x_n), P_F u) = d(p, q) = \|p' - q'\|.$$

Recall from (3.1) that $x_{n+1} = \exp_u(1 - \alpha_n) \exp_u^{-1} T(x_n) = \exp_u(1 - \alpha_n) \exp_u^{-1} p$. The comparison point of x_{n+1} is $x'_{n+1} = \alpha_n u' + (1 - \alpha_n)p'$. Let β and β' denote the angles at q and q', respectively. Then $\beta \leq \beta'$ by Lemma 3.5 (1) and so $\cos\beta' \leq \cos\beta$. Then, by Lemma 3.5 (2) we have

$$
\begin{aligned}
& d^2(x_{n+1}, P_F u) \\
&\leq \|x'_{n+1} - q'\|^2 \\
&= \|\alpha_n(u' - q') + (1 - \alpha_n)(p' - q')\|^2 \\
&= \alpha_n^2\|u' - q'\|^2 + (1 - \alpha_n)^2\|p' - q'\|^2 + 2\alpha_n(1 - \alpha_n)\|u' - q'\|\|p' - q'\|\cos\beta' \\
&\leq \alpha_n^2 d^2(u, P_F u) + (1 - \alpha_n)^2 d^2(T(x_n), P_F u) \\
&\quad + 2\alpha_n(1 - \alpha_n)d(u, P_F u)d(T(x_n), P_F u)\cos\beta \\
&\leq (1 - \alpha_n)d^2(x_n, P_F u) + \alpha_n\big(\alpha_n d(u, P_F u) + \langle \exp_{P_F(u)}^{-1} u, \exp_{P_F(u)}^{-1} T(x_n)\rangle\big).
\end{aligned}
$$

Hence (3.9) is proved and the proof is complete.   ∎

4. Mann's Algorithm in Hadamard Manifolds

In the case when K is a closed convex subset of a normed linear space X and $T : K \to K$ a nonexpansive mapping, for $\{\alpha_n\} \subset [0, 1]$, the iteration process $\{x_n\}$ defined by the algorithm

$$(4.1) \qquad x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T(x_n), \quad \forall n \geq 0$$

was introduced by Mann in [19] and is known as Mann's iteration for finding fixed points of nonexpansive mappings. Ishikawa proved in [13] that if $0 < a \leq \alpha_n < 1$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, then $\|x_n - T(x_n)\| \to 0$ as $n \to \infty$, which implies the convergence of $\{x_n\}$ to a fixed point of T if the range of T lies in a compact subset of X. In 1979, Reich obtained in [23] the weak convergence of the sequence in a uniformly convex space with a Fréchet differentiable norm under the assumption that $\{\alpha_n\} \subset (0, 1)$ and $\sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = \infty$. For further results see [8] and references therein.

An extension of Mann's iteration (4.1) and its convergence results to the framework of metric spaces (X, d) is due to Goebel and Kirk [15, 10] and Reich-Shafrir [25], who provided an iterative method for finding fixed points of nonexpansive mappings on spaces of *hyperbolic type*, which include Hadamard manifolds as special cases. The iteration algorithm is defined by

$$(4.2) \quad x_{n+1} \in [x_n, T(x_n)] \quad \text{such that} \quad d(x_n, T(x_n)) = (1 - \alpha_n) d(x_n, x_{n+1}),$$

where $\{\alpha_n\} \subset (0, 1)$. In particular, under the assumption that $\{\alpha_n\}$ is bounded away from 0 and 1, Reich and Shafrir proved the convergence of this iteration to a fixed point of T. Goebel and Reich [11] studied the behavior of the sequence of the iterates $x_{n+1} = T(x_n)$ in Hyperbolic metric spaces for firmly nonexpansive mappings.

Motivated by these results, we continue to consider Mann's iteration (4.2) in Hadamard manifolds M, which has the following form

$$(4.3) \qquad x_{n+1} = \exp_{x_n}(1 - \alpha_n) \exp_{x_n}^{-1} T(x_n), \quad \forall n \geq 0,$$

and study the convergence of Mann's iteration (4.2) for nonexpansive mappings $T : M \to M$ but without assuming that $\{\alpha_n\}$ is bounded away from 0 and 1. In fact, we will prove in Theorem 4.3 that the sequence $\{x_n\}$ generated by Mann's iteration (4.2) in Hadamard manifolds converges to a fixed point of T only if $\{\alpha_n\}$ satisfies the following condition:

$$(4.4) \qquad \sum_{n=0}^{\infty} \alpha_n (1 - \alpha_n) = \infty.$$

For this purpose, some definitions and results are necessary at first.

Definition 4.1. Let X be a complete metric space and $F \subset X$ be a nonempty set. A sequence $\{x_n\} \subset X$ is called *Fejér convergent* to F if

$$d(x_{n+1}, y) \leq d(x_n, y)$$

for all $y \in F$ and $n \geq 0$.

Lemma 4.2. [7]. *Let X be a complete metric space. If $\{x_n\} \subset X$ is Fejér convergent to a nonempty set $F \subset X$, then $\{x_n\}$ is bounded. Moreover, if a cluster point x of $\{x_n\}$ belongs to F, then $\{x_n\}$ converges to x.*

Now we are ready to prove the main theorem of this section.

Theorem 4.3. *Let K be a closed convex subset of M and $T : K \to K$ a nonexpansive mapping with $F := Fix(T) \neq \emptyset$. Suppose that $\{\alpha_n\} \subset (0, 1)$ satisfy the condition (4.4). Let $x_0 \in M$ and let $\{x_n\}$ be the sequence generated by the algorithm (4.3). Then $\{x_n\}$ converges to a fixed point of T.*

Proof. Note that K itself is a complete metric space. Thus, by Lemma 4.2, it suffices to verify that $\{x_n\}$ is Fejér convergent to F and that all cluster points of $\{x_n\}$ belong to F. For this purpose, let $n \geq 0$ and $p \in F$ be fixed. Let $\gamma : [0, 1] \to M$ denote the geodesic joining x_n to $T(x_n)$. Then $x_{n+1} = \gamma(1 - \alpha_n)$. By the convexity of the distance function and the nonexpansivity of T, we have

$$d(x_{n+1}, p) = d(\gamma(1 - \alpha_n), p) \leq \alpha_n d(x_n, p) + (1 - \alpha_n)d(T(x_n), p) \leq d(x_n, p).$$

Hence $\{x_n\}$ is Fejér convergent to F. Now let x be a cluster point of $\{x_n\}$. Then there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $x_{n_k} \to x$. Below we prove that

$$(4.5) \qquad\qquad \lim_{n \to \infty} d(x_n, T(x_n)) = 0.$$

Granting this, one has that

$$d(x, T(x)) \leq d(x, x_{n_k}) + d(x_{n_k}, T(x_{n_k})) + d(T(x_{n_k}), T(x))$$
$$\leq 2d(x_{n_k}, x) + d(x_{n_k}, T(x_{n_k})),$$

Taking limit, we obtain that $d(x, Tx) = 0$, which means that $x \in F$, and completes the proof.

To show (4.5), let $p \in F$ and $n \geq 0$. Let $\Delta(x_n, q, p)$ be the geodesic triangle with vertices x_n, $q := T(x_n)$ and p. From Lemma 3.3 there exists a comparison triangle $\Delta(x'_n, q', p')$ which conserve the length of the edge. Recall

that $x_{n+1} = \gamma(1 - \alpha_n)$. Set $x'_{n+1} := (1 - \alpha_n)x'_n + \alpha_n T(x_n)' = (1 - \alpha_n)x'_n + \alpha_n q'$ be its comparison point. By Lemma 3.5 (2),

$$
\begin{aligned}
d^2(x_{n+1}, p) &\leq \|x'_{n+1} - p'\|^2 \\
&= \|\alpha_n(x'_n - p') + (1 - \alpha_n)(q' - p')\|^2 \\
&= \alpha_n\|x'_n - p'\|^2 + (1 - \alpha_n)\|q' - p'\|^2 - \alpha_n(1 - \alpha_n)\|x'_n - q'\|^2 \\
&= \alpha_n d^2(x_n, p) + (1 - \alpha_n)d^2(T(x_n), p) - \alpha_n(1 - \alpha_n)d^2(x_n, T(x_n)) \\
&\leq d^2(x_n, p) - \alpha_n(1 - \alpha_n)d^2(x_n, T(x_n)).
\end{aligned}
$$

It follows that

$$
\alpha_n(1 - \alpha_n)d^2(x_n, T(x_n)) \leq d^2(x_n, p) - d^2(x_{n+1}, p)
$$

and

$$
\text{(4.6)} \qquad \sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n)d^2(x_n, T(x_n)) < \infty,
$$

which implies that

$$
\text{(4.7)} \qquad \liminf_{n \to \infty} d(x_n, T(x_n)) = 0
$$

because otherwise, $d(x_n, T(x_n)) \geq a \; \forall n \geq 0$ for some $a > 0$, and then,

$$
\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n)d(x_n, T(x_n)) \geq a\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty
$$

which is a contradiction with (4.6).

On the other hand, using the nonexpansivity of T and the convexity of the distance function,

$$
\begin{aligned}
d(x_{n+1}, T(x_{n+1})) &\leq d(x_{n+1}, T(x_n)) + d(T(x_n), T(x_{n+1})) \\
&\leq d(x_{n+1}, T(x_n)) + d(x_n, x_{n+1}) \\
&\leq \alpha_n d(x_n, T(x_n)) + (1 - \alpha_n)d(x_n, T(x_n)) \\
&= d(x_n, T(x_n)).
\end{aligned}
$$

This means that $\{d(x_n, T(x_n))\}$ is a monotone sequence. Combining this and (4.7) completes the proof of (4.5). ∎

5. NUMERICAL EXAMPLE

Let $\mathbb{E}^{m,1}$ denote the vector space $\mathbb{R}^{m+1}$ endowed with the symmetric bilinear form (which is called the Lorentz metric) defined by

$$\langle x, y \rangle = \sum_{i=1}^{m} x_i y_i - x_{m+1} y_{m+1}, \quad \forall x = (x_i),\ y = (y_i) \in \mathbb{R}^{m+1}.$$

The *hyperbolic m-space* $\mathbb{H}^m$ is defined by

$$\{x = (x_1, ..., x_{m+1}) \in \mathbb{E}^{m,1} : \langle x, x \rangle = -1, x_{m+1} > 0\},$$

that is the upper sheet of the hyperboloid $\{x \in \mathbb{E}^{m,1} : \langle x, x \rangle = -1\}$. Note that $x_{m+1} \geq 1$ for any $x \in \mathbb{H}^m$, with equality if and only if $x_i = 0$ for all $i = 1, ..., m$. The metric of $\mathbb{H}^m$ is induced from the Lorentz metric $\langle \cdot, \cdot \rangle$ and it will be denoted by the same symbol. Then $\mathbb{H}^m$ is a Hadamard manifold with sectional curvature -1 (cf. [1] and [6]). Furthermore, the normalized geodesic $\gamma : \mathbb{R} \to \mathbb{H}^m$ starting from $x \in \mathbb{H}^m$ is given by

$$(5.1) \qquad\qquad \gamma(t) = (\cosh t)x + (\sinh t)v, \quad \forall t \in \mathbb{R},$$

where $v \in T_x \mathbb{H}^m$ is a unit vector; while the distance d on $\mathbb{H}^m$ is

$$(5.2) \qquad\qquad d(x, y) = \text{arccosh}(-\langle x, y \rangle), \quad \forall x, y \in \mathbb{H}^m.$$

Then, the exponential map can be expressed as

$$\exp_x(rv) = (\cosh r)x + (\sinh r)v,$$

for any $r \in \mathbb{R}^+$, $x \in \mathbb{H}^m$ and any unit vector $v \in T_x \mathbb{H}^m$. To get the expression of the inverse exponential map, we write for any $x, y \in \mathbb{H}^m$,

$$y = \exp_x\left(d(x, y)\frac{\exp_x^{-1} y}{d(x, y)}\right) = (\cosh d(x, y))x + (\sinh d(x, y))\frac{\exp_x^{-1} y}{d(x, y)}.$$

Therefore, by definition of the distance (5.2), we obtain

$$\exp_x^{-1} y = \text{arccosh}(-\langle x, y \rangle)\frac{y + \langle x, y \rangle x}{\sqrt{\langle x, y \rangle^2 - 1}}, \quad \forall x, y \in \mathbb{H}^m.$$

Thus, using these expressions, both algorithms in previous sections can be formulated in a simple way in the hyperbolic space $\mathbb{H}^m$. Write

$$r(x, y) = \text{arccosh}(-\langle x, T(y) \rangle)$$

and

$$V(x, y) = \frac{T(y) + \langle x, T(y) \rangle x}{\sqrt{\langle x, T(y) \rangle^2 - 1}}.$$

Then Halpern's algorithm (3.1) has the form

$$x_{n+1} = (\cosh(1 - \alpha_n)r(u, x_n))u + (\sinh(1 - \alpha_n)r)V(u, x_n), \quad \forall n \geq 0;$$

while Mann's algorithm (4.3) has the form

$$x_{n+1} = (\cosh(1 - \alpha_n)r(x_n, x_n)x_n + (\sinh(1 - \alpha_n)r)V(x_n, x_n), \quad \forall n \geq 0.$$

We end this paper with an example of $\mathbb{H}^3$, where these methods are implemented for two concrete operators.

Table 5.1

T_1		x_0^1	x_0^2	x_0^3
	Step	Error		
	d_0	1.23849936794859	1.43016443780138	0.96560216993144
	d_1	0.61924968397430	0.71508221890069	0.48280108496572
	d_2	0.37154981038458	0.42904933134041	0.28968065097943
	d_3	0.24769987358972	0.28603288756028	0.19312043398629
	d_4	0.17692848113551	0.20430920540020	0.13794316713307
	d_5	0.13269636085164	0.15323190405015	0.10345737534980
	d_6	0.10320828066239	0.11918036981678	0.08046684749429
MANN	d_7	0.08256662452991	0.09534429585343	0.06437347799544
	d_8	0.06755451097902	0.07800896933462	0.05266920926899
	d_9	0.05629542581585	0.06500747444552	0.04389100772417
	d_{10}	0.04763459107495	0.05500632453083	0.03713854499737
	Error	Step		
	10^{-2}	d_{17}	d_{19}	d_{15}
	10^{-3}	d_{59}	d_{64}	d_{52}
	10^{-4}	d_{191}	d_{205}	d_{168}
	10^{-5}	d_{608}	d_{654}	d_{537}
	Step	Error		
	d_0	1.23849936794859	1.43016443780138	0.96560216993144
	d_1	0.79109087958575	0.94324711126741	0.56347787068548
	d_2	0.72937999179719	0.83691943183081	0.57904469220055
	d_3	0.51644150342379	0.61225961625570	0.37537998889839
	d_4	0.51460518499467	0.58768625280562	0.41346210869283
	d_5	0.38173470848514	0.45056824179018	0.28140064643068
	d_6	0.39667994376475	0.45136530320607	0.32150418565275
HALPERN	d_7	0.30206081644163	0.35527907593590	0.22504211840976
	d_8	0.32231407126604	0.36567764790811	0.26300078945845
	d_9	0.24953770871435	0.29265581960747	0.18748437135137
	d_{10}	0.27120338594617	0.30694720797307	0.22250671712339
	Error	Step		
	10^{-2}	d_{162}	d_{176}	d_{141}
	10^{-3}	d_{1593}	d_{1710}	d_{1439}
	10^{-4}	d_{15652}	d_{16630}	d_{14421}
	10^{-5}	d_{123278}	d_{132268}	d_{111932}

Example 5.1. Let $M = \mathbb{H}^3$ and let $T_1, T_2 : M \to M$ be the nonexpansive mappings respectively defined by

$$T_1(x) = (-x_1, -x_2, -x_3, x_4)$$

and

$$T_2(x) = (-x_1, x_2, x_3, x_4),$$

for any $x = (x_1, x_2, x_3, x_4) \in \mathbb{H}^3$. Then $\text{Fix}(T_1) = \{(0, 0, 0, 1)\}$ and

$$\text{Fix}(T_2) = \{(x_1, x_2, x_3, x_4) \in \mathbb{H}^3 : x_1 = 0, x_2^2 + x_3^2 = x_4^2 - 1\}.$$

Table 5.2

T_2		x_0^1	x_0^2	x_0^3
	Step		Error	
	d_1	1.85774905192289	2.14524665670207	1.44840325489716
	d_2	0.99079949435887	1.14413155024110	0.77248173594515
	d_3	0.61924968397430	0.71508221890069	0.48280108496572
	d_4	0.42462835472523	0.49034209296047	0.33106360111935
	d_5	0.30962484198715	0.35754110945034	0.24140054248286
	d_6	0.23590464151402	0.27241227386693	0.18392422284409
MANN	d_7	0.18577490519229	0.21452466567021	0.14484032548972
	d_8	0.15012113550892	0.17335326518805	0.11704268726442
	d_9	0.12384993679486	0.14301644378014	0.09656021699315
	d_{10}	0.10393001689079	0.12001379897634	0.08102955272153
	Error		Step	
	10^{-2}	d_{26}	d_{28}	d_{23}
	10^{-3}	d_{85}	d_{91}	d_{75}
	10^{-4}	d_{271}	d_{291}	d_{239}
	10^{-5}	d_{862}	d_{926}	d_{761}
	Step		Error	
	d_1	0.99837317911716	1.16992727222576	1.31340494917273
	d_2	0.71833797458614	0.83426922900920	0.97908824565578
	d_3	0.59032246197087	0.67290452208375	0.81786161285239
	d_4	0.47974944268260	0.54618472938029	0.67524807759001
	d_5	0.42202799259708	0.47635447878610	0.59506375226788
	d_6	0.36239751805204	0.40927663184813	0.51596963770002
HALPERN	d_7	0.32928308480446	0.37005540327231	0.46796492723635
	d_8	0.29184728056858	0.32831699473056	0.41767189977732
	d_9	0.27026706878293	0.30305136210601	0.38570372433189
	d_{10}	0.24453550260074	0.27450025725934	0.35090005098115
	Error		Step	
	10^{-2}	d_{151}	d_{168}	d_{218}
	10^{-3}	d_{1524}	d_{1702}	d_{2198}
	10^{-4}	d_{15266}	d_{17050}	d_{22001}
	10^{-5}	d_{152673}	d_{170513}	d_{220033}

For both algorithms we are going to consider the sequence of parameters $\alpha_n = \frac{1}{n+3}$ for each $n = 0, 1, \cdots$, and the point

$$u = (0.60379247919382, 0.27218792496996, 0.19881426776106, 1.21580374135624)$$

for Halpern's iteration. We consider three random initial points x_0:

$$x_0^1 = (0.69445440978475, 1.01382609280137, 0.99360871330745, 1.87012527625153);$$

$$x_0^2 = (0.82054041398189, 1.78729906182707, 0.11578260956854, 2.20932797928782);$$

$$x_0^3 = (0.93181457846166, 0.46599434167542, 0.41864946772751, 1.50356127641534).$$

The numerical results are listed in the following tables, where $d_n = d(x_n, x^*)$ denote the error of the nth step with the unique fixed point $x^* = (0, 0, 0, 1)$ in the first table for T_1; while, in the second one for T_2, $d_n = d(x_{n+1}, x_n)$ denote the distance between two consecutive iterates x_{n+1} and x_n. In both tables we first compute, for both algorithm, the errors in the steps 1-10 of the iteration and later we provide the step where the errors get the lower bounds $10^{-2}, 10^{-3}, 10^{-4}$ and 10^{-5}, respectively.

From the numerical results, one can observe that Mann's algorithm is much quicker than Halpern's algorithm. Moreover, as it is predicted by the theoretical results, the measure of the errors for the steps 1-10 in the first table show that the sequence of iterates is Fejer monotone just in the case of Mann algorithm.

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Chong Li
Department of Mathematics,
Zhejiang University,
Hangzhou 310027,
P. R. China
and
Department of Mathematics,
College of Sciences,
King Saud University,
P. O. Box 2455, Riyadh 11451,
Saudi Arabia
E-mail: cli@zju.edu.cn

Genaro López and Victoria Martín-Márquez
Departamento de Anlisis Matemático,
Universidad de Sevilla,
Apdo. 1160, 41080-Sevilla,
Spain
E-mail: glopez@us.es
         victoriam@us.es