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ON A RESULT OF W. A. KIRK

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Abstract. W. A. Kirk has recently proved a constructive fixed point theorem for continuous mappings in compact hyperconvex metric spaces [6]. In the present work we use the concept of hyperconvex hull of a metric space to obtain a noncompact counterpart of Kirk's result.

1. Introduction

A metric space (M, d) is said to be *hyperconvex* if

$$\bigcap_{\alpha \in \mathcal{A}} B(x_\alpha, r_\alpha) \neq \emptyset$$

for any indexed class of closed balls $\{B(x_\alpha, r_\alpha) : \alpha \in \mathcal{A}\}$ in M such that $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ for all α and β in $\mathcal{A}$.

N. Aronszajn and P. Panitchpakdi [1] proved that a metric space M is hyperconvex if and only if every nonexpansive mapping T from any metric space D into M has, for any metric space Y containing D metrically, a nonexpansive extension $\hat{T}$ from Y into M.

The intersection of two hyperconvex spaces may not be hyperconvex. But if (X_i) is a decreasing chain of nonempty spaces, then the intersection is also hyperconvex. Baillon [3] has shown that if M is a hyperconvex metric space and (X_i) is a decreasing chain of nonempty bounded hyperconvex subsets of X, then $\cap_i X_i$ is nonempty and hyperconvex.

For a bounded subset D of a metric space set:

$$cov(D) = \cap\{B : B \text{ is a closed ball and } D \subset B\}.$$

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Let $\mathcal{A}(M) = \{B \subset M : B = cov(B)\}$. Thus $\mathcal{A}(M)$ denotes the collection of all ball intersection sets of M.

The following abstraction of an interval analysis result appeared in [10] has been proved by W. A. Kirk in [6].

Theorem 1.1. *Let M be a compact hyperconvex space, and let $f\colon M \to M$ be a continuous mapping. Define the mapping $\overline{f}$ from $\mathcal{A}(M)$ to itself by*

$$\begin{array}{rl} \overline{f}: & \mathcal{A}(M) \longrightarrow \mathcal{A}(M) \\ & D \quad \longmapsto cov(f(D)) \end{array}$$

Set $D_0 = M$, $D_n = \overline{f}(D_{n-1}) = \overline{f}^n(M)$, and suppose $D = \bigcap_{n\in\mathbb{N}} D_n$. Then

$$\overline{f}(D) = D \neq \emptyset \text{ and } D = \lim_{n\to\infty} D_n,$$

where the limit is taken with respect to the Hausdorff metric H in $\mathcal{A}(M)$. In particular, if $f(x) = x$ then $x \in D$.

Our aim is to give a counterpart of this theorem for the noncompact case.

For a bounded subset A of a metric space M the Kuratowski measure of noncompactness of A, $\alpha(A)$, is defined by

$$\alpha(A) = \inf\{\varepsilon :\ A \subset \cup_{i=1}^n A_i \text{ with diam } A_i \leq \varepsilon\}.$$

The Hausdorff measure of noncompactness of A, $\chi(A)$, is defined by

$$\chi(A) = \inf\{r :\ A \subset \cup_{i=1}^n B(x_i, r) \text{ with } x_i \in M\}.$$

Henceforth, ϕ denotes either the Kuratowski or the Hausdorff measure of noncompactness.

Given a metric space M and $D \subset M$, a continuous map $T : D \to M$ is said to be $k - \phi$-condensing if $\phi(T(A)) \leq k\phi(A)$ for every bounded subset A of D.

In order to define the hyperconvex hull of a subset of a hyperconvex space we will need the concept of injective envelope introduced by Isbell in [4].

Definition 1. A mapping of metric spaces $e : X \longrightarrow M$ is called an *injective envelope* of X if M is hyperconvex, e is an isometric embedding, and no hyperconvex proper subspace of X contains $e(X)$.

Lemma 1.2. *Let $e : X \longrightarrow M$ and $f : X \longrightarrow N$ be injective envelopes of X. Then, there exists an isometry $i : N \longrightarrow M$.*

Given a hyperconvex metric space M, we will denote by $\mathcal{F}$ the family of all the hyperconvex subsets of M. From Zorn's lemma it is easy to deduce that the set $\mathcal{F}(A) = \{B \in \mathcal{F}; A \subset B\}$ has minimal elements.

Definition 2. Let M be a hyperconvex metric space and $A \subset M$. We will say that a set, $h(A)$, is a *hyperconvex hull* of A if $h(A)$ is a minimal element of the set $\mathcal{F}(A)$.

Remark. From Lemma 1.2 and the previous observation every subset of a hyperconvex space has a hyperconvex hull and all its hyperconvex hulls are related by isometries.

2. Main Results

We begin this section with the following proposition where we summarize the main properties of hyperconvex hulls.

Proposition 2.1. *Let M be a hyperconvex metric space and A a bounded subset of M. Then*

1. *If $B \subset A$ there exists $h(A)$ and $h(B)$ such that $h(B) \subset h(A)$.*
2. $\alpha(A) = 2\chi(A)$.
3. $\phi(A) = \phi(h(A))$.
4. *If $h_1(A)$ and $h_2(A)$ are hyperconvex hulls of A then there exists an isometry $i : h_1(A) \longrightarrow h_2(A)$ such that $i(x) = x$ for all $x \in A$.*

Proof. The proof of statements 1., 2. and 3. can be found in [5]. So it suffices to prove 4.

If $h_1(A)$ and $h_2(A)$ are hyperconvex hulls of A, by the properties of extension for hyperconvex spaces, there exist two nonexpansive mappings $r : h_1(A) \longrightarrow h_2(A)$ and $s : h_2(A) \longrightarrow h_1(A)$ such that $r(x) = s(x) = x$ for all $x \in A$. We will complete this proof by proving that r is an isometry.

Consider $r \circ s : h_2(A) \longrightarrow h_2(A)$. This mapping is a nonexpansive mapping such that the set of its fixed points, $Fix(r \circ s)$, contains A and is hyperconvex (see [3] Theorem 5.). Then, by minimality of $h_2(A)$, we have $Fix(r \circ s) = h_2(A)$ and so $r \circ s$ is the identity. Now bearing in mind that r and s are nonexpansive we may deduce that r is an isometry and the proof is complete. □

From now on, if M is a metric space, $N_\rho(D)$ will denote the set

$$N_\rho(D) = \{z \in M : \text{dist}(z, D) \leq \rho\},$$

where $D \subseteq M$ y $\rho \geq 0$. The Hausdorff metric between two subsets A and B of M, may be described as follows:

$$H(A,B) = \inf\{\rho \geq 0 : \ A \subseteq N_\rho(B) \text{ and } B \subseteq N_\rho(A)\}.$$

Throughout this work we will understand that a sequence of closed subsets of a metric space converges to another subset of this metric space if the convergence is with respect to the Hausdorff metric.

We begin by introducing some technical lemmas.

Lemma 2.2. *Let M be a hyperconvex metric space, and suppose A is a hyperconvex subset of M. Then, given $\varepsilon > 0$ there exists a hyperconvex subset of M, $A(\varepsilon)$, such that $N_\varepsilon(A) \subseteq A(\varepsilon) \subseteq N_{2\varepsilon}(A)$.*

Proof. Since A is a hyperconvex set there exists a nonexpansive retraction

$$r\colon M \longrightarrow A.$$

Let us fix $A(\varepsilon) = \{x \in M : d(r(x), x) \leq 2\varepsilon\}$. This set $A(\varepsilon)$ is called the 2ε-fixed point set of r and is hyperconvex (see [9]). We are going to show that

$$N_\varepsilon(A) \subseteq A(\varepsilon) \subseteq N_{2\varepsilon}(A).$$

It is clear $A(\varepsilon) \subseteq N_{2\varepsilon}(A)$.

Given $\eta > 0$ and $x \in N_\varepsilon(A)$ we fix $y \in A$ such that $d(x,y) \leq \varepsilon + \eta$. Now since $r(y) = y$,

$$\begin{gathered} d\left(x, r(x)\right) \leq d(x,y) + d\left(y, r(x)\right) \leq \\ \leq \varepsilon + \eta + d\left(r(y), r(x)\right) \leq \varepsilon + \eta + d(y,x) \leq 2\varepsilon + 2\eta. \end{gathered}$$

Finally, since η is arbitrary the conclusion follows. □

Lemma 2.3. *Suppose (D_n) is a nonincreasing sequence of nonempty bounded closed subsets of a metric space M such that $\lim_{n\to\infty} \phi(D_n) = 0$. Then*

$$\lim_{n\to\infty} D_n = \bigcap_{n\in\mathbb{N}} D_n \neq \emptyset.$$

Proof. Let $D = \bigcap_{n\in\mathbb{N}} D_n$. Since $\lim_{n\to\infty} \phi(D_n) = 0$, D is nonempty and compact (see [2]).

Suppose D is not the limit of D_n. Then, since $D \subset D_n$ for all $n \in N$ and (D_n) is a decreasing sequence, given $\varepsilon > 0$ there exists $x_n \in D_n \setminus N_\varepsilon(D)$ for all $n \in N$. Since $\phi(\{x_k : \ k \geq n\}) \leq \phi(D_n)$, the sequence is precompact and so has a convergent subsequence to a point which is necessarily in D. This is a contradiction with the fact that $x_n \in D_n \setminus N_\varepsilon(D)$. □

We omit the proof of the following lemma.

Lemma 2.4. *Let (D_n) be the sequence of the previous lemma and D its limit. If $f : M \to M$ is a continuous mapping, then the sequence $\left(\overline{f(D_n)}\right)$ converges to $f(D)$.*

Definition 3. Let M be a hyperconvex space, and $f\colon M \longrightarrow M$ a mapping. We say that a sequence of subsets of M, (D_n), is a *proper sequence of hyperconvex hulls* of M relative to the mapping f, if (D_n) is defined in the following way

1. $D_0 = M$,
2. $D_n = h\left(f(D_{n-1})\right)$,

where $h\left(f(D_{n-1})\right)$ is a hyperconvex hull of $f(D_{n-1})$ contained in D_{n-1}.

Remark. From the properties of the hyperconvex hulls one can easily deduce that such sequences always exist under the conditions of the definition.

Theorem 2.5. *Let M be a bounded hyperconvex space and $f : M \to M$ a $k-\phi-$condensing mapping with $k < 1$. Let (D_n) be a proper sequence of hyperconvex hulls of M relative to f, and suppose $D = \bigcap_{n\in\mathbb{N}} D_n$. Then D is a hyperconvex hull of $f(D)$. In particular, if $f(x) = x$ then $x \in D$.*

Proof. Since ϕ is $k-\phi-$condensing with $k < 1$ and

$$\phi(D_n) = \phi(h(f(D_{n-1}))) = \phi(f(D_{n-1})) < k\phi(D_{n-1}),$$

the sequence satisfies the hypothesis of Lemma 2.3. So D is a compact hyperconvex set and $D = \lim_{n\to\infty} D_n$. It is also clear that, by definition of D, D contains the set of the fixed points of f.

Now we want to prove that $D = h(f(D))$. Since D is hyperconvex and, by definition of D_n, $D \supseteq f(D)$ there exists a hyperconvex hull of $f(D)$, $h(f(D))$, contained in D. We will prove that $D = h(f(D))$.

It is clear that

$$\varepsilon_0 = \inf\{\delta \geq 0 : f(D_0) \subseteq N_\delta(h(f(D)))\} \leq H(\overline{f(D_0)}, f(D))$$

and

$$f(D_0) \subseteq N_{\varepsilon_0}(h(f(D))).$$

By Lemma 2.2, there exists a hyperconvex set $A(\varepsilon_0)$ such that

$$N_{\varepsilon_0}(h(f(D))) \subseteq A(\varepsilon_0) \subseteq N_{2\varepsilon_0}(h(f(D))).$$

Therefore we can choose a hyperconvex hull of $f(D_0)$, denoted by $\tilde{h}(f(D_0))$, such that

$$\tilde{h}(f(D_0)) \subseteq A(\varepsilon_0) \subseteq N_{2\varepsilon_0}(h(f(D))).$$

We continue in this fashion obtaining a sequence $(\tilde{h}(f(D_n)))$ of hyperconvex hulls of $(f(D_n))$ such that

$$\tilde{h}(f(D_n)) \subseteq N_{2\varepsilon_n}(h(f(D))),$$

where

$$\varepsilon_n \le H(\overline{f(D_n)}, f(D))$$

for all $n \in N$.

But from Lemma 2.4, $(\overline{f(D_n)})$ converges to $f(D)$. Hence $\lim_{n\to\infty} \varepsilon_n = 0$.

Consequently, we have just proved that if (x_n) is a convergent sequence such that $x_n \in \tilde{h}(f(D_n))$ for all $n \in \mathbb{N}$, then its limit must belong to $h(f(D))$.

According to Proposition 2.1, for each $n \in N$ there exists an isometry

$$i_n : h(f(D_n)) \to \tilde{h}(f(D_n))$$

such that its restriction to $f(D_n)$ coincides with the identity and hence, so does its restriction to $f(D)$.

Since D is a compact set we can fix a dense sequence in D, (x_m) . From the compactness of D we can follow a diagonalization argument so that we obtain a subsequence of (i_n), which for the sake of simplicity we denote as (i_n), such that the sequence$\{i_n(x_m)\}_{n=1}^{\infty}$ is convergent for all $m \in \mathbb{N}$. Since $D \subseteq h(f(D_n))$ for all $n \in \mathbb{N}$ their limits are in $h(f(D))$. Therefore we can define

$$\begin{array}{rccc} i: & \{x_m\}_{m=1}^{\infty} & \longrightarrow & h(f(D)) \\ & x_m & \longmapsto & \lim_{n\to\infty} i_n(x_m) \end{array}$$

Since (x_m) is dense in D, we can extend this mapping to the whole D in such a way that i restricted to $h(f(D))$ is the identity.

On the other hand, since $h(f(D)) \subseteq D$, it is defined the natural embedding

$$j : h(f(D)) \longrightarrow D.$$

Consider now the isometry

$$i \circ j : h(f(D)) \longrightarrow H \subseteq h(f(D)),$$

where H stands for the range of $i \circ j$. Since H is hyperconvex and $f(D) \subset H$ we have $H = h(f(D))$. Consequently $i(D) = h(f(D))$ and, hence, D is a hyperconvex hull of $h(f(D))$. Finally, by minimality of the hyperconvex hull, we obtain that $D = h(f(D))$. □

The following corollary is the compact version of this result.

Corollary 2.6. *Let M be a compact hyperconvex space, and suppose $f : M \to M$ is continuous. Let (D_n) be a proper sequence of hyperconvex hulls of M relative to f, and suppose $D = \bigcap_{n\in N} D_n$,then $D = \lim_{n\to\infty} D_n$ and D is a hyperconvex hull of $f(D)$. In particular, if $f(x) = x$ then $x \in D$.*

Remark. We may state this result in a more similar way to Kirk's result. Let M be a compact hyperconvex space and $\mathcal{H}(M)$ the set of all hyperconvex subsets of M. Suppose $f : M \to M$ is continuous. We may fix a hyperconvex hull for all subset of M, A,

$$\begin{array}{rcl} \overline{f}: & \mathcal{P}(M) & \longrightarrow \mathcal{H}(M) \\ & A & \longmapsto h(f(A)) \end{array}$$

in such a way that we obtain a proper sequence of hyperconvex hulls for the mapping f. Assuming (D_n) is a sequence as in Kirk's theorem (hence, it coincides with our proper sequence), we have

$$D = \bigcap_{n\in\mathbb{N}} D_n = \lim_{n\to\infty} D_n$$

and $D = \overline{f}(D)$.

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