

## Geometric Properties of Banach Spaces and Metric Fixed Point Theory

TOMÁS DOMÍNGUEZ BENAVIDES

*Facultad de Matemáticas, P.O. Box 1160, 41080 Sevilla, Spain*  
*e-mail: tomasd@us.es*

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### 1. INTRODUCTION

Let  $T$  a mapping from a set  $X$  into itself. The mapping  $T$  has a fixed point if there exists  $x_0 \in X$  such that  $Tx_0 = x_0$ . The most known fixed point theorem is the Contraction Mapping Principle, due to S. Banach [3]. The classical statement is the following:

**THEOREM.** (Banach Fixed Point Theorem [3]) *Let  $X$  be a complete metric space and  $T : X \rightarrow X$  a contractive mapping, i.e. there exists  $k \in [0, 1)$  such that*

$$d(Tx, Ty) \leq kd(x, y)$$

*for every  $x, y \in X$ . Then  $T$  has a (unique) fixed point  $x_\omega$ . Furthermore  $x_\omega = \lim_n T^n x_0$  for any  $x_0 \in X$ .*

The proof of the Banach Theorem is rarely simple (probably because the wide diffusion of Banach techniques). Indeed, let  $x_0$  be an arbitrary point and set  $x_n = T^n x_0$ . We have

$$d(x_{n+1}, x_n) \leq kd(x_n, x_{n-1}) \leq \dots \leq k^n d(x_1, x_0).$$

Thus,  $\{x_n\}$  is a Cauchy sequence, because

$$\begin{aligned} d(x_{n+j}, x_n) &\leq d(x_{n+j}, x_{n+j-1}) + \dots + d(x_{n+1}, x_n) \\ &\leq (k^{n+j-1} + \dots + k^n)d(x_1, x_0) \leq \frac{k^{n+j-1}}{1-k} < \epsilon \end{aligned}$$

if  $n$  is large enough. Obviously, the fixed point is unique, because if  $x = Tx$ ,  $y = Ty$  we have  $d(x, y) = d(Tx, Ty) \leq kd(x, y)$  which implies  $d(x, y) = 0$ .

Banach Theorem has proved to be very useful to solve different theoretical and practical problems. For instance, it is used to prove the Picard-Lindelöf Theorem about existence of solution of differential equations and to prove the Inverse (or Implicit) Function Theorem in infinite dimensional spaces. A big number of generalizations of the Banach Theorem have appeared in the literature, trying to weak some assumptions (see, for instance, [11, Cap. 3]). However, the most natural improvement would be to let the constant  $k$  attain the value 1, but in this case the result is not more true. Indeed, a translation in  $\mathbb{R}$  satisfies the assumptions and it is fixed point free. Even the mild assumption  $d(Tx, Ty) < d(x, y)$  does not assure the existence of a fixed point. Indeed, consider  $X = [1, \infty)$  and  $Tx = x + 1/x$ . We have

$$d(Tx, Ty) = \left| x + \frac{1}{x} - y - \frac{1}{y} \right| = |y - x| - \left| \frac{1}{x} - \frac{1}{y} \right| < |y - x|.$$

However  $T$  is fixed point free. The fail of the existence of fixed point in this example occurs because, in addition, the interval  $[1, \infty)$  is unbounded. Indeed, otherwise we could apply Brouwer's Theorem:

**THEOREM.** (Brouwer's Theorem (1912) [5]) *Let  $M$  a bounded convex closed subset of  $\mathbb{R}^n$  and  $T : M \rightarrow M$  a continuous mapping. Then  $T$  has a fixed point.*

Having in mind both Banach and Brouwer Theorems, a question seems to be natural: Assume that  $M$  is a convex, closed bounded subset of an arbitrary Banach space and  $T : M \rightarrow M$  is a nonexpansive mapping, i.e.,  $\|Tx - Ty\| \leq \|x - y\|$  for every  $x, y \in M$ . Does  $T$  has a fixed point? We must realize that the assumption on the space has been weakened (the dimension is not necessarily finite) but stronger conditions are assumed on the mapping (nonexpansiveness instead of continuity). The answer is again negative:

**EXAMPLE.** (Kakutani (1943) [9]). Let  $B$  the unit ball in  $c_0$  and  $T : B \rightarrow B$  defined by  $T(x_1, x_2, \dots) = (1 - \|x\|, x_1, x_2, \dots)$ . It is easy to check that  $T$  is nonexpansive and fixed point free.

As a consequence of these facts, the problems about existence of fixed point for nonexpansive mappings were relegated. However in 1965, two surprising theorems appeared:

**THEOREM.** (Browder's Theorem [6]) *Let  $C$  be a convex bounded closed subset of a uniformly convex Banach space (a preliminary version was given for Hilbert spaces) and  $T : C \rightarrow C$  a nonexpansive mapping. Then  $T$  has a fixed point.*

**THEOREM.** (Kirk's Theorem [10]) *Let  $C$  be a convex bounded closed subset of a reflexive Banach space with normal structure. If  $T : C \rightarrow C$  is nonexpansive, then  $T$  has a fixed point.*

In the second section we will include a proof of the latter theorem (which includes Browder's Theorem). It is noteworthy that these results state a bridge between notions which had usually been considered in Linear Functional Analysis (uniform convexity, reflexivity, normal structure, etc) and problems about existence of fixed point for nonlinear operators. From this starting point a big number of fixed point results have been obtained for different classes of mappings using geometric properties of Banach spaces. In forthcoming sections we will review some of these results.

## 2. NORMAL STRUCTURE. EXISTENCE OF FIXED POINTS FOR NONEXPANSIVE OPERATORS

We recall some definitions yielding to the notion of normal structure.

**DEFINITION 1.** Let  $X$  be a Banach space,  $A$  a bounded subset of  $X$  and  $B$  an arbitrary subset of  $X$ . The Chebyshev radius of  $A$  with respect to  $B$  is defined by

$$r(A, B) = \inf\{\sup\{\|x - y\| : x \in A\} : y \in B\}$$

where we write  $r(A)$  instead of  $r(A, \text{co}(A))$ . The Chebyshev center of  $A$  with respect to  $B$  is defined by

$$Z(A, B) = \{y \in B : \sup\{\|x - y\| : x \in A\} = r(A, B)\}$$

where we write  $Z(A)$  instead of  $Z(A, \text{co}(A))$ .

*Remark 1.* Roughly speaking, we can say that the Chebyshev radius  $r(A, B)$  is the radius of the smallest ball centered at a point in  $B$  and covering the set  $A$ , the Chebyshev center  $Z(A, B)$  being the set formed by all centers of these smallest balls. However, since the infimum appearing in the definition

is not, necessarily attained, the set  $Z(A, B)$  can be empty. In opposition, if for every  $\varepsilon > 0$  we consider the set

$$Z_\varepsilon(A, B) = \{y \in B : r(A, y) \leq r(A, B) + \varepsilon\},$$

then  $Z_\varepsilon(A, B)$  is a nonempty, convex, bounded and closed set if  $B$  satisfies the same properties. Thus,  $Z_\varepsilon(A, B)$  is convex, nonempty and weakly compact if so is  $B$ . Since

$$\bigcap_{\varepsilon > 0} Z_\varepsilon(A, B) = Z(A, B),$$

the finite intersection property implies that  $Z(A, B)$  is nonempty when  $B$  is a convex and weakly compact set.

**DEFINITION 2.** A bounded convex closed subset  $A$  of a Banach space  $X$  is said to be diametral if  $\text{diam}(A) = r(A)$ . Equivalently, if  $Z(A) = A$ . We say that a Banach space  $X$  has normal structure (respectively weak normal structure) if every convex closed nonempty (respectively convex weakly compact) diametral subset of  $X$  is a singleton.

*Remark 2.* According to the above definition, a Banach space has normal structure if every convex set  $A$  which is not a singleton can be covered by a ball whose radius is less than the diameter of  $A$  and centered at a point in  $A$ . We could think that this is the case of every Banach space, and in fact, this occurs for every uniformly convex space (we will see the definition in the next section), for instance,  $\ell^p$  and  $L^p(\Omega)$ ,  $1 < p < +\infty$ . However the sequence space  $c_0$  fails to have both normal structure and weak normal structure. Indeed, consider the set  $A = \overline{\text{co}}(\{e_n : n \in \mathbb{N}\})$  where  $\{e_n\}$  is the standard basis. We have  $\text{diam}(A) = 1$  and  $r(A) = 1$  because  $\lim_{n \rightarrow \infty} \|x - e_n\| = 1$  for every  $x \in c_0$ . Furthermore, since the sequence  $\{e_n\}$  is weakly null,  $A$  is a weakly compact set. The same set can be considered in the sequence space  $\ell^1$ , giving us that  $\ell^1$  fails to have normal structure either. However, we will show in the next section that  $\ell^1$  (and any Banach space with the Schur property) has weak normal structure.

**THEOREM 1.** *Let  $X$  be a Banach space with weak normal structure,  $C$  a convex weakly compact subset of  $X$  and  $T : C \rightarrow C$  a nonexpansive mapping. Then  $T$  has a fixed point.*

*Proof.* Let  $\mathcal{B}$  be the collection of all convex weakly compact subsets of  $C$  which are  $T$ -invariant. It is easy to check that  $\mathcal{B}$ , ordered by inclusion is

an inductive family. Zorn's Lemma assures the existence of a minimal set  $K$ . Since  $T(K) \subset K$  we have  $\overline{\text{co}}(T(K)) \subset K$ . Thus,  $\overline{\text{co}}(T(K))$  is a convex weakly compact subset of  $K$  which is  $T$ -invariant. The minimality of  $K$  implies  $K = \overline{\text{co}}(T(K))$ . Since  $K$  is a weakly compact convex set, we know from Remark 1 that  $Z(K)$  is a nonempty convex weakly compact set. We will prove that  $Z(K)$  is  $T$ -invariant. Indeed, take  $x \in Z(K)$ , i.e.  $r(K, x) = r(K)$ . For every  $y \in K$  we have  $\|Ty - Tx\| \leq \|y - x\| \leq r(K)$ . Hence  $T(K)$  is covered by the closed ball  $\overline{B}(Tx, r(K))$  which implies  $\overline{\text{co}}(T(K)) = K \subset \overline{B}(Tx, r(K))$ . Therefore  $r(K, Tx) \leq r(K)$  which implies  $Tx \in Z(K)$ . The minimality of  $K$  implies  $Z(K) = K$  and thus  $\text{diam}(K) = 0$  because  $X$  has weak normal structure. Hence  $K$  is a singleton and contains a fixed point of  $T$ . ■

### 3. GEOMETRIC PROPERTIES WHICH IMPLY NORMAL STRUCTURE

In order to study some geometric properties implying normal structure we recall a geometric coefficient defined by Bynum [7] in 1980 (a preliminary form had been studied by por Jüing in 1901).

DEFINITION 3. Let  $X$  be a Banach space. The normal structure coefficient of  $X$  is defined by

$$N(X) = \inf \left\{ \frac{\text{diam}(A)}{r(A)} : A \subset X \text{ convex closed and bounded with } \text{diam}(A) > 0 \right\}.$$

It is clear that  $X$  has normal structure if  $N(X) > 1$ . However spaces with normal structure exist which satisfy  $N(X) = 1$ . This coefficient can be considered as a measure of the "worst" possible relationship between the diameter and the Chebyshev radius of a subset of  $X$ . For instance, in the euclidean plane  $\ell_2^2$  this "worst" relationship is attained at the equilateral triangle and its value is  $\sqrt{3}$ . In the tridimensional euclidean space  $\ell_2^3$  it is attained at the tetrahedron with a value equal to  $2\sqrt{2}/\sqrt{3}$  and, in general, for  $\ell_2^n$  the worst value corresponds to the "hipertetrahedron" with a value equal to  $\sqrt{2}\sqrt{(n+1)/n}$ . It is not easy to evaluate  $N(X)$  for a determined space  $X$ , and, in fact, its value is unknown in many cases. The following connection between the value of  $N(X)$  and the reflexivity of the space is important:

THEOREM 2. ([12]) *Let  $X$  be a Banach space such that  $N(X) > 1$ . Then  $X$  is reflexive.*

*Proof.* If  $X$  is not reflexive, for every  $\varepsilon > 0$  there exists a sequence  $\{x_n\}$  (see [13]) such that  $1 - \varepsilon \leq \|u_{1,n} - u_{n,\omega}\| \leq 1 + \varepsilon$  for every  $u_{1,n} \in \text{co}(\{x_j\}_{1 \leq j \leq n})$ ,  $u_{n,\omega} \in \text{co}(\{x_j\}_{j > n})$ , and for each  $n$ . Thus  $\text{diam}(\{x_n\}) \leq 1 + \varepsilon$ . Furthermore, if  $v$  belongs to  $\text{co}(\{x_n\})$  and  $n$  is large enough we have  $\|x_n - v\| \geq 1 - \varepsilon/2$ . Since  $\varepsilon$  is arbitrary, we obtain  $N(X) = 1$ . ■

The normal structure coefficient is useful to study the stability of the normal structure under renorming. Recall that if  $X$  and  $Y$  are isomorphic Banach spaces, the Banach-Mazur distance between  $X$  and  $Y$  is defined by

$$d(X, Y) = \inf \{ \|T\| \|T^{-1}\| : T \in \text{Isom}(X, Y) \}.$$

Clearly,  $d(X, Y) = 1$  when  $X$  and  $Y$  are isometric spaces.

**THEOREM 3.** *Let  $X$  and  $Y$  be isomorphic Banach spaces. Then*

$$N(X) \leq d(X, Y)N(Y).$$

*Proof.* Let  $C$  be a bounded convex closed subset of  $Y$ . If  $U : Y \rightarrow X$  is an isomorphism we have

$$\begin{aligned} r(C) &\leq \|U^{-1}\| r(U(C)) \leq \|U^{-1}\| \text{diam}(U(C))/N(X) \\ &\leq \|U^{-1}\| \|U\| \text{diam}(C)/N(X). \end{aligned}$$

Thus  $r(C) \leq d(X, Y) \text{diam}(C)/N(X)$  and this inequality implies the result. ■

The first geometric property, related to normal structure, which will be considered is the uniform convexity. Let us recall that a Banach space is said to be strictly convex if the unit sphere does not contain a segment. Equivalently:

**DEFINITION 4.** We say that a Banach space  $X$  is strictly convex if for every vectors  $x$  and  $y$  in  $X$  which are not collinear, we have

$$\|x + y\| < \|x\| + \|y\|.$$

A stronger notion appears if we assume this property in a uniform sense, that is, roughly speaking, assuming that there is no segment with a predetermined length as close to the unit sphere as wanted.

DEFINITION 5. We say that a Banach space  $X$  is uniformly convex, if for every  $\varepsilon \in (0, 2]$  there exists  $\delta > 0$  such that for  $x, y \in X$  with

$$\left. \begin{array}{l} \|x - y\| \geq \varepsilon \\ x, y \in \overline{B}(0, 1) \end{array} \right\} \Rightarrow 1 - \left\| \frac{x + y}{2} \right\| > \delta.$$

EXAMPLE 1. Hilbert spaces are uniformly convex as a consequence of the parallelogram identity. Indeed, if  $x, y \in \overline{B}(0, 1)$  and  $\|x - y\| \geq \varepsilon$ , we have

$$\left\| \frac{x + y}{2} \right\| \leq \sqrt{1 - \left(\frac{\varepsilon}{2}\right)^2}.$$

Considering  $\delta = \sqrt{1 - (\varepsilon/2)^2}$  we obtain the uniform convexity of the space. To prove that  $\ell_p$  spaces are uniformly convex is more technical (see, for instance, [4]). On the other hand, recall that a Banach space  $X$  is finitely representable in another Banach space  $Y$  if for every finite dimensional subspace  $E$  of  $X$  and every  $\varepsilon > 0$  there exists a subspace  $F$  of  $Y$  such that  $d(E, F) < 1 + \varepsilon$ . It is not difficult to prove that  $L^p(\Omega)$  is finitely representable in  $\ell^p$ . Indeed, if  $E$  is an  $n$ -dimensional subspace of  $L^p(\Omega)$  and  $\{f_1, f_2, \dots, f_n\}$  is a normalized basis of  $E$ , with basic constant  $c$ , for every  $\varepsilon > 0$  we can find simple functions  $\{s_1, s_2, \dots, s_n\}$  such that  $\|f_k - s_k\| < \varepsilon/nc(2 + \varepsilon)$  for  $k = 1, \dots, n$ . If  $f = \sum_{k=1}^n a_k f_k$ , we define  $Tf = \sum_{k=1}^n a_k s_k$ . Then  $T$  is an isomorphism from  $E$  onto  $\text{span}\{s_1, \dots, s_n\}$  and  $\|T\| \|T^{-1}\| < 1 + \varepsilon$ . Since  $\text{span}\{s_1, \dots, s_n\}$  can be isometrically embedded in  $\ell^p$  (by discretization of the measure), there exists a subspace  $F$  of  $\ell^p$  such that  $d(E, F) < 1 + \varepsilon$ . Since the definition of the uniform convexity only depends on 2-dimensional subspaces and  $L_p([0, 1])$  contains isometrically to  $\ell_p$ , we can assure that  $L_p([0, 1])$  spaces are uniformly convex for the same choice of  $\delta$  as in  $\ell_p$ .

We will need a measure of the uniform convexity of the space:

DEFINITION 6. Let  $X$  be a Banach space. The modulus of convexity of  $X$ ,  $\delta_X(\varepsilon)$ , is defined by

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : x, y \in \overline{B}(0, 1), \|x - y\| \geq \varepsilon \right\}.$$

THEOREM 4. Let  $X$  be a Banach space with modulus of convexity  $\delta_X$ . Then  $N(X) \geq (1 - \delta_X(1))^{-1}$ .

*Proof.* Let  $A$  be a closed convex bounded subset of  $X$  which is not a singleton and choose  $\varepsilon > 0$ . Denote  $d = \text{diam}(A)$  and  $r = r(A)$ . Choose  $x$  and  $y$  in  $A$  such that  $\|x - y\| \geq d - \varepsilon$ . Write  $w = (x + y)/2$ , and choose  $z$  in  $A$  such that  $\|z - w\| \geq r - \varepsilon$ . Since  $\|(z - x)/d\| \leq 1$ ,  $\|(z - y)/d\| \leq 1$  and  $\|(z - x)/d - (z - y)/d\| > (d - \varepsilon)/d$  from the definition of  $\delta_X$  we obtain

$$\|z - w\| \leq d \left( 1 - \delta_X \left( \frac{d - \varepsilon}{d} \right) \right).$$

Thus

$$r \leq \varepsilon + d \left( 1 - \delta_X \left( \frac{d - \varepsilon}{d} \right) \right)$$

which implies the result using the continuity of the norm. ■

*Remark 3.* Notice that uniform convexity implies normal structure, but this property is also shared by every space such that segments with length equal to 1 are separated from the unit sphere. On the other hand, it is well known that uniformly convex spaces are reflexive. From theorems 2 and 4 a stronger result is obtained: the condition  $\delta_X(1) > 0$  implies reflexivity.

Next we will review another coefficient which implies weak normal structure. We will use the notion of asymptotically equilateral sequence

**DEFINITION 7.** Let  $X$  be a metric space. A sequence  $\{x_n\}$  in  $X$  is said to be asymptotically equilateral if  $\lim_{n,m; n \neq m} d(x_n, x_m)$  exists, i.e., a number  $d$  exists such that for every  $\varepsilon > 0$  there is a nonnegative integer  $n_0$  such that  $d - \varepsilon < d(x_n, x_m) < d + \varepsilon$  if  $n, m > n_0$  and  $n \neq m$ .

To study the existence of asymptotically equilateral sequence we will use a simple version of Ramsey Lemma. Let us fix the notation: By  $\mathbb{N}$  we denote the set of nonnegative integers,  $[\mathbb{N}]$  the collection of its infinite subsets and for every set  $C$  in  $[\mathbb{N}]$ ,  $[C]^2$  will denote the set formed by all ordered pair formed with numbers in  $C$ .

**LEMMA.** (Ramsey Lemma) *Let  $f : [\mathbb{N}]^2 \rightarrow \{1, 2\}$  be a function. Then, there exists  $C \in [\mathbb{N}]$  such that the restriction of  $f$  to  $[C]^2$  is a constant.*

**THEOREM 5.** *Let  $\{x_n\}$  be a bounded sequence in a metric space. Then  $\{x_n\}$  contains an asymptotically equilateral subsequence.*



*Proof.* For every subsequence  $\{y_n\}$  of  $\{x_n\}$  we denote  $\phi(\{y_n\}) = \inf\{\varepsilon > 0 : \{y_n\} \text{ can be covered by finitely many sets with diameter } \leq \varepsilon\}$ . Claim. "There exists a subsequence  $\{y_n\}$  of  $\{x_n\}$  such that  $\phi(\{z_n\}) = \phi(\{y_n\})$  for every subsequence  $\{z_n\}$  of  $\{y_n\}$ ." To prove the claim, define by induction  $\{z_n^0\} = \{x_n\}$  and

$$\phi_{m+1} = \inf\{\phi(\{z_n\}) : \{z_n\} \text{ subsequence of } \{z_n^m\}\}.$$

Let  $\{z_n^{m+1}\}$  be a subsequence of  $\{z_n^m\}$  such that

$$\phi(\{z_n^{m+1}\}) < \phi_{m+1} + \frac{1}{m+1}.$$

Consider the diagonal subsequence  $\{z_n^n\}$ . We will show that this sequence satisfies the required condition. Since  $\{z_n^n\}$  is a subsequence of  $\{z_n^m\}$  for  $n > m$  we have  $\phi(\{z_n^n\}) \leq \phi(\{z_n^m\})$  for each  $m$ . Assume that  $\{z_n\}$  is a subsequence of  $\{z_n^n\}$ . Hence  $\{z_n\}$  is a subsequence of  $\{z_n^m\}$  for  $n > m$ . Thus

$$\phi(\{z_n\}) \leq \phi(\{z_n^m\}) < \phi_m + \frac{1}{m} \leq \phi(\{z_n\}) + \frac{1}{m}.$$

Since  $m$  is arbitrary we obtain

$$\phi(\{z_n\}) \leq \phi(\{z_n\}) \leq \phi(\{z_n^n\})$$

and the claim is proved. Choose now an arbitrary  $\varepsilon > 0$  and a subsequence  $\{y_n\}$  of  $\{x_n\}$  satisfying the property in the claim. Taking a subsequence (which "a fortiori" satisfies the same property) we can assume  $\phi(\{y_n\}) + \varepsilon \geq \|y_n - y_m\|$  for every  $n, m$ . Define the following function from  $[\mathbb{N}]^2$  into  $\{1, 2\}$ :  $f(n, m) = 1$  if  $\|y_n - y_m\| > \phi(\{y_n\}) - \varepsilon$  and  $f(n, m) = 2$  if  $\|y_n - y_m\| \leq \phi(\{y_n\}) - \varepsilon$ . By Ramsey's Lemma, there exists a subsequence  $\{z_n\}$  of  $\{y_n\}$  satisfying either  $\|z_n - z_m\| > \phi(\{y_n\}) - \varepsilon$  for every  $n, m; n \neq m$  or  $\|z_n - z_m\| \leq \phi(\{y_n\}) - \varepsilon$  for every  $n, m$ . Since the second possibility is a contradiction according to the property satisfied by  $\{y_n\}$ , we deduce that the first possibility always holds and we have

$$\phi(\{y_n\}) - \varepsilon \leq \|z_n - z_m\| < \phi(\{y_n\}) + \varepsilon$$

for every  $n, m; n \neq m$ . Choosing  $\varepsilon = 1/n$  we can conclude the proof by a diagonal argument. ■

To introduce a coefficient for weak normal structure we need some previous definitions.

DEFINITION 8. The asymptotic diameter, radius and center of a sequence  $\{x_n\}$  in a Banach space  $X$  are defined by:

$$\text{diam}_a(\{x_n\}) = \limsup_{k \rightarrow \infty} \{\|x_n - x_m\| : n, m \geq k\},$$

$$r_a(\{x_n\}, B) = \inf \{\limsup_{n \rightarrow \infty} \|x_n - y\| : y \in B\},$$

$$Z_a(\{x_n\}, B) = \{y \in B : \limsup_{n \rightarrow \infty} \|x_n - y\| = r_a(\{x_n\}, B)\},$$

where  $B$  is an arbitrary subset of  $X$ . Whenever  $B = \overline{\text{co}}(\{x_n\})$  we will write  $r_a(\{x_n\})$  and  $Z_a(\{x_n\})$  (resp.) for  $r_a(\{x_n\}, \text{co}(\{x_n\}))$  and  $Z_a(\{x_n\}, \text{co}(\{x_n\}))$ .

DEFINITION 9. Let  $X$  a Banach space without the Schur property, i.e., weakly convergent sequences exist which are not norm converging. The weakly convergent sequence coefficient of  $X$  is defined by

$$WCS(X) = \inf \left\{ \frac{\text{diam}_a(\{x_n\})}{r_a(\{x_n\})} : \{x_n\} \text{ is a weakly convergent sequence which is not norm converging} \right\}.$$

Since the maximum value for  $WCS(X)$  in Definition 9 is 2, we can say that  $WCS(X) = 2$  if  $X$  satisfies the Schur property.

THEOREM 6. *Let  $X$  be a Banach space with  $WCS(X) > 1$ . Then  $X$  has weak normal structure.*

*Proof.* Assume that  $X$  fails to have weak normal structure. Thus, it contains a weakly compact convex diametral set  $A$  with is not a singleton. Denote  $d = \text{diam}(A) > 0$  and let  $\varepsilon < d$  be an arbitrary positive number. Choose an arbitrary  $x_1$  in  $A$ . Recursively we can construct a sequence  $\{x_n\}$  such that

$$\|y_n - x_{n+1}\| > d - \frac{\varepsilon}{n^2}$$

where  $y_n = \sum_{i=1}^n x_i/n$ . Assume that  $x$  is an arbitrary point in the convex hull of  $\{x_1, \dots, x_n\}$ , i.e.,  $x = \sum_{j=1}^n \alpha_j x_j$  where  $\alpha_j \geq 0$  and  $\sum_{j=1}^n \alpha_j = 1$ . If  $\alpha = \max\{\alpha_1, \dots, \alpha_n\}$ , we have

$$y_n = \frac{x}{n\alpha} + \sum_{j=1}^n \left( \frac{1}{n} - \frac{\alpha_j}{n\alpha} \right) x_j.$$

Since

$$\frac{1}{n\alpha} + \sum_{j=1}^n \left( \frac{1}{n} - \frac{\alpha_j}{n\alpha} \right) = 1 \quad ; \quad \frac{1}{n} - \frac{\alpha_j}{n\alpha} \geq 0$$

we have

$$\begin{aligned} d - \frac{\varepsilon}{n^2} < \|y_n - x_{n+1}\| &\leq \frac{1}{n\alpha} \|x - x_{n+1}\| + \sum_{j=1}^n \left( \frac{1}{n} - \frac{\alpha_j}{n\alpha} \right) \|x_j - x_{n+1}\| \\ &\leq \frac{1}{n\alpha} \|x - x_{n+1}\| + \left( 1 - \frac{1}{n\alpha} \right) d. \end{aligned}$$

Thus

$$\|x - x_{n+1}\| \geq \left( \frac{d}{n\alpha} - \frac{\varepsilon}{n^2} \right) n\alpha = d - \frac{\varepsilon\alpha}{n} \geq d - \frac{\varepsilon}{n}.$$

Hence  $\lim_{n \rightarrow \infty} d(x_{n+1}, \text{co}(\{x_1, \dots, x_n\})) = d$ . Since  $A$  is a weakly compact set and every subsequence of  $\{x_n\}$  satisfies the above condition, we can assume that  $\{x_n\}$  is weakly convergent. In particular  $\text{diam}_a(\{x_n\}) \leq d$ . If  $X$  satisfies the Schur property,  $\{x_n\}$  is convergent and we obtain the contradiction  $d = 0$ . Otherwise, if  $y$  belongs to the convex hull of  $\{x_n\}$  we know that  $y$  belongs to  $\text{co}(\{x_1, \dots, x_k\})$  for some  $k$ . If  $n > k$  we have  $\|y - x_n\| \geq d - \varepsilon/n$ . Thus  $r_a(\{x_n\}) \geq d$ . Since  $\text{diam}_a(\{x_n\}) \leq d$  we obtain  $WCS(X) \leq 1$ . ■

We will show some equivalent useful forms for  $WCS(X)$ :

**THEOREM 7.** *Let  $X$  be a Banach space without the Schur property. Then*

(a)

$$WCS(X) = \inf \left\{ \frac{\text{diam}_a(\{x_n\})}{\limsup_{n \rightarrow \infty} \|x_n\|} : \{x_n\} \text{ converges weakly to zero} \right\}.$$

(b)

$$\begin{aligned} WCS(X) = \inf \left\{ \frac{\lim_{n,m;n \neq m} \|x_n - x_m\|}{\lim_{n \rightarrow \infty} \|x_n\|} : \{x_n\} \text{ converges weakly to zero} \right. \\ \left. \text{and } \lim_{n,m;n \neq m} \|x_n - x_m\| \text{ and } \lim_{n \rightarrow \infty} \|x_n\| \text{ exist} \right\}. \end{aligned}$$

(c)

$$\begin{aligned} WCS(X) = \inf \left\{ \lim_{n,m;n \neq m} \|x_n - x_m\| : \{x_n\} \text{ converges weakly to zero,} \right. \\ \left. \|x_n\| = 1 \text{ and } \lim_{n,m;n \neq m} \|x_n - x_m\| \text{ exists} \right\}. \end{aligned}$$

*Proof.* Let  $\{x_n\}$  be a weakly null sequence. For each  $k \geq 1$ ,  $A_k$  will denote the closed convex hull of  $\{x_n\}_{n \geq k}$ . It is easy to check that  $\bigcap_{k=1}^{\infty} A_k = \{0\}$ . Since the function  $\Phi(x) = \limsup_{n \rightarrow \infty} \|x_n - x\|$  is lower weakly semicontinuous and  $A_k$  is a weakly compact set, this function attains a minimum on  $A_k$ . Thus the asymptotic Chebyshev center  $Z(\{x_n\}, A_k)$  is nonempty. Choose  $z_k \in Z(\{x_n\}, A_k)$ . Since  $\{z_k\}$  lies in a weakly compact set and 0 is the unique cluster point of  $\{z_k\}$ , the sequence  $\{z_k\}$  converges weakly to 0. Furthermore  $\{\Phi(z_k)\}$  is a nondecreasing sequence bounded by  $\Phi(0)$ . Thus  $\lim_{k \rightarrow \infty} \Phi(z_k) \leq \Phi(0)$ . Moreover, the lower weak semicontinuity of  $\Phi$  implies  $\lim_{k \rightarrow \infty} \Phi(z_k) \geq \Phi(0)$  and so  $\lim_{k \rightarrow \infty} \Phi(z_k) = \Phi(0)$ . Since

$$\Phi(z_k) = \min_{z \in A_k} \limsup_{n \rightarrow \infty} \|x_n - z\| = r_a(\{x_n\}_{n \geq k})$$

we obtain

$$\Phi(z_k) WCS(X) \leq \text{diam}_a(\{x_n\}).$$

Taking limits as  $k$  goes to infinity we obtain

$$\limsup_{n \rightarrow \infty} \|x_n\| WCS(X) \leq \text{diam}_a(\{x_n\})$$

and (a) can be deduced from this inequality. Statements in (b) and (c) can be obtained from (a) because any bounded sequence has an asymptotically equilateral subsequence. ■

Next, we will consider another geometric coefficient related to normal structure.

**DEFINITION 10.** A Banach space  $X$  is said to be smooth if any point  $x$  in the unit sphere supports a unique functional tangent, i.e., there exists a unique  $f \in X^*$  such that  $\|f\| = 1$  and  $f(x) = 1$ .

It is known that  $X$  is smooth if and only if the norm is Gateaux-differentiable, i.e., for every  $x \in X$  and  $h \neq 0$  in  $X$  there exists

$$\lim_{t \rightarrow 0} \frac{\|x + th\| - \|x\|}{t}.$$

*Remark 4.* Smoothness is in some sense a dual notion of strict convexity. In fact, it can be proved the following: (a) If  $X^*$  is smooth, then  $X$  is strictly convex. (b) If  $X^*$  is strictly convex, then  $X$  is smooth. In general, inverse assertions of (a) and (b) do not hold (see [2, Example IV.3]).

We will consider a stronger notion: uniform smoothness.

DEFINITION 11. A Banach space  $X$  is said to be uniformly smooth if

$$\lim_{t \rightarrow 0^+} \rho_X(t)/t = 0$$

where  $\rho_X(t)$  is the smoothness modulus defined by,

$$\rho_X(t) = \sup \left\{ \frac{1}{2}(\|x + ty\| + \|x - ty\|) - 1 : \|x\| \leq 1, \|y\| \leq t \right\}.$$

It is easy to check that every uniformly smooth space is smooth and to find examples of smooth spaces which are not uniformly smooth. Furthermore, uniform smoothness is equivalent to a stronger differentiability condition:  $X$  is uniformly smooth if the norm is uniformly Fréchet differentiable (see [4] for details). On the other hand some relationships can be stated between the modulus of convexity of  $X$  and the smoothness modulus of  $X^*$  and vice-versa. The proof of the following result, implying, in particular, that uniform convexity and uniform smoothness are dual notions can be found in [4, page 208].

THEOREM 8. (Lindenstrauss Duality Formula) *For every Banach space  $X$  and every  $t \geq 0$  one has*

(a)

$$\rho_{X^*}(t) = \sup_{0 \leq \varepsilon \leq 2} \left\{ \frac{t\varepsilon}{2} - \delta_X(\varepsilon) \right\}.$$

(b)

$$\rho_X(t) = \sup_{0 \leq \varepsilon \leq 2} \left\{ \frac{t\varepsilon}{2} - \delta_{X^*}(\varepsilon) \right\}.$$

Denote  $\rho_0(X)$  the characteristic of smoothness of  $X$ , defined by

$$\rho_0(X) = \lim_{t \rightarrow 0^+} \frac{\rho_X(t)}{t},$$

and  $\varepsilon_0(X)$  the characteristic of convexity

$$\varepsilon_0(X) = \sup\{\varepsilon \geq 0 : \delta_X(\varepsilon) = 0\}.$$

From the Lindenstrauss formula we can deduce:

THEOREM 9. For every Banach space  $X$  one has

(a)  $\rho_0(X^*) = \varepsilon_0(X)/2$ .

(b)  $\rho_0(X) = \varepsilon_0(X^*)/2$ .

*Proof.* We will prove (a); the same argument, using the dual formula, proves (b). First, we must note that

$$\begin{aligned} \rho_0(X) &= \lim_{t \rightarrow 0^+} \frac{\rho_X(t)}{t} = \lim_{t \rightarrow 0^+} \frac{1}{t} \sup_{0 \leq \varepsilon \leq 2} \left\{ \frac{t\varepsilon}{2} - \delta_X(\varepsilon) \right\} \\ &= \lim_{t \rightarrow 0^+} \sup_{0 \leq \varepsilon \leq 2} \left\{ \frac{\varepsilon}{2} - \frac{\delta_X(\varepsilon)}{t} \right\} \geq \varepsilon_0(X)/2. \end{aligned}$$

On the other hand, denote

$$a = \lim_{t \rightarrow 0^+} \frac{\rho_X(t)}{t}.$$

For any  $\eta > 0$  there exists  $t_0 > 0$  such that for every  $t$ ,  $0 < t < t_0$ , there exists  $\varepsilon(t)$  satisfying

$$a - \eta < \frac{\varepsilon(t)}{2} - \frac{\delta_X(\varepsilon(t))}{t}.$$

From this inequality we can deduce  $\varepsilon(t) > 2(a - \eta)$ . If we assume  $\delta_X(2(a - \eta)) > 0$  we obtain  $\delta_X(\varepsilon(t)) \geq \delta_X(2(a - \eta))$  and taking limit as  $t \rightarrow 0^+$ , we obtain the contradiction  $a = -\infty$ . Hence  $\delta_X(2(a - \eta)) = 0$  and since  $\eta$  is arbitrary we deduce  $\varepsilon_0(X)/2 \geq a$ . ■

As a consequence of Theorem 9 and Remark 3 we obtain a sufficient condition for reflexivity.

THEOREM 10. Let  $X$  be a Banach space with  $\rho_0(X) < 1/2$ . Then  $X$  is reflexive.

THEOREM 11. Let  $X$  be a Banach space with  $\rho_0(X) < 1/2$ . Then  $X$  has normal structure.

*Proof.* We have shown that the condition  $\rho_0(X) < 1/2$  implies that  $X$  is reflexive. We will prove that  $WCS(X) > 1$ . Let  $\tau$  be a number in  $(0, 1/2]$  and  $\{x_n\}$  a normalized weakly null sequence in  $X$ . We assume that  $d = \lim_{n,m;n \neq m} \|x_n - x_m\|$  exists and consider a normalized functional sequence  $\{x_n^*\}$  such that  $x_n^*(x_n) = 1$ . Since  $X^*$  is reflexive we can assume

that  $\{x_n^*\}$  converges weakly to a vector, say  $x^* \in X^*$ . Let  $\eta > 0$  be an arbitrary number and choose  $n$  large enough so that  $|x^*(x_n)| < \eta/2$  and  $d - \eta < \|x_n - x_m\| < d + \eta$  for  $m > n$ . For  $m > n$  large enough we have

$$|(x_m^* - x^*)(x_n)| < \eta/2 \quad \text{and} \quad |x_n^*(x_m)| < \eta.$$

Therefore  $|x_m^*(x_n)| < \eta$  and if  $l = \|x_n - x_m\| \leq 2$  we have

$$\begin{aligned} \rho_X(\tau) &\geq \frac{1}{2} \left( \left\| \frac{x_n - x_m}{l} + \tau x_n \right\| + \left\| \frac{x_n - x_m}{l} - \tau x_n \right\| \right) - 1 \\ &\geq \frac{1}{2} \left( x_n^* \left( \left( \frac{1}{l} + \tau \right) x_n - \frac{x_m}{l} \right) + x_m^* \left( \frac{x_m}{l} - \left( \frac{1}{l} - \tau \right) x_n \right) \right) - 1 \\ &\geq \frac{1}{2} \left( \frac{1}{l} + \tau - \frac{\eta}{l} + \frac{1}{l} - \left( \frac{1}{l} - \tau \right) \eta \right) - 1 \geq \frac{1}{d + \eta} + \frac{\tau}{2} - \frac{\eta}{d - \eta} - 1. \end{aligned}$$

Since  $\eta$  is arbitrary, we obtain

$$\rho_X(\tau) \geq \frac{1}{d} + \frac{\tau}{2} - 1.$$

Hence  $1/d \leq \rho_X(\tau) - \tau/2 + 1$  for every  $\tau \in (0, 1/2]$  and

$$WCS(X) \geq \frac{1}{\rho_X(\tau) - \tau/2 + 1}.$$

Since  $\rho_0(X) < 1/2$  there exists  $\tau_0 \in (0, 1/2]$  such that  $\rho_X(\tau_0)/\tau_0 < 1/2$ . For this value of  $\tau_0$  we have

$$WCS(X) \geq \frac{1}{\rho_X(\tau_0) - \tau_0/2 + 1} > 1$$

and  $X$  has normal structure. ■

*Remark 5.* From Theorem 11 and some duality results in this section we could guess that normal structure is preserved under pass to the dual space. The following example shows that this conjecture is not true.

EXAMPLE 2. Consider the sequence space  $\ell_2$  with the following norms

$$\begin{aligned} \|x\|_{2,1} &= (\|x^+\|_2 + \|x^-\|_2) \\ \|x\|_{2,\infty} &= \max\{\|x^+\|_2, \|x^-\|_2\} \end{aligned}$$

where  $x^+$  and  $x^-$  are vectors whose coordinates are

$$(x^+)^i = \max\{x^i, 0\} = \frac{|x^i| + x^i}{2}$$

$$(x^-)^i = \max\{-x^i, 0\} = \frac{|x^i| - x^i}{2}.$$

It is not difficult to prove that  $WCS(\ell^2, \|\cdot\|_{2,1}) = \sqrt{2}$  (see [2, Theorem VI.3.11]) and so this space has normal structure. Furthermore, its conjugate space is  $(\ell_2, \|\cdot\|_{2,\infty})$ . We will show that this space fails to have normal structure. Indeed, the standard basic sequence is a diametral sequence because  $\|e_n - e_m\|_{p,\infty} = 1$  if  $n \neq m$  and for every point  $u = \sum_{i=1}^n \alpha_i e_i$ ,  $\alpha_i \geq 0$ ,  $\sum_{i=1}^n \alpha_i = 1$  we have  $\|e_{n+1} - u\|_{p,\infty}^p = \sup\{1, \sum \alpha_i^p\} = 1$ .

*Remark 6.* Many other geometric properties of Banach spaces implying normal structure can be found in the books [8] and [2, Chapter VI].

#### 4. EXISTENCE OF FIXED POINT IN ABSENCE OF NORMAL STRUCTURE

Until now, we have studied the existence of fixed point for nonexpansive mappings as a consequence of the normal structure. In this section we will show some geometric properties which imply the existence of fixed points in absence of normal structure. The most important case corresponds to the space  $c_0$  which fails to have weak normal structure. However, we will see that Theorem 1 still holds for this space. We first recall a “classical” result in Metric Fixed Point Theory.

LEMMA. (Goebel-Karlovitc Lemma) *Let  $K$  be a convex weakly compact subset of a Banach space  $X$ , and  $T : K \rightarrow K$  a nonexpansive mapping. Assume that  $K$  is minimal with these properties and let  $\{x_n\}$  be an approximated fixed point sequence for  $T$  in  $K$ , i.e.  $\lim_n \|x_n - Tx_n\| = 0$ . Then*

$$\lim_{n \rightarrow \infty} \|y - x_n\| = \text{diam}(K)$$

for every  $y \in K$ .

*Proof.* From the proof of Theorem 1 we know that  $K = Z(K)$  which implies that  $K$  is a diametral set. We will prove that  $Z_a(\{x_n\}, K) = K$ . Let

$$Z_{a,\varepsilon}(\{x_n\}, K) = \{y \in K : \limsup_{n \rightarrow \infty} \|x_n - y\| \leq r_a(\{x_n\}, K) + \varepsilon\}.$$



It is easy to check that  $Z_{a,\varepsilon}(\{x_n\}, K)$  is a nonempty closed convex and  $T$ -invariant set. Thus  $Z_{a,\varepsilon}(\{x_n\}, K) = K$  and  $Z_a(\{x_n\}, K) = \bigcap_{\varepsilon>0} Z_{a,\varepsilon}(\{x_n\}, K) = K$ . We claim that  $\limsup_{n\rightarrow\infty} \|y - x_n\| = \text{diam}(K)$  for every  $y \in K$ . Indeed, assume that there exists  $y \in K$  such that  $\limsup_{n\rightarrow\infty} \|y - x_n\| < \text{diam}(K)$ . Denote  $r = \limsup_{n\rightarrow\infty} \|y - x_n\|$ ,  $d = \text{diam}(K)$  and consider the collection  $\{\overline{B}(z, (r+d)/2) \cap K : z \in K\}$ . Choose an arbitrary positive number  $\varepsilon$  such that  $\varepsilon < (d-r)/2$ . From the first part in the proof we know that  $\limsup_{n\rightarrow\infty} \|x_n - z\| = r$  for every  $z \in K$ . Thus, for every finite subset  $\{z_1, \dots, z_k\}$  of  $K$  there exists a nonnegative integer  $N$  such that  $\|x_N - z_i\| \leq r + \varepsilon = (r+d)/2$  for  $i = 1, \dots, k$ . Hence  $x_N$  belongs to  $\bigcap_{i=1}^k \overline{B}(z_i, (r+d)/2)$ . The weak compactness of  $K$  implies the existence of  $x_0 \in \bigcap_{z \in K} \overline{B}(z, (r+d)/2) \cap K$  and this point is not diametral because

$$\sup_{z \in K} \|z - x_0\| < \frac{r+d}{2} < d = \text{diam}(K).$$

This contradiction proves the claim. If  $\liminf_{n\rightarrow\infty} \|y - x_n\| < \text{diam}(K)$  for some  $y \in K$  there exists a subsequence  $\{y_n\}$  of  $\{x_n\}$  such that  $\limsup_{n\rightarrow\infty} \|y_n - y\| = \liminf_{n\rightarrow\infty} \|x_n - y\| < \text{diam}(K)$ , which is a contradiction according to the claim applied to the sequence  $\{y_n\}$  which is again an approximated fixed point sequence. ■

**THEOREM 12.** *Let  $K$  be a convex weakly compact subset of a Banach space  $X$ , and  $T : K \rightarrow K$  a nonexpansive mapping. Assume that  $K$  is minimal for these conditions,  $\text{diam}(K) = 1$  and  $\{x_n\}$  is an approximated fixed point sequence which is weakly null. Then, for every  $\varepsilon > 0$  and  $t \in [0, 1]$ , there exists a sequence  $\{z_n\}$  in  $K$  such that: (i)  $\{z_n\}$  is weakly convergent to a point  $z \in K$ . (ii)  $\|z_n\| > 1 - \varepsilon$  for every  $n \in \mathbb{N}$ . (iii)  $\|z_n - z_m\| \leq t$  for every  $n, m \in \mathbb{N}$ . (iv)  $\limsup_n \|z_n - x_n\| \leq 1 - t$ .*

*Proof.* Since  $\{w_n\}$  is an approximated fixed point sequence in  $K$ ,  $\text{diam}(K) = 1$  and  $0$  lies in  $K$ , from Goebel-Karlovitz Lemma we deduce  $\lim_n \|w_n\| = 1$ . Hence, for every  $\varepsilon > 0$  there exists  $\delta(\varepsilon) > 0$  such that  $\|x\| > 1 - \varepsilon$  if  $x \in K$  and  $\|Tx - x\| < \delta(\varepsilon)$ . Indeed, otherwise, there exists  $\varepsilon_0 > 0$  such that we can find  $w_n \in K$  satisfying  $\|Tw_n - w_n\| < 1/n$  and  $\|w_n\| \leq 1 - \varepsilon_0$  for every  $n \in \mathbb{N}$ . Therefore, the sequence  $\{w_n\}$  is an approximated fixed point sequence in  $K$  satisfying  $\limsup_n \|w_n\| \leq 1 - \varepsilon_0$ . Let  $\varepsilon > 0$  and  $t \in [0, 1]$ . Choose  $\gamma < \min\{1, \delta(\varepsilon)\}$  and for any  $n \in \mathbb{N}$  define the contraction  $S_n : K \rightarrow K$  by

$$S_n(x) = (1 - \gamma)T(x) + \gamma tx_n.$$

The Contractive Mapping Principle assures that there exists a (unique) fixed point  $z_n$  of  $S_n$ . Since  $K$  is a weakly compact set, we can assume, taking a subsequence if necessary, that  $\{z_n\}$  satisfies (i). Since  $\|z_n - Tz_n\| < \gamma$  we know that  $\{z_n\}$  satisfies (ii). Condition (iii) is easily obtained and (iv) is a consequence of the inequalities

$$\begin{aligned} \|z_n - x_n\| &\leq \|(1 - \gamma)Tz_n + \gamma tx_n - x_n\| \\ &\leq (1 - \gamma)\|Tz_n - Tx_n\| + (1 - \gamma)\|Tx_n - x_n\| + \gamma(1 - t)\|x_n\|. \end{aligned}$$

Thus,

$$\|z_n - x_n\| \leq 1 - t + \frac{1 - \gamma}{\gamma}\|Tx_n - x_n\|.$$

Taking limsup as  $n$  tends to infinity, we obtain (iv). ■

DEFINITION 12. Let  $X$  be a Banach space. We define the coefficient

$$R(X) = \sup\{\liminf_{n \rightarrow \infty} \|x_n + x\|\}$$

where the supremum is taken over all  $x \in X$  with  $\|x\| \leq 1$  and over all weakly null sequences in  $\overline{B}(0, 1)$ .

THEOREM 13. Let  $X$  be a Banach space with  $R(X) < 2$ . If  $C$  is a convex weakly compact subset of  $X$  and  $T : C \rightarrow C$  is a nonexpansive mapping, then  $T$  has a fixed point.

*Proof.* Otherwise, we can find a convex weakly compact  $T$ -invariant subset of  $X$  which is not a singleton and which is minimal for these conditions. By multiplication we assume that its diameter is 1. Furthermore, from the Contractive Mapping Principle it is easy to construct an approximated fixed point sequence  $\{x_n\}$  in  $K$ . We can assume that  $\{x_n\}$  is weakly convergent and, by translation, that  $\{x_n\}$  is weakly null. Consider a sequence  $\{z_n\}$  satisfying (i)-(iv) in Theorem 12 for  $t = 1/2$ . Taking again a subsequence, if necessary, we can assume that  $\lim_n \|z_n - z\|$  exists. Furthermore,  $\lim_n \|z_n - z\| \leq \lim_n \lim_m \|z_n - z_m\| \leq 1/2$ . We can choose  $\eta > 0$  such that  $\eta R(X) < 1 - R(X)/2$ . For  $n$  large enough, we have  $\|z_n - z\| \leq 1/2 + \eta$ . Furthermore  $\|z\| \leq \liminf_{n \rightarrow \infty} \|z_n - x_n\| \leq 1/2$ . Thus

$$\left\| \frac{z_n}{1/2 + \eta} \right\| = \left\| \frac{z_n - z}{1/2 + \eta} + \frac{z}{1/2 + \eta} \right\| \leq R(X).$$

Therefore  $\limsup_{n \rightarrow \infty} \|z_n\| \leq R(X)(1/2 + \eta) < 1$  which is a contradiction because  $0 \in K$ . ■

*Remark 7.* For  $X = c_0$  it is easy to check that  $R(c_0) = 1$ . Thus Theorem 1 holds for  $c_0$  even though this space fails to have weak normal structure.

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