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COMPOSITIONAL HYPERCYCLICITY EQUALS SUPERCYCLICITY

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Abstract. In this note it is proved that the sequence of composition operators generated by automorphisms of a simply connected domain strictly contained in the complex plane is hypercyclic –that is, possesses some dense orbit– if and only if it is supercyclic –i.e., possesses some dense projective orbit–. When the domain is the full complex plane, a result in this direction is also obtained. In addition, a number of statements about the corresponding cyclicity properties of single composition operators are either proved directly or extracted as a consequence.

1. Introduction

The main aim of this paper is to show that a sequence of composition operators (C_{φ_n}) on $H(G)$ is hypercyclic if and only if it is supercyclic, at least when G is simply connected. Here G is a domain of the complex plane $\mathbb{C}$, that is, G is a nonempty connected open subset of $\mathbb{C}$; the class $H(G)$ is the Fréchet space of all holomorphic functions on G, endowed with the compact-open topology; the functions φ_n ($\mathbb{N} := \{1, 2, ...\}$) are members of $\mathrm{Aut}(G) = \{$automorphisms of $G\} = \{\varphi \in H(G) : \varphi$ is one-to-one, $\varphi(G) = G\}$, and $C_\varphi f := f \circ \varphi$ for $f \in H(G)$. Recall that G is simply connected whenever its complement in the extended complex plane $\mathbb{C}_\infty$ is connected.

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Let us also recall some terminology coming from Linear Dynamics of operators, see [11] for concepts, results and history. Assume that X is a topological vector space over the scalar field $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ (:= the real line) and that $T_n : X \to X$ ($n \in \mathbb{N}$) is a sequence in $L(X) = \{\text{operators on } X\} := \{\text{continuous linear selfmappings on } X\}$. Then the sequence (T_n) is said to be *hypercyclic* (*supercyclic*, resp.) provided that there exists some vector $x \in X$ –called *hypercyclic* (*supercyclic*, resp.) for (T_n)– whose orbit $\{T_n x : n \in \mathbb{N}\}$ (projective orbit $\{\lambda T_n x : n \in \mathbb{N}, \lambda \in \mathbb{K}\}$, resp.) under (T_n) is dense in X. If $T \in L(X)$ then T is called *hypercyclic* (*supercyclic*, resp.) whenever the sequence (T^n) of its iterates is hypercyclic (supercyclic, resp.); in this case every vector $x \in X$ whose orbit $\{T^n x : n \in \mathbb{N}\}$ (projective orbit $\{\lambda T^n x : n \in \mathbb{N}, \lambda \in \mathbb{K}\}$, resp.) under T is dense in X will be called hypercyclic (supercyclic, resp.) for T. The sets of hypercyclic vectors for (T_n) or T and of supercyclic vectors for (T_n) or T will be respectively denoted by $HC((T_n))$, $HC(T)$, $SC((T_n))$, $SC(T)$. In order to clarify the difference between supercyclicity and supercyclicity, we point out that, if X is a Fréchet space, then a sequence $(T_n) \subset L(X)$ is supercyclic if and only if $(\alpha_n T_n)$ is hypercyclic for some sequence (α_n) of scalars, see [3, p. 50].

Apparently, the notions of hypercyclicity and supercyclicity were respectively introduced by Beauzamy [2] and Hilden and Wallen [13]. It is evident that the first notion is stronger than the second one. The backward shift $B(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$ on the space l_1 of absolutely summable sequences is an example of a supercyclic operator which is not hypercyclic.

The organization of this paper is as follows. It will be proved in the Section 3 that, if $G \neq \mathbb{C}$ is simply connected, then both concepts are equivalent for the aforementioned sequences of composition operators (Theorem 3.3). In the case $G = \mathbb{C}$, we will demonstrate a corresponding statement for a large class of automorphisms (Theorem 3.7). Either as a consequence of the results for sequences or directly, it will also be shown that, for any simply connected domain, the hypercyclity of a single composition operator C_φ is equivalent to its supercyclicity (Corollary 3.4 and Theorem 3.6). Before all of this, a rather general necessary condition for supercyclicity is provided (Theorem 3.1). Section 2 is devoted to give some additional terminology together with a number of preparatory results.

2. Preliminary results

In our terminology, Birkhoff [5] constructed in 1929 a hypercyclic entire function with respect to the translation operator C_φ generated by $\varphi(z) := z + b$

($b \in \mathbb{C} \setminus \{0\}$). In 1941 Seidel and Walsh [14] were able to construct a (C_{φ_n})-hypercyclic function $f \in H(\mathbb{D})$, where $\mathbb{D}$ is the open unit disc, φ_n is the non-euclidean translation $z \mapsto \frac{z+a_n}{1+\overline{a}_n z}$ ($n \in \mathbb{N}$) and $|a_n| \to 1$ ($n \to \infty$). Much progress has been done since then, including other domains G, outstanding subspaces of $H(G)$ (mainly if $G := \mathbb{D}$ or $\mathbb{C}$) and selfmappings φ which may not be automorphisms of G, see [11].

Concerning the full space $H(G)$, in 1995 Montes and the first author [4] gave a complete characterization of the hypercyclicity of a sequence (C_{φ_n}), where $(\varphi_n) \subset \mathrm{Aut}(G)$. The following assertion collects some of their findings. Recall that $\mathrm{Aut}(\mathbb{C}) = \{$the similarities $z \mapsto az + b : \ a, b \in \mathbb{C}, \ a \neq 0\}$ and $\mathrm{Aut}(\mathbb{D}) = \{$the Möbius transformations $z \mapsto k \, \frac{z-a}{1-\overline{a}z} : \ |a| < 1 = |k|\}$.

Theorem 2.1. *Let $G \subset \mathbb{C}$ be a domain, which is not isomorphic to $\mathbb{C} \setminus \{0\}$. Consider the sequence $(C_{\varphi_n}) \subset L(H(G))$ generated by a sequence $(\varphi_n) \subset \mathrm{Aut}(G)$. Consider also respective automorphisms $\varphi \in \mathrm{Aut}(\mathbb{C})$, $\phi \in \mathrm{Aut}(\mathbb{D})$. Then we have:*

- (a) *The following properties are equivalent:*
  - (i) *(C_{φ_n}) is hypercyclic.*
  - (ii) *(φ_n) acts properly discontinuously on G, that is, given a compact set $K \subset G$, there is $n \in \mathbb{N}$ such that $K \cap \varphi_n(K) = \emptyset$.*
- (b) *If $G = \mathbb{C}$ and $\varphi_n(z) \equiv a_n z + b_n$ (a_n, $b_n \in \mathbb{C}$; $a_n \neq 0$; $n \in \mathbb{N}$), then (C_{φ_n}) is hypercyclic if and only if the sequence $\{\min\{|b_n|, |b_n/a_n|\} : \ n \in \mathbb{N}\}$ is unbounded.*
- (c) *In particular, C_φ is hypercyclic on $H(\mathbb{C})$ if and only if φ is a translation, that is, $\varphi(z) := z + b$ for some $b \in \mathbb{C} \setminus \{0\}$.*
- (d) *If $G = \mathbb{D}$ and $\varphi_n(z) := k_n \, \frac{z-a_n}{1-\overline{a}_n z}$ ($|k_n| = 1 > |a_n|$, $n \in \mathbb{N}$), then (C_{φ_n}) is hypercyclic if and only if $\sup_{n\in\mathbb{N}} |\varphi_n(0)| = \sup_{n\in\mathbb{N}} |a_n| = 1$.*
- (e) *In particular, C_ϕ is hypercyclic on $H(\mathbb{D})$ if and only if ϕ is not elliptic, that is, it has no fixed point in $\mathbb{D}$.*

We point out that if $\phi(z) := e^{i\theta}(z-a)/(1-\overline{a}z) \in \mathrm{Aut}(\mathbb{D})$ then ϕ is elliptic if and only if $|a| < |\sin(\theta/2)|$, see [15, page 7].

As for the supercyclicity of composition operators, several important advances have been made, mainly on Hardy–type or Dirichlet–type spaces on $\mathbb{D}$ (see for instance [1], [7, Chapter 5], [8]) but, as far as we know, a complete characterization of compositional supercyclicity on the full space $H(G)$ has not been performed. In order to motivate the problem, let us say that by using that no isometry on a Banach space can be supercyclic, Ansari and Bourdon [1] showed that if φ is elliptic then C_φ is not supercyclic on the Hardy space H^2. But this does *not* imply the non-supercyclicity of C_φ on $H(\mathbb{D})$, because the norm-topology of H^2 is stronger than the one inherited from $H(\mathbb{D})$. In fact, the nonsupercyclicity of C_φ on a wide class of spaces of analytic functions is true, and this will be proved in Section 3, but through a very different approach.

Finally, the following auxiliary assertion, which has a purely topological nature, will reveal useful in the next section. Recall that a mapping $F : X \to Y$ between two topological spaces X, Y is open whenever the image $F(A)$ of every open subset A of X is open in Y.

Lemma 2.2. *Let X be a topological space and $F : X \to \mathbb{K}$ be a continuous open function. Assume that $\Phi_k : X \to X$ $(k \in \mathbb{N})$ is a sequence of mappings converging pointwise to a function $\Phi : X \to X$, such that Φ is continuous and open. Then there is no sequence of scalars $(c_k) \subset \mathbb{K}$ for which*

$$\lim_{k\to\infty} c_k F(\Phi_k(z)) = 1 \qquad \text{for all } z \in X.$$

Proof. By way of contradiction, let us start with the assumption that there exists a sequence $(c_k) \subset \mathbb{K}$ satisfying $c_k F(\Phi_k(z)) \to 1$ as $k \to \infty$ for every $z \in X$. We have that $\Phi_k \to \Phi$ pointwise in X. Let $A := (F \circ \Phi)^{-1}(\{0\})$. Since Φ and F are open, the composite mapping $F \circ \Phi$ also is, so the set A has empty interior. Hence $X \setminus A$ is dense in X (in particular, it is nonempty). Fix $b \in X \setminus A$. We have that $F(\Phi(b)) \neq 0$. Therefore, by the continuity of F and by the fact that $\Phi_k(b) \to \Phi(b)$, there must exist $k_0 \in \mathbb{N}$ such that $F(\Phi_k(b)) \neq 0$ for all $k \geq k_0$. Hence

$$c_k = \frac{c_k F(\Phi_k(b))}{F(\Phi_k(b))} \to \frac{1}{F(\Phi(b))} \qquad (k \to \infty).$$

Since the last display holds for each $b \in X \setminus A$, the uniqueness of the limit forces $F \circ \Phi$ to be constant in $X \setminus A$. But this set is dense in X and $F \circ \Phi$ is continuous (because F and Φ are). Consequently, $F \circ \Phi$ is constant on X, which is a contradiction because $F \circ \Phi$ is open. □

3. Hypercyclicity versus Supercyclicity

Our first statement extends widely Theorem 5.2 of [7] (take $G = \mathbb{D}$, $E = H^2(\beta)$ = a generalized Hardy space in part (b) of Theorem 3.1 below), which in turn improves Proposition 2.3 of [1]. We denote $\varphi^n = \varphi \circ \cdots \circ \varphi$ (n times, $n \in \mathbb{N}$) for every holomorphic selfmapping $\varphi : G \to G$. Observe that $(C_\varphi)^n = C_{\varphi^n}$ and, if $\varphi \in \mathrm{Aut}(G)$, $(C_\varphi)^{-1} = C_{\varphi^{-1}}$.

Theorem 3.1. *Let G be a domain in $\mathbb{C}$, $\varphi : G \to G$ be a holomorphic selfmapping and E be a metrizable topological vector space over $\mathbb{C}$ with $E \subset H(G)$ such that* *each evaluation functional* $f \in E \mapsto f(a) \in \mathbb{C}$ $(a \in G)$ *is continuous and* C_φ *acts continuously on* E*. Assume that at least one of the following conditions holds:*

(a) *The map* φ *is not one-to-one and* E *separates points in* G*, that is, given* $a, b \in G$ *with* $a \neq b$*, there is* $f \in E$ *such that* $f(a) \neq f(b)$*.*

(b) *The domain* G *is simply connected,* $G \neq \mathbb{C}$*,* φ *fixes some point in* G *and* E *does not collapse at any point of* G*, that is, given* $a \in G$*, there exist functions* $g, h \in E$ *such that* $g(a) = 0 \neq h(a)$ *and* $g \not\equiv 0$*.*

Then the composition operator C_φ *is not supercyclic on* E*.*

Proof. The hypothesis on the evaluation functionals means that convergence of a sequence of E implies pointwise convergence in G.

Under the condition (a), there exist a, $b \in G$ with $a \neq b$ and $\varphi(a) = \varphi(b)$. Since E separates points we can find $g \in E$ such that $g(a) \neq g(b)$. Assume, by way of contradiction, that $f \in SC(C_\varphi)$. Then, since E is metrizable, there are sequences $(c_k) \subset \mathbb{C}$ and $(n_k) \subset \mathbb{N}$ satisfying

$$c_k(f \circ \varphi^{n_k}) \to g \qquad (k \to \infty) \text{ in } E.$$

Then $\lim_{k\to\infty} c_k f(\varphi^{n_k}(z)) = g(z)$ for all $z \in G$. In particular,

$$\lim_{k\to\infty} c_k f(\varphi^{n_k}(a)) = g(a) \neq g(b) = \lim_{k\to\infty} c_k f(\varphi^{n_k}(b)),$$

which is a contradiction, because $\varphi^{n_k}(a) = \varphi^{n_k}(b)$ for all $k \in \mathbb{N}$.

As for (b), let us observe firstly that we can suppose without loss of generality that $G = \mathbb{D}$ and that the origin is a fixed point for φ. Indeed, if this were not the case, we could apply Riemann's isomorphism theorem to get an isomorphism (= bijective, holomorphic function) $\Phi : G \to \mathbb{D}$ with $\Phi(z_0) = 0$, where $z_0 \in G$ is such that $\varphi(z_0) = z_0$. Consider the associated space E_1 of E over $\mathbb{D}$, $E_1 := \{f \circ \Phi^{-1} : \ f \in E\}$, endowed with the distance $d_1(f_1, f_2) := d(f_1 \circ \Phi, f_2 \circ \Phi)$, where d is a translation-invariant distance on E compatible with its topology. Then $C_\Phi : f_1 \in E_1 \mapsto f_1 \circ \Phi \in E$ is, trivially, a linear isometric isomorphism. In addition, $\widetilde{\varphi} := \Phi \circ \varphi \circ \Phi^{-1}$ would be a selfmapping of $\mathbb{D}$ fixing the origin and $C_{\widetilde{\varphi}}$ $(= (C_\Phi)^{-1} \circ C_\varphi \circ C_\Phi)$ would be an operator on E_1. Trivially, $\dim(E_1) \geq 2$ and E_1 collapses at no point of $\mathbb{D}$. Finally, using $C_{\widetilde{\varphi}}^n = (C_\Phi)^{-1} \circ C_\varphi^n \circ C_\Phi$ $(n \in \mathbb{N})$ we derive that C_φ is supercyclic if and only if $C_{\widetilde{\varphi}}$ is. The details are left to the reader.

Thus we are already assuming $G = \mathbb{D}$ and $\varphi(0) = 0$. Let us suppose, by way of contradiction, that there is $f \in SC(C_\varphi)$. Necessarily, $f(0) \neq 0$, because $\{\lambda f(\varphi^n(0)) : n \in \mathbb{N}, \lambda \in \mathbb{C}\} = \{\lambda f(0) : \lambda \in \mathbb{C}\}$ should be dense in $\{h(0) : h \in E\} = \mathbb{C}$; note that the last set equals $\mathbb{C}$ since, by hypothesis, there is $h_0 \in E$ with

$h_0(0) \neq 0$. Again from the hypothesis there is $g \in E \setminus \{0\}$ such that $g(0) = 0$. Let $a \in G$ with $g(a) \neq 0$. Since f is supercyclic, one can find sequences $(c_k) \subset \mathbb{C}$ and $\{n_1 < n_2 < \cdots\} \subset \mathbb{N}$ for which $c_k(f \circ \varphi^{n_k})$ tends in E (so pointwise on G) to g. Then $c_k f(\varphi^{n_k}(0)) = c_k f(0) \to g(0) = 0$, so $c_k \to 0$ as $k \to \infty$. On the other hand, $c_k f(\varphi^{n_k}(a)) \to g(a) \neq 0$ as $k \to \infty$. But, since $\varphi(0) = 0$, the Schwarz Lemma yields

$$\{\varphi^k(a) : k \in \mathbb{N}\} \subset \{|z| \leq |a|\}.$$

Therefore the sequence $\{f(\varphi^k(a))\}_{n \geq 1}$ is bounded, so $c_k f(\varphi^{n_k}(a)) \to 0$, which is absurd. □

Of course, all "usual" spaces of analytic functions (as $H(G)$, Hardy spaces, Bergman spaces, Dirichlet spaces,...; see [6] or [16] for a description of them) satisfy all hypotheses in the last theorem, that is, continuity of the evaluation functionals, point separation (S), and non-collapsing (C). As for the two purely algebraical properties, (S) and (C), we observe that if E contains a one-to-one function then (S) holds; the converse is not true: take for instance $G = \mathbb{C}$ and $E = \text{span}\{e^z, e^{iz}\}$. Moreover, (S) and (C) are not comparable. Indeed, consider $G := \mathbb{C}$, $E_1 := \{f \in H(\mathbb{C}) : f(0) = 0\}$ and $E_2 := \text{span}\{1, z(z-1)\}$. Then E_1 satisfies (S) but not (C), while E_2 satisfies (C) but not (S).

The following special instance of Theorem 3.1, which is the case $E = H(G)$, will be used several times in the remainder. Because of this, we establish it separately.

Corollary 3.2. *Let $G \subset \mathbb{C}$ be a domain and $\varphi : G \to G$ be a holomorphic selfmapping such that C_φ is supercyclic on $H(G)$. Then we have:*
(a) *The map φ is one-to-one.*
(b) *If G is simply connected and $G \neq \mathbb{C}$, then φ fixes no point in G.*

For the sake of convenience, we introduce the following definition. If G is a domain of $\mathbb{C}$ and (z_n) is a sequence in G, then we say that (z_n) *approximates the boundary of G* whenever, given a compact set $K \subset G$, there is $n \in \mathbb{N}$ such that $z_n \notin K$. For instance, if $G = \mathbb{D}$, the last property means $\sup_{n \in \mathbb{N}} |z_n| = 1$. With this we can state our next assertion, which shows the equivalence of hypercyclicity and supercyclicity for sequences of automorphisms on simply connected domains different from $\mathbb{C}$.

Theorem 3.3. *Let $G \neq \mathbb{C}$ be a simply connected domain, and $(\varphi_n) \subset \text{Aut}(G)$. Then the following properties are equivalent:*

(a) *$(\varphi_n(z_0))$ approximates the boundary of G for some (all, resp.) $z_0 \in G$.*
(b) *(C_{φ_n}) is hypercyclic on $H(G)$.*

(c) (C_{φ_n}) *is supercyclic on* $H(G)$.

Proof. We can use Riemann's isomorphism theorem to find an isomorphism $\Phi : G \to \mathbb{D}$ with $\Phi(z_0) = 0$, where z_0 is any given point in G. Then an argument similar to one used in the proof of Theorem 3.1 reveals that we can suppose $z_0 = 0$, $G = \mathbb{D}$ (irrespective of whether the condition in (a) is "for some z_0" or "for all z_0"). Therefore (a) can be reduced to one statement, namely "$\sup_{n\in\mathbb{N}} |\varphi_n(0)| = 1$", so by Theorem 2.1 we have that (a) and (b) are equivalent. Since hypercyclicity implies supercyclicity, it is enough to prove "(c) $\Rightarrow$ (a)".

Assume, by way of contradiction, that there is $f \in SC((C_{\varphi_n}))$ and $(\varphi_n(0))$ does not approximate the boundary of $\mathbb{D}$. Then, if $\varphi_n(z) \equiv \alpha_n \frac{z-a_n}{1-\overline{a}_n z}$ (where $|a_n| < 1 = |\alpha_n|$ for all $n \in \mathbb{N}$), we derive the existence of an $r \in (0, 1)$ such that $|a_n| = |\varphi_n(0)| \leq r$ for all $n \in \mathbb{N}$. On the other hand, there must be sequences $(c_k) \subset \mathbb{C}$ and $\{n_1 < n_2 < \cdots\} \subset \mathbb{N}$ such that $c_k(f \circ \varphi_{n_k}) \to 1$ $(k \to \infty)$ uniformly on compacta (in particular, pointwise) in $\mathbb{D}$. By considering a subsequence of (n_k) if necessary, we may suppose that there are α_0 with $|\alpha_0| = 1$ and $a_0 \in \mathbb{D}$ with $\lim_{k\to\infty} \alpha_{n_k} = \alpha_0$ and $\lim_{k\to\infty} a_{n_k} = a_0$. Now it is elementary to check that (Φ_k) tends to Φ in $H(\mathbb{D})$ (so pointwise) as $k \to \infty$, where $\Phi_k = \varphi_{n_k}$ $(k \in \mathbb{N})$ and $\Phi(z) \equiv \alpha_0 \frac{z-a_0}{1-\overline{a}_0 z} \in \mathrm{Aut}(\mathbb{D})$. Evidently, f and Φ are holomorphic and nonconstant, hence both of them are continuous and open. An application of Lemma 2.2 (with $\mathbb{K} = \mathbb{C}$ and $X = \mathbb{D}$) yields the desired contradiction. □

Corollary 3.4. *Let $G \neq \mathbb{C}$ be a simply connected domain and $\varphi \in \mathrm{Aut}(G)$. Then the following properties are equivalent:*

(a) φ *has no fixed point in* G.
(b) C_φ *is hypercyclic on* $H(G)$.
(c) C_φ *is supercyclic on* $H(G)$.

Proof. By Theorem 3.3, (b) and (c) are equivalent. By Corollary 3.2(b), (c) implies (a). Finally, assume that (a) holds. Then if we fix any point $z_0 \in G$ we have that $(\varphi^n(z_0))$ approximates the boundary of G (this is evident if $G = \mathbb{D}$ and $z_0 = 0$; otherwise, use again a Riemann isomorphism $\Phi : G \to \mathbb{D}$ with $\Phi(z_0) = 0$). Hence Theorem 3.3 shows that (b) is true. □

Remarks 3.5. 1. Recall that a noneuclidean translation (NET) is an automorphism of $\mathbb{D}$ of the form $\varphi(z) = \frac{z+a}{1+\overline{a}z}$ $(0 < |a| < 1)$. It is easy to see that $\varphi \circ \psi$ is a NET whenever φ and ψ are NETs. Gorkin and Mortini [10, Theorem 3.1] have recently proved that if $\{\varphi_n(z) := \frac{z+a_n}{1+\overline{a}_n z} : n \in \mathbb{N}\}$ is a sequence of NETs with $\lim_{n\to\infty} |a_n| = 1$, then there exists a Blaschke product B such that the set $\{B \circ \varphi_n : n \in \mathbb{N}\}$ is locally uniformly dense in the set

$\mathcal{B} := \{f \in H(\mathbb{D}) : \ |f(z)| \le 1 \text{ for all } z \in \mathbb{D}\}$. It is clear that $\mathcal{B}_l := \{$Blaschke products$\} \subset \mathcal{B} \subset H^\infty := \{f \in H(\mathbb{D}) : \ f \text{ is bounded on } \mathbb{D}\}$. If P is a polynomial and B, (φ_n) are as before, then the closure in $H(\mathbb{D})$ of $\{B \circ \varphi_n : \ n \in \mathbb{N}\}$ contains $P/(1 + \sup_{\mathbb{D}} |P|)$, so the projective (C_{φ_n})-orbit of B contains the set of polynomials, which in turn is dense in $H(\mathbb{D})$. Consequently, under the assumption that φ, φ_n $(n \in \mathbb{N})$ are NETs, we have:

(1.1) Conditions (a) to (c) in Theorem 3.3 (with $G = \mathbb{D}$) are equivalent to each of the following:
  (d) $SC((C_{\varphi_n})) \cap H^\infty \neq \emptyset$.
  (e) $SC((C_{\varphi_n})) \cap \mathcal{B}_l \neq \emptyset$.

(1.2) $SC(C_\varphi) \cap \mathcal{B}_l \neq \emptyset$. This complements Corollary 3.4 (with $G = \mathbb{D}$).

Notice that, in spite of the equivalence of hyper- and supercyclicity of (C_φ) (or of (C_{φ_n})), there may be supercyclic functions which are not hypercyclic. Indeed, a hypercyclic function cannot be bounded.

2. We can extend Corollary 3.4 to other selfmappings when the domain is $\mathbb{D}$. For this, we are going to consider the group $LFT(\mathbb{D})$ of linear fractional transformations $\varphi(z) := \frac{az+b}{cz+d}$ $(ad - bc \neq 0)$ of $\mathbb{C}_\infty$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$. Obviously, $LFT(\mathbb{D}) \supset \mathrm{Aut}(\mathbb{D})$. Then if $\varphi \in LFT(\mathbb{D})$ we have that properties (a) to (c) in Corollary 3.4 (with $G = \mathbb{D}$) are equivalent. Indeed, that (b) implies (c) is trivial. Corollary 3.2(b) tells us that (c) implies (a). Finally, if (a) is satisfied then the Seidel-Walsh theorem still holds for φ (see [15, pages 123–124]), that is, (b) is true.

In the case $G = \mathbb{C}$ we will also obtain that the hypercyclicity of a composition operator is equivalent to its supercyclicity. Observe that this time we are able to handle *any* selfmapping φ to generate C_φ (compare to Corollary 3.4).

Theorem 3.6. *Let φ be an entire function. Then the following properties are equivalent:*

(a) *φ is a translation.*
(b) *C_φ is hypercyclic on $H(\mathbb{C})$.*
(c) *C_φ is supercyclic on $H(\mathbb{C})$.*

PROOF. It is trivial that (b) implies (c). On the other hand, Corollary 3.2(a) yields that φ must be one-to-one if (c) holds (so if (b) holds). But recall that $\mathrm{Aut}(\mathbb{C}) = \{$similarities of $\mathbb{C}\} = \{$one-to-one entire functions$\}$, hence by Theorem 2.1 the properties (a) and (b) are equivalent. Therefore, we should prove only that if $\varphi(z) := az + b \in \mathrm{Aut}(\mathbb{C})$ $(a, b \in \mathbb{C};\ a \neq 0)$ and C_φ is supercyclic then $a = 1$ and $b \neq 0$. But notice that if $[a = 1,\ b = 0]$ then $\varphi(z) \equiv z$, which obviously does not generate supercyclicity. Consequently, our goal is to show that if C_φ is supercyclic then $a = 1$.

By way of contradiction, suppose that $f \in SC(C_\varphi)$ and $a \neq 1$. Then C_φ is similar to the composition operator defined by a rotation-dilation. Specifically, if $c := b/(1-a)$ is the fixed point of φ and $\psi(z) := z+c$ then $C_\varphi = C_\psi \circ C_{\widetilde{\varphi}} \circ (C_\psi)^{-1}$, where $\widetilde{\varphi}(z) := az$. But C_φ is supercyclic if and only if $C_{\widetilde{\varphi}}$ is. Thus, we may leave with the assumption that $\varphi(z) \equiv az$, $a \neq 1$. Observe that the projective orbit of f is $\{\lambda f(a^n z) : \lambda \in \mathbb{C}, n \in \mathbb{N}\}$. Since it is dense in $H(G)$, we have that f is not a polynomial. We distinguish three cases: $|a| < 1$, $|a| = 1$, $|a| > 1$.

Assume that $|a| < 1$. Then $f(a^n z) \to f(0)$ $(n \to \infty)$ locally uniformly in $\mathbb{C}$, so the projective orbit of f can only approximate constant functions, which is absurd.

Let $|a| = 1$ now. By supercyclicity, there are scalars $(c_k) \subset \mathbb{C}$ and integers $\{n_1 < n_2 < \cdots\}$ for which $c_k f(a^{n_k} z) \to 1$ $(k \to \infty)$ in $H(\mathbb{C})$. By extracting a subsequence if necessary, we may suppose that $a^{n_k} \to a_0$, where $|a_0| = 1$. Then an application of Lemma 2.2 (with $\mathbb{K} = \mathbb{C} = X$, $F = f$, $\Phi_k(z) := a^{n_k} z$, $\Phi(z) := a_0 z$) drives us again to a contradiction.

Finally, suppose that $|a| > 1$. Since f is transcendental, Picard's theorem (see e.g. [9, Chapter 9]) guarantees that there is an $\alpha \in \mathbb{C}$ such that, for any $A \in \mathbb{C} \setminus \{\alpha\}$, there is a sequence $z_k \to \infty$ such that $f(z_k) = A$ for all $k \in \mathbb{N}$. Choose any values A, B with $A \neq B \neq 0$ and $A \neq \alpha \neq B$, and consider their corresponding sequences (z_k), (w_k); so $f(z_k) = A$, $f(w_k) = B$ $(k \in \mathbb{N})$. By supercyclicity, there are sequences $(c_k) \subset \mathbb{C}$ and $\{n_1 < n_2 < \cdots\} \subset \mathbb{N}$ with $c_k f(a^{n_k} z) \to 1$ $(k \to \infty)$ uniformly on $K_0 := \{z : \ |z| \leq 1\}$. By extracting subsequences if necessary, we can assume that $c_k \neq 0$, $z_k' := a^{-n_k} z_k \in K_0$ and $w_k' := a^{-n_k} w_k \in K_0$ for all $k \in \mathbb{N}$ (the fact $|a| > 1$ has been used here). Then $c_k f(a^{n_k} z_k') \to 1$ and $c_k f(a^{n_k} w_k') \to 1$. Therefore

$$\frac{A}{B} = \lim_{k \to \infty} \frac{c_k f(z_k)}{c_k f(w_k)} = 1,$$

which is absurd. □

We remark that from Theorem 3.6 it is derived that Corollary 3.2(b) also holds for $G = \mathbb{C}$.

In the case of sequences of automorphisms, we get the next partial extension of the last theorem. Observe that the following final result covers the case of translations $z \mapsto z + b_n$ $(n \in \mathbb{N})$. It turns out that three dynamical properties –hypercyclicity, supercyclicity and non-equicontinuity– are equivalent.

Theorem 3.7. *Let* $(\varphi_n(z) := a_n z + b_n) \subset \mathrm{Aut}(\mathbb{C})$ *such that* $0 < \inf_n |a_n| \leq \sup_n |a_n| < +\infty$. *Then the following properties are equivalent:*

(a) (C_{φ_n}) *is hypercyclic on* $H(\mathbb{C})$.
(b) (C_{φ_n}) *is supercyclic on* $H(\mathbb{C})$.
(c) *The sequence* (b_n) *is unbounded.*
(d) (C_{φ_n}) *is not equicontinuous in* $H(\mathbb{C})$.

Proof. Under our hypothesis on the a_n's we have that (c) holds if and only if the sequence $(\min\{|b_n|, |b_n/a_n|\})$ is unbounded. Thus, according to Theorem 2.1, properties (a) and (c) are equivalent. Once more, the implication (a) $\Rightarrow$ (b) is trivial. If (b_n) is bounded, we can take $M \in (0, +\infty)$ with $|b_n| \leq M$ for all $n \in \mathbb{N}$. Given a basic neighborhood $V(R, \varepsilon) := \{f \in H(\mathbb{C}) : \ |f(z)| \leq \varepsilon \text{ if } |z| \leq R\}$ of the origin in $H(\mathbb{C})$ $(R, \varepsilon > 0)$, it is clear that $\bigcup_{n\in\mathbb{N}} C_{\varphi_n}(V(\mu, \delta)) \subset V(R, \varepsilon)$, where $\delta := \varepsilon$ and $\mu := M + R \sup_n |a_n|$. Hence (C_{φ_n}) is equicontinuous. Conversely, if (C_{φ_n}) is equicontinuous then there are $\mu, \delta \in (0, +\infty)$ such that $\bigcup_{n\in\mathbb{N}} C_{\varphi_n}(V(\mu, \delta)) \subset V(1, 1)$. Since $f(z) := (\delta/\mu)z \in V(\mu, \delta)$, we get $|f(a_n z + b_n)| \leq 1$ for all $n \in \mathbb{N}$ if $|z| \leq 1$. Setting $z = 0$ we obtain $|b_n| \leq \mu/\delta$ $(n \in \mathbb{N})$. So (c) and (d) are equivalent.

It remains to show that if (b_n) is bounded then (C_{φ_n}) is not supercyclic. Assume that (b_n) is bounded and that, in addition, there is $F \in SC((C_{\varphi_n}))$. Then $c_k F(a_{n_k} z + b_{n_k}) \to 1$ $(k \to \infty)$ pointwise in $\mathbb{C}$ for certain sequences $(c_k) \subset \mathbb{C}$ and $\{n_1 < n_2 < \cdots\} \subset \mathbb{N}$. Since (a_{n_k}), (b_{n_k}) are bounded and (a_{n_k}) is away from zero, by extracting subsequences if necessary one can suppose that $a_{n_k} \to a_0$ and $b_{n_k} \to b_0$ for certain complex numbers a_0, b_0 with $a_0 \neq 0$. Consequently, $\Phi_k \to \Phi$ pointwise in X and $\lim_{k\to\infty} c_k F(\Phi_k(z)) = 1$ for all $z \in X$, where $X := \mathbb{C}$, $\Phi(z) := a_0 z + b_0$ and $\Phi_k(z) := a_{n_k} z + b_{n_k}$. The sought-after contradiction is taken out again from Lemma 2.2. □

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