Spectral properties of certain tridiagonal matrices

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Abstract

We study spectral properties of irreducible tridiagonal $k-$Toeplitz matrices and certain matrices which arise as perturbations of them.

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1 Introduction

In this paper we focusing on the spectral properties of general irreducible tridiagonal $k-$Toeplitz matrices and certain perturbations of them. Recall that a tridiagonal $k-$Toeplitz matrix is an irreducible tridiagonal matrix such that the entries along the diagonals are sequences of period k (see M. J. C. Gover [15]). Apart its own theoretical interest, the study of this type of matrices appears to be useful, for instance, in the study of sound propagation problems [4, 16], as well as in the description of several models of coupled quantum oscillators which may be described by using appropriate perturbations of tridiagonal $k-$Toeplitz matrices (see [1, 2]). We will focus on the localization of the eigenvalues of such matrices, as well as on the distance between two consecutive eigenvalues. The matrix perturbations to be considered here have the form

$$J_n^{\mu,\lambda} := \begin{pmatrix} \beta_0+\mu & 1 & 0 & \dots & 0 & 0 \\ \gamma_1 & \beta_1 & 1 & \dots & 0 & 0 \\ 0 & \gamma_2 & \beta_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \beta_{n-2} & 1 \\ 0 & 0 & 0 & \dots & \gamma_{n-1} & \beta_{n-1}+\lambda \end{pmatrix}, \tag{1.1}$$

where, by varying n, the sets of entries $(\beta_s)_s$ and $(\gamma_s)_s$ are sequences of real numbers such that $\gamma_s > 0$ for all s, and λ and μ are given real numbers (the perturbation parameters). When $(\beta_s)_s$ and $(\gamma_s)_s$ are $k-$periodic sequences, so that

$$\beta_s = a_{j+1} \text{ if } s \equiv j \pmod{k}, \quad \gamma_{s+1} = b_{j+1}^2 \text{ if } s \equiv j \pmod{k} \tag{1.2}$$

for all $s = 0, 1, 2, \ldots, n-1$, with $a_j \in \mathbb{R}$ and $b_j > 0$ for all $j = 1, 2, \ldots, k$, one obtains the mentioned perturbed $k-$Toeplitz matrix. (The $k-$Toeplitz matrix corresponds to the choice $\lambda = \mu = 0$, subject to the periodicity conditions (1.2)).

Recently, these perturbed matrices $J_n^{\mu,\lambda}$ subject to the periodicity conditions (1.2), i.e., the perturbed $k-$Toeplitz matrices, where investigated by several authors for some special choices of the period k, among which we distinguish S. Kouachi [20] for the case $k = 2$, and A. R. Willms [27] for the case $k = 1$ (notice that in this case the entries along each diagonal are constant, up to the entries in the left upper corn and in the lower right corn). These authors have studied the eigenvalues of these matrices by considering a trigonometric equation whose solution yields the eigenvalues, focusing their contributions in several special cases corresponding to situations when these trigonometric equations have explicit solutions, and exact expressions for the eigenvalues and eigenvectors were obtained. Regarding the case of a $2-$Toeplitz matrix (hence, in particular, of a $1-$Toeplitz matrix) a trigonometric equation whose solution yields the eigenvalues was stated by F. Marcellán and J. Petronilho in [21]. This equation has been deduced on the basis of the fact that the characteristic polynomial of a $2-$Toeplitz matrix may be expressed in terms of Chebyshev polynomials of the second kind (via a quadratic polynomial mapping) and, as it is well known, these polynomials admit trigonometric representations. Notice, however, that the explicit expressions for the eigenvalues of a tridiagonal $2-$Toeplitz matrix have been given previously by M. J. C. Gover [15], without using orthogonal polynomial theory. Therefore, since, by making some basic operations on determinants, the characteristic polynomial of the perturbed $k-$Toeplitz matrix can be expressed in terms of the characteristic polynomial of the non-perturbed $k-$Toeplitz matrix, it is clear that a trigonometric equation yielding the eigenvalues of the perturbed $2-$Toeplitz matrix can be established. In fact, more generally, by using similar arguments and the results in [10, 13, 18], a trigonometric equation whose solution yields the eigenvalues of the general perturbed $k-$Toeplitz matrix defined by (1.1)-(1.2) can be easily established.

Concerning the mentioned works [20] and [27], Kouachi and Willms studied the spectral properties of the perturbed $1-$Toeplitz and $2-$Toeplitz matrices exhibiting explicit nice formulae for the eigenvalues and eigenvectors of such matrices for appropriately choices of the parameters λ and μ. By contrast, our study in the present paper will not be focused on the determination of explicit formulae. Instead, our aim will be the location of the eigenvalues for general perturbed and non-perturbed $k-$Toeplitz matrices of large order. Roughly speaking, we will state that, for large n, the eigenvalues of a perturbed $k-$Toeplitz matrix of order $nk + j - 1$ $(1 \leq j \leq k-1)$ may be approximated by the eigenvalues of the corresponding non-perturbed $k-$Toeplitz matrix of order $nk - 1$, up to a finite number of them, and we remark that this number depends on k but it is independent of n.

The analytical study of infinite tridiagonal matrices (infinite Jacobi matrices, regarded as operators acting in ℓ^2, the space of the square summable sequences of complex numbers) was considered before by several authors. For instance, in [8, 9, 11] in connection with the Theory of Toda Lattices, as well as in [14, 18, 19, 23, 25, 26], where the spectrum of the corresponding Jacobi operators was studied. We also point out that a matrix theoretic approach to the problem concerning the study of the spectral properties of $k-$Toeplitz matrices has been presented in works by S. Serra Capizzano and D. Fasino [12, 24]. The computation of the orthogonal polynomials corresponding to the $k-$Toeplitz matrices (hence of the characteristic polynomials of such matrices) may be done by reducing the study to a problem involving an appropriate polynomial mapping. Concerning the spectral measure associated to such Jacobi matrices it is known [14] that it is given by a polynomial transformation on the Chebyshev measure of the second kind, plus a finite number of mass points, and a connection with a polynomial mapping have been done in [13, 21, 22] in order to describe the orthogonal polynomials. This mapping is essential to describe the results in the work we present here. Further, it allow us to give some interlacing properties from which we deduce some upper bounds for the distance between consecutive eigenvalues of the involved matrices.

The structure of the paper is as follows. In section 2 some mathematical results concerning the spectral properties of tridiagonal $k-$Toeplitz matrices are presented. In section 3 we apply the results in section 2 in order to obtain interlacing properties for certain perturbed tridiagonal $k-$Toeplitz matrices. Finally, in section 4 we include some numerical experiments.

2 Spectral properties of $k-$Toeplitz matrices

In this section we will present some properties of the eigenvalues of a tridiagonal $k-$Toeplitz matrix, i.e., a matrix of the form

$$J_m := \begin{pmatrix} \beta_0 & 1 & 0 & \dots & 0 & 0 \\ \gamma_1 & \beta_1 & 1 & \dots & 0 & 0 \\ 0 & \gamma_2 & \beta_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \beta_{m-2} & 1 \\ 0 & 0 & 0 & \dots & \gamma_{m-1} & \beta_{m-1} \end{pmatrix} . \tag{2.1}$$

where the entries $(\beta_s)_s$ and $(\gamma_s)_s$ are k-periodic sequences, say

$$\beta_s = a_{j+1} \text{ if } s \equiv j \pmod{k} , \quad \gamma_{s+1} = b_{j+1}^2 \text{ if } s \equiv j \pmod{k} \tag{2.2}$$

for all $s = 0, 1, 2, \dots, m-1$, with $a_j \in \mathbb{R}$ and $b_j > 0$ for all $j = 1, 2, \dots, k$.

Theorem 2.1 *Fix $k \in \mathbb{N}$. Assume that J_m is a tridiagonal $k-$Toeplitz matrix such that (2.2) holds, with $a_j \in \mathbb{R}$ and $b_j > 0$ for all $j = 1, 2, \dots, k$. If $1 \le i < j \le k$, define*

$$\Delta_{i,j}(x) := \begin{vmatrix} x-a_i & 1 & 0 & \cdots & 0 & 1 \\ b_i^2 & x-a_{i+1} & 1 & \cdots & 0 & 0 \\ 0 & b_{i+1}^2 & x-a_{i+2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x-a_{j-1} & 1 \\ 0 & 0 & 0 & \cdots & b_{j-1}^2 & x-a_j \end{vmatrix} ,$$

so that $\Delta_{i,j}(x)$ is a polynomial of degree $j-i+1$ in x; and if $j \le i \le k$, define

$$\Delta_{i,j}(x) := \begin{cases} 0 & \text{if } j < i-1 , \\ 1 & \text{if } j = i-1 , \\ x-a_i & \text{if } j = i . \end{cases}$$

Let π_k and Δ_{k-1} be the polynomials of degrees k and $k-1$ (resp.) defined by

$$\pi_k(x) := \Delta_{1,k}(x) - b_k^2 \Delta_{2,k-1}(x) , \quad \Delta_{k-1}(x) := \Delta_{1,k-1}(x) .$$

Furthermore, set

$$\Sigma_k := \pi_k^{-1}([-\alpha, \alpha]) , \quad \alpha := 2b_1 b_2 \cdots b_k .$$

Then the following holds:

(i) *The set* Σ_k *is an union of* k *intervals* $I_1, \cdots, I_k$*, such that any two of these intervals intersect at most at a single point (i.e.,* Σ_k *is indeed an union of at most* k *disjoint intervals), so that*

$$\Sigma_k = \pi_k^{-1}([-\alpha, \alpha]) = I_1 \cup \cdots \cup I_k \ . \tag{2.3}$$

(ii) *Except for at most* $k-1$ *ones, all the eigenvalues of* J_{kn+k-1} *are located in the set* Σ_k*, for all* $n = 1, 2, \cdots$*. More precisely, each interval* $I_1, \ldots, I_k$ *contains exactly* n *eigenvalues of* J_{kn+k-1} *in its interior, and the remaining* $k-1$ *eigenvalues are located between these* k *intervals, so that between* I_ℓ *and* $I_{\ell+1}$ $(\ell = 1, \cdots, k-1)$ *there exists exactly one eigenvalue of* J_{kn+k-1}*. These* $k-1$ *eigenvalues are the* $k-1$ *solutions of the algebraic equation* $\Delta_{k-1}(x) = 0$.

(iii) *For each* $j = 0, 1, \cdots, k$*, all the eigenvalues of* J_{kn+j-1} $(n = 1, 2, \cdots)$ *are contained in the convex hull of the set* Σ_k*. Furthermore, between two consecutive intervals* I_ℓ *and* $I_{\ell+1}$ $(\ell = 1, \cdots, k-1)$ *the number of eigenvalues of* J_{kn+j-1} *is at most* N_j*, where*

$$N_j := \begin{cases} j+1 & \text{if} \quad 0 \le j \le \lfloor k/2 \rfloor \ , \\ k-j+1 & \text{if} \quad \lfloor k/2 \rfloor + 1 \le j \le k \ . \end{cases} \tag{2.4}$$

For each $\ell = 1, \cdots, k$*, let* $n_{j,n}(\ell)$ *denote the number of eigenvalues of* J_{nk+j-1} *inside the interval* I_ℓ*. Then*

$$n - L_j \le n_{j,n}(\ell) \le n + M_j - 1 \ , \quad j = 0, 1, \cdots, k \ , \tag{2.5}$$

where

$$L_j := \begin{cases} j+1 & \text{if} \quad 0 \le j \le \lfloor k/2 \rfloor - 1 \ , \\ k-j & \text{if} \quad \lfloor k/2 \rfloor \le j \le k \ , \end{cases}$$

and

$$M_j := \begin{cases} j & \text{if} \quad 0 \le j \le \lfloor k/2 \rfloor \ , \\ k-j+1 & \text{if} \quad \lfloor k/2 \rfloor + 1 \le j \le k \ . \end{cases}$$

Proof. The sequences $\{\beta_s\}_{s\ge0}$ and $\{\gamma_s\}_{s\ge1}$ satisfying the periodicity conditions (2.2) generate a MOPS, $(P_n)_n$, defined by the three-term recurrence relation

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x) \ , \quad n = 0, 1, 2, \cdots$$

with initial conditions $P_{-1} = 0$ and $P_0 = 1$. It follows from very well known facts in the Theory of Orthogonal Polynomials that the zeros of P_n are the eigenvalues of the matrix J_n [6, Ex. 5.7], which are all real and simple [6, Th. 5.2], and the zeros of P_n interlace with those of P_{n-1} [6, Th. 5.3]. Under the given hypothesis, it is known (see e.g. [14, 18]) that the support of the measure with respect to which the MOPS $(P_n)_n$ is orthogonal consists of an union of k intervals such that any two of these intervals may intersect at a single point, plus at most $k-1$ isolated points between them. Furthermore (see e.g. [13, 18]) these k intervals are defined by the inverse polynomial mapping $[-\alpha, \alpha] \mapsto \pi_k^{-1}([-\alpha, \alpha])$ and they are separated by the points $\xi_1, \cdots, \xi_{k-1}$ which are the solutions of the algebraic equation $\Delta_{k-1}(x) = 0$ (see Figure 1 for the case $k = 3$, where the inverse polynomial mapping $[-\alpha, \alpha] \mapsto \pi_3^{-1}([-\alpha, \alpha])$ is illustrated). This justifies statement (i) in the Proposition. In order to prove (ii) notice first that (cf. e.g. [13])

$$P_{nk+k-1}(x) = \left(\tfrac{\alpha}{2}\right)^n \Delta_{k-1}(x) U_n\left(\frac{\pi_k(x)}{\alpha}\right) \ , \quad n = 0, 1, 2, \cdots \tag{2.6}$$

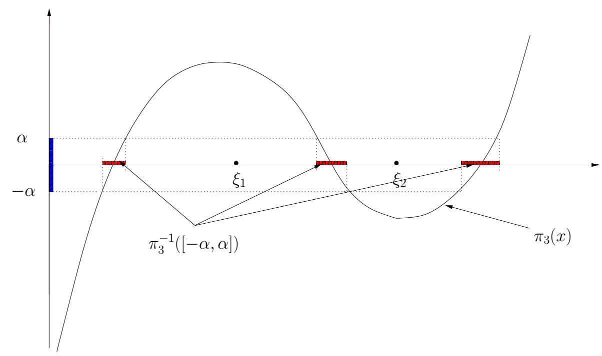

Figure 1: Inverse polynomial mapping

where U_n is the Chebyshev polynomial of the second kind of degree n,

$$U_n(x) := \frac{\sin(n+1)\theta}{\sin\theta}, \quad x = \cos\theta. \tag{2.7}$$

Thus the zeros of P_{nk+k-1} (hence the eigenvalues of J_{nk+k-1}) are the above $k-1$ real numbers $\xi_1, \cdots, \xi_{k-1}$, which are located between the k intervals $I_1, \cdots, I_k$, together with the kn real numbers x such that $U_n\left(\frac{\pi_k(x)}{\alpha}\right) = 0$, i.e.,

$$\pi_k(x) = \alpha \cos\frac{j\pi}{n+1}\,, \quad j = 0, 1, \cdots, k-1\,. \tag{2.8}$$

Moreover, these nk eigenvalues lie inside the k intervals $I_1, \cdots, I_k$, and each interval contains exactly n eigenvalues of J_{kn+k-1} in its interior. This follows from the fact that π_k is monotone in each interval I_ℓ ($\ell = 1, \cdots, k$), as follows from the proof of [18, Theorem 5.1]. This proves statement (ii).

To prove (iii), notice first that for $j = 0$ or $j = k$ (2.4) gives $N_0 = N_k = 1$, which is true by (ii), so we may assume $1 \le j \le k-1$. For the sake of simplicity we assume that k is even (the case when k is odd can be treated in a similar way). Denote by Γ_ℓ the set between two consecutive intervals I_ℓ and $I_{\ell+1}$ ($\ell = 1, \cdots, k-1$). Notice that Γ_ℓ may reduce to a single point in case that the intervals I_ℓ and $I_{\ell+1}$ toch each other. By (ii) we know that the polynomial P_{nk-1} has exactly one zero in each Γ_ℓ. Then, by the interlacing property, P_{nk} has at most two zeros in each Γ_ℓ. Then, again by the interlacing property, P_{nk+1} has at most three zeros in each Γ_ℓ, and so one. Hence, at step $k/2$, we see that in each Γ_ℓ the polynomial $P_{nk+k/2-1}$ has at most $k/2+1$ zeros. This proves that P_{nk+j-1} has at most $j+1$ zeros in each Γ_ℓ if $1 \le j \le k/2$. To prove that P_{nk+j-1} has at most $k-j+1$ zeros in each Γ_ℓ if $k/2 < j \le k-1$, we argue by contradiction. We know by (ii) that P_{nk+k-1} has exactly one zero in each Γ_ℓ. Then, by the interlacing property, P_{nk+k-2} should have no more than two zeros in each Γ_ℓ. Then, again by the interlacing property, P_{nk+k-3} should have no more than three zeros in each Γ_ℓ, and so one. Continuing in this way, at step $k/2$, we would conclude that $P_{nk+k/2}$ should have no more than $k/2$ zeros in each Γ_ℓ. We therefore proved that J_{nk+j-1} has no more than N_j eigenvalues in any set Γ_ℓ, for every $j = 0, 1, \cdots, k$. To prove the first inequality in (2.5) notice that, by (ii), J_{nk-1} has $n-1$ eigenvalues on each interval I_ℓ, and using again the interlacing property (arguing in a similar way as before, by counting successively the minimum possible number of eigenvalues of J_{kn-1}, J_{kn}, J_{kn+1},

..., $J_{kn+\lfloor k/2\rfloor-2}$ in an interval I_ℓ, and then the minimum possible number of eigenvalues of J_{kn+k-1}, J_{kn+k-2}, J_{kn+k-3}, ..., $J_{kn+\lfloor k/2\rfloor-1}$ in the same interval) we may conclude that in each interval I_ℓ the number of eigenvalues of J_{kn+j-1} is at least $n-L_j$. The second inequality in (2.5) can be proved by a similar reasoning, by counting successively the maximum possible number of eigenvalues of J_{kn-1}, J_{kn}, J_{kn+1}, ..., $J_{kn+\lfloor k/2\rfloor-2}$ in an interval I_ℓ, and then the maximum possible number of eigenvalues of J_{kn+k-1}, J_{kn+k-2}, J_{kn+k-3}, ..., $J_{kn+\lfloor k/2\rfloor-1}$ in the same interval. This completes the proof. □

Remark 2.2 *Notice that when $\ell=1$ and $\ell=k$ (corresponding to the intervals I_1 and I_k) the following more precise estimates hold*

$$n_{0,n}(1)=n_{0,n}(k)=n-1\ ,\quad n_{k,n}(1)=n_{k,n}(k)=n\ ,$$
$$n-1\le n_{j,n}(1),n_{j,n}(k)\le n\quad\text{for}\quad 2\le j\le k-1\ ,$$

which are also a consequence of the interlacing properties.

3 Interlacing properties for certain perturbed tridiagonal $k-$Toeplitz matrices

In what follows, we consider a set of numbers $\{\beta_s,\gamma_{s+1}\}_{s\ge0}$, with $\gamma_{s+1}\neq0$ for all $s=0,1,2,\dots$, and a pair of parameters μ and λ. For every integer number $n\ge2$ we will denote by $J_n^{\mu,\lambda}$ the tridiagonal matrix of order n

$$J_n^{\mu,\lambda}:=\begin{pmatrix}\beta_0+\mu & 1 & 0 & \dots & 0 & 0\\ \gamma_1 & \beta_1 & 1 & \dots & 0 & 0\\ 0 & \gamma_2 & \beta_2 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & \beta_{n-2} & 1\\ 0 & 0 & 0 & \dots & \gamma_{n-1} & \beta_{n-1}+\lambda\end{pmatrix}.$$

By varying n we can associate to the matrices $J_n^{\mu,\lambda}$ a monic orthogonal polynomial sequence (MOPS), which will be denoted by $(P_n^{\mu,\lambda})_n$. In particular, when $\lambda=0$, to the family of tridiagonal matrices $J_n^{\mu,0}$ (by varying n) it can be associated the MOPS $(P_n^\mu)_n$ which is generated by the three-term recurrence relation

$$xP_n^\mu(x)=P_{n+1}^\mu(x)+\beta_nP_n^\mu(x)+\gamma_nP_{n-1}^\mu(x)\ ,\quad n=1,2,3,\dots\tag{3.1}$$

with $P_0^\mu(x)=1$ and $P_1^\mu(x)=x-\beta_0-\mu$. Then [6, Ex. 4.12]

$$P_n^\mu(x)=\begin{vmatrix}x-\beta_0-\mu & 1 & 0 & \dots & 0 & 0\\ \gamma_1 & x-\beta_1 & 1 & \dots & 0 & 0\\ 0 & \gamma_2 & x-\beta_2 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & x-\beta_{n-2} & 1\\ 0 & 0 & 0 & \dots & \gamma_{n-1} & x-\beta_{n-1}\end{vmatrix}$$

for all $n=2,3,\dots$, and each zero of P_n^μ is an eigenvalue of the corresponding tridiagonal matrix $J_n^{\mu,0}$. If $\lambda=\mu=0$ we write $J_n\equiv J_n^{0,0}$ and $P_n\equiv P_n^0$. The MOPS $(P_n^\mu)_n$ is called the co-recursive sequence with parameter μ associated to the sequence $(P_n)_n$. The co-recursive polynomials were introduced and studied by T. S. Chihara [5].

Proposition 3.1 *Assume that $\beta_s \in \mathbb{R}$ and $\gamma_{s+1} > 0$ for all $s = 0, 1, 2, \cdots$. Then, for all $\mu, \lambda \in \mathbb{R}$ and every integer number $n \geq 2$, the following holds:*

(i) *The eigenvalues of $J_n^{\mu,\lambda}$ are real and simple.*

(ii) *If $\mu \neq 0$, then the eigenvalues of the matrices $J_n^{\mu,0}$ and J_n interlace.*

(iii) *If $\lambda \neq 0$, then the eigenvalues of the matrices $J_n^{\mu,\lambda}$ and $J_n^{\mu,0}$ interlace. More precisely, the following holds:*

$$\lambda < 0 \quad \Rightarrow \quad x_{n,j}^{\mu,\lambda} < x_{n,j}^{\mu,0} < x_{n,j+1}^{\mu,\lambda} < x_{n,n}^{\mu,0} \quad (1 \leq j \leq n-1)$$
$$\lambda > 0 \quad \Rightarrow \quad x_{n,j}^{\mu,0} < x_{n,j}^{\mu,\lambda} < x_{n,j+1}^{\mu,0} < x_{n,n}^{\mu,\lambda} \quad (1 \leq j \leq n-1)\ ,$$

where $x_{n,j}^{\mu,0}$ and $x_{n,j}^{\mu,\lambda}$ denote the eigenvalues of the matrices $J_n^{\mu,0}$ and $J_n^{\mu,\lambda}$ (resp.). As a consequence, there exists at most one eigenvalue of $J_n^{\mu,\lambda}$ out of the interval $[x_{n,1}^{\mu,0}, x_{n,n}^{\mu,0}]$.

(iv) *Between two consecutive eigenvalues of J_n there exists at most two eigenvalues of $J_n^{\mu,\lambda}$, and conversely. Furthermore, there exist at most two eigenvalues of $J_n^{\mu,\lambda}$ out of the interval $[x_{n,1}, x_{n,n}]$, where $x_{n,1}$ and $x_{n,n}$ denote the smallest and the greatest eigenvalues of J_n (resp.).*

Proof. The statement in (i) is a well known fact, which follows at once from the fact that, under the conditions of the proposition, the matrix $J_n^{\mu,\lambda}$ is similar to a symmetric tridiagonal matrix with positive entries along the upper and sub diagonals. Further, since the zeros of the co-recursive polynomial P_n^{μ} interlace with those of P_n (see [5]) then we may conclude that the eigenvalues of $J_n^{\mu,0}$ and J_n interlace, which proves (ii). In order to prove (iii), recall first that, since the zeros of the orthogonal polynomials are real and distinct, then for each m we may denote by $x_{m,1} < x_{m,2} < \cdots < x_{m,m}$ the eigenvalues of $J_m \equiv J_m^{0,0}$. Define a polynomial sequence $(Q_m)_m \equiv (Q_m(\cdot;\lambda))_m$ by

$$Q_m(x) := P_m(x) - \lambda P_{m-1}(x)$$

for all $m = 0, 1, 2, \ldots$. Notice the relations

$$Q_n(x_{n,j}) = -\lambda P_{n-1}(x_{n,j})\ ,$$
$$Q_n(x_{n,j+1}) = -\lambda P_{n-1}(x_{n,j+1})$$

for all $j = 1, \ldots, n-1$. Then, since the zeros of P_n and P_{n-1} interlace [6, p. 28], it follows that the quantities $P_{n-1}(x_{n,j})$ and $P_{n-1}(x_{n,j+1})$ have opposite signs for all $j = 1, \ldots, n-1$ (see Figure 2).

Thus the polynomial Q_n has n real zeros and between two consecutive zeros of P_n there is exactly one zero of Q_n. This gives the location of $n-1$ zeros of Q_n. The remainder zero is less than $x_{n,1}$ if $\lambda < 0$ and it is greater than $x_{n,n}$ if $\lambda > 0$. To prove this let us assume $\lambda < 0$. Then, $Q_n(x_{n,1})$ has the same sign as $P_{n-1}(x_{n,1})$, which is the opposite one of $\lim_{x\to-\infty} P_n(x)$, which in turn has the same sign as $\lim_{x\to-\infty} Q_n(x)$. In other words, denoting by $x_{n,1}(\lambda) < x_{n,2}(\lambda) < \cdots < x_{n,n-1}(\lambda) < x_{n,n}(\lambda)$ the zeros of Q_n, the interlacing property

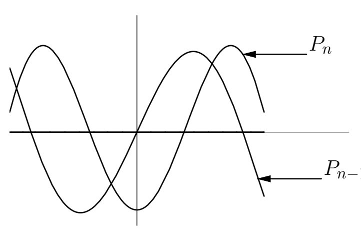

Figure 2: P_n and P_{n-1}

$$x_{n,1}(\lambda) < x_{n,1} < x_{n,2}(\lambda) < x_{n,2} < \cdots < x_{n,n}(\lambda) < x_{n,n}$$

holds. For the case when $\lambda > 0$ we can use the same reasoning but with the greatest zero $x_{n,n}$ instead of $x_{n,1}$ to obtain

$$x_{n,1} < x_{n,1}(\lambda) < x_{n,2} < x_{n,2}(\lambda) < \cdots < x_{n,n} < x_{n,n}(\lambda) \; .$$

Next, notice that the polynomial Q_n can be written as

$$Q_n(x) = \begin{vmatrix} x-\beta_0 & 1 & 0 & \ldots & 0 & 0 \\ \gamma_1 & x-\beta_1 & 1 & \ldots & 0 & 0 \\ 0 & \gamma_2 & x-\beta_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & x-\beta_{n-2} & 1 \\ 0 & 0 & 0 & \ldots & \gamma_{n-1} & x-\beta_{n-1}-\lambda \end{vmatrix} .$$

This follows by expanding this determinant by its last row and taking into account (3.1) for $\mu = 0$ as well as the definition of Q_n. Notice that Q_n is the (monic) characteristic polynomial of the matrix $J_n^{0,\lambda}$, hence $Q_n = P_n^{0,\lambda}$. Now, introduce a new sequence $(R_m)_m \equiv (R_m(\cdot, \mu, \lambda))_n$ by

$$R_m(x) := P_m^\mu(x) - \lambda P_{m-1}^\mu(x) \; , \quad m = 0, 1, 2, \cdots \; .$$

Since R_n is defined from the sequence $(P_m^\mu)_m$ by the same way as Q_n was defined from $(P_m)_m$, we have that the zeros of R_n and P_n^μ must interlace and

$$R_n(x) = \begin{vmatrix} x-\beta_0-\mu & 1 & 0 & \ldots & 0 & 0 \\ \gamma_1 & x-\beta_1 & 1 & \ldots & 0 & 0 \\ 0 & \gamma_2 & x-\beta_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & x-\beta_{n-2} & 1 \\ 0 & 0 & 0 & \ldots & \gamma_{n-1} & x-\beta_{n-1}-\lambda \end{vmatrix} ,$$

so that R_n is the (monic) characteristic polynomial of $J_n^{\mu,\lambda}$, i.e., $R_n = P_n^{\mu,\lambda}$. Hence the eigenvalues of $J_n^{\mu,\lambda}$ and $J_n^{\mu,0}$ interlace, which proves (iii). Finally, (iv) is an immediate consequence of (ii) and (iii). □

Theorem 3.2 *Let J_m be the tridiagonal $k-$Toeplitz matrix* (2.1) *whose entries $(\beta_s)_s$ and $(\gamma_s)_s$ are k-periodic sequences fulfilling* (2.2)*. Let Σ_k be the set defined in* (2.3)*. Then, for every $n = 1, 2, \cdots$ and each $j = 0, 1, \cdots, k-1$, the following properties hold:*

- (i) *There exists at most two eigenvalues of the perturbed matrix $J_{nk+j-1}^{\mu,\lambda}$ out of the convex hull of Σ_k.*
- (ii) *There exists at most $N_j + 2$ eigenvalues of $J_{nk+j-1}^{\mu,\lambda}$ in between two consecutive intervals I_ℓ and $I_{\ell+1}$ ($\ell = 1, \cdots, k-1$), where N_j is given by* (2.4)*.*
- (iii) *There are at most $(k-1)N_j + 2k$ eigenvalues of the perturbed matrix $J_{nk+j-1}^{\mu,\lambda}$ out of the set Σ_k.*

Proof. Statement (i) follows from Theorem 2.1-(iii) and Proposition 3.1-(iv). Statement (iii) is an immediate consequence of (i) and (ii). To prove (ii) we notice first that, by Proposition 3.1-(ii) and Theorem 2.1-(iii), in between two consecutive intervals I_ℓ and $I_{\ell+1}$ ($\ell = 1, \cdots, k-1$) there exists at most $N_j + 1$ eigenvalues of the perturbed matrix $J_{kn+j-1}^{\mu,0}$. (In Figure 3 an illustrative situation is presented.) Thus using Proposition 3.1-(iii) it follows that in between I_ℓ

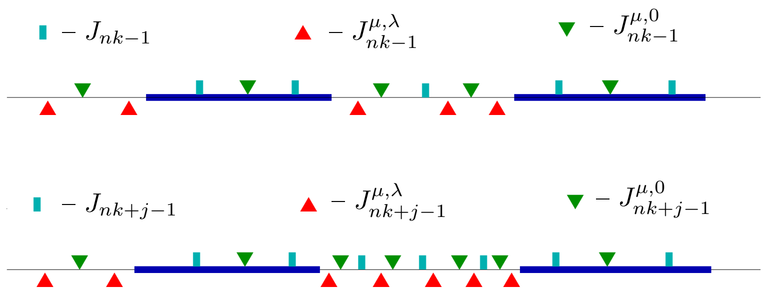

Figure 3: The typical distribution of the eigenvalues of the matrices J_{kn-1}, $J^{\mu,0}_{nk-1}$, and $J^{\mu,\lambda}_{nk-1}$ (upper picture) and J_{kn+j-1}, $J^{\mu,0}_{nk+j-1}$, and $J^{\mu,\lambda}_{nk+j-1}$ (lower picture).

and $I_{\ell+1}$ there are at most N_j+2 eigenvalues of the perturbed matrix $J^{\mu,\lambda}_{kn+j-1}$. □

The next proposition gives a bound for the distance between two consecutive eigenvalues of J_{kn+k-1} inside each interval I_ℓ ($\ell = 1,\dots,k$), assuming that all these intervals are disjoint.

Theorem 3.3 *Assume the conditions of Theorem* 2.1 *as well as the conditions* $\pi_k(\xi_i) \neq \pm\alpha$ *for all* $i = 1,\cdots,k-1$. *Let* $z_{\ell,n,1} < z_{\ell,n,2} < \cdots < z_{\ell,n,n}$ *be the* n *eigenvalues of* J_{kn+k-1} *that lie in the interior of the interval* I_ℓ ($\ell = 1,2,\cdots k$). *Then*

$$|z_{\ell,n,\nu+1} - z_{\ell,n,\nu}| \le \frac{\varrho_\ell}{n+1}, \quad \varrho_\ell := \frac{\alpha\pi}{\min_{x\in I_\ell}|\pi_k'(x)|} \tag{3.2}$$

for all $\ell = 1,2,\cdots,k$ *and* $\nu = 1,2,\cdots,n-1$. *Moreover, the interlacing property*

$$z_{\ell,n,\nu} < z_{\ell,n-1,\nu} < z_{\ell,n,\nu+1} < z_{\ell,n-1,\nu+1} < z_{\ell,n,\nu+2} \tag{3.3}$$

holds for all $\ell = 1,2,\cdots,k$, $\nu = 1,2,\cdots,n-2$, *and* $n \ge 3$.

Proof. According to (2.6) the eigenvalues of J_{nk+k-1} inside the interior of Σ_k are the kn roots of the equations

$$\pi_k(x) = y_{n,\nu} := \alpha\cos\frac{\nu\pi}{n+1} \quad (\nu = 1,2,\cdots,n)\,.$$

Notice that for each fixed ν this equation has k roots which are distributed over the k intervals I_ℓ in such a way that there is exactly one root in each I_ℓ. By the mean value Theorem we know that for all real numbers a and b we may write $\pi_k(a) - \pi_k(b) = \pi_k'(\xi)(a-b)$ for some ξ such that $a < \xi < b$. Hence taking $a = z_{\ell,n,\nu+1}$ and $b = z_{\ell,n,\nu}$ we deduce

$$|z_{\ell,n,\nu+1} - z_{\ell,n,\nu}| \le \frac{|y_{n,\nu+1} - y_{n,\nu}|}{\min_{x\in I_\ell}|\pi_k'(x)|} \le \frac{1}{\min_{x\in I_\ell}|\pi_k'(x)|}\,\frac{\alpha\pi}{n+1} = \frac{\varrho_\ell}{n+1}\,.$$

Let us point out that π_k is monotone in each interval I_ℓ ($\ell = 1,\cdots,k$). This fact together with the hypothesis $\pi_k(\xi_i) \neq \pm\alpha$ for every $i = 1,\cdots,k-1$ implies $\pi_k(\xi_i) \notin [-\alpha,\alpha]$ for all $i = 1,\cdots,k-1$, hence $\pi_k'(x) \neq 0$ for all $x \in I_\ell$, $\ell = 1,\cdots,k$.

Finally, (3.3) follows from (2.6), taking into account the interlacing property of the zeros of the Chebyshev polynomials $U_n(x)$, as well as the monotonicity of the function π_k in I_ℓ, for every $\ell = 1,\cdots,k$. □

Remark 3.4 *It is clear that every* ϱ_ℓ *in* (3.2) *may be replaced by a uniform upper bound (independent of* ℓ*), say,* ϱ*. For instance,*

$$\varrho_\ell \leq \varrho := \alpha\pi / \min_{x\in\Sigma_k} |\pi_k'(x)| \ , \quad \ell = 1, \cdots, k \ . \tag{3.4}$$

Remark 3.5 *For the case* $k = 1$ *and* $k = 2$ *the above estimates can be easily sharpenned. In fact from (2.8) and after some straigthforward computations using the mean value Theorem, one can obtain the following results.*

1. *The* $(1-)$*Toeplitz matrix* J_n*. Let* $\beta_s = a$ *and* $\gamma_s = b^2$*,* $a, b \in \mathbb{R}$*,* $b > 0$ *for all* s*. Then all the eigenvalues* $z_{n,\nu} := z_{1,n,\nu}$ *of* J_n *lie inside the interval* $I_1 = [a - 2b, a + 2b]$*, and*

$$|z_{n,\nu+1} - z_{n,\nu}| \leq \frac{2b\pi}{n+1}, \quad \nu = 1, \ldots, n-1.$$

2. *The* $2-$*Toeplitz matrix* J_{2n+1}*. Let*

$$\beta_s = \begin{cases} a & \text{if } s \text{ is even} \\ b & \text{if } s \text{ is odd} \end{cases} \quad \textit{and} \quad \gamma_s = \begin{cases} c^2 & \text{if } s \text{ is even} \\ d^2 & \text{if } s \text{ is odd} \end{cases}$$

for all s*, with* $a, b \in \mathbb{R}$*,* $c, d > 0$ *and* $|a - b| + |c - d| \neq 0$*, and let* $r = \sqrt{|c-d|^2 + |(a-b)/2|^2}$ *and* $s = \sqrt{|c+d|^2 + |(a-b)/2|^2}$*. Then, there are* n *eigenvalues* $z_{1,n,\nu}$ *inside the interval* $I_1 = \left[\frac{a+b}{2} - s, \frac{a+b}{2} - r\right]$*,* n *eigenvalues* $z_{2,n,\nu}$ *inside* $I_2 = \left[\frac{a+b}{2} + r, \frac{a+b}{2} + s\right]$*, and the remaining eigenvalue,* a*, lies between these two intervals. Furthermore, the distance between two consecutive eigenvalues inside the intervals* I_1 *or* I_2 *satisfy the inequality*

$$|z_{\ell,n,\nu+1} - z_{\ell,n,\nu}| \leq \frac{cd}{\sqrt{\left(\frac{a-b}{2}\right)^2 + (c-d)^2}} \frac{\pi}{n+1},$$

for all $\ell = 1, 2$ *and* $\nu = 1, 2, \ldots, n-1$*.*

From Theorem 3.3 and Theorem 2.1-(iii) we can obtain a bound similar to (3.2) for the distance of the eigenvalues of the matrices J_{nk+j-1} $(j = 1, \cdots, k-1)$ inside Σ_k. Denote by $z^{(j)}_{\ell,n,1} < \cdots < z^{(j)}_{\ell,n,n_{j,n}(\ell)}$ the eigenvalues of J_{kn+j-1} inside the interval I_ℓ $(\ell = 1, 2, \cdots, k)$. Notice that, according to Theorem 2.1-(iii), the numbers N_j, L_j and M_j satisfy the uniform bounds

$$1 \leq N_j \leq \lfloor k/2 \rfloor + 1 \ , \quad 0 \leq L_j, M_j \leq \lfloor (k+1)/2 \rfloor$$

for all $j = 0, 1, \cdots, k$, hence from (2.5) one sees that the number $n_{j,n}(\ell)$ of eigenvalues of J_{nk+j-1} inside I_ℓ satisfies

$$n - \lfloor (k+1)/2 \rfloor \leq n - L_j \leq n_{j,n}(\ell) \leq n + M_j - 1 \leq n + \lfloor (k-1)/2 \rfloor$$

for all $j = 0, 1, \cdots, k$ and $\ell = 1, \cdots, k$. This implies

$$|n_{j,n}(\ell) - n| \leq \max\{L_j, M_j - 1\} \leq \lfloor (k+1)/2 \rfloor$$

for all $j = 0, 1, \cdots, k$ and $\ell = 1, \cdots, k$.

Theorem 3.6 *Under the conditions of Theorem* 3.3*,*

$$\left| z^{(j)}_{\ell,n,\nu} - z_{\ell,n,\nu} \right| \leq \frac{j\varrho_\ell}{n+1} < \frac{k\varrho}{n+1} \tag{3.5}$$

for all $j = 0, 1, \cdots, k-1$*,* $\ell = 1, \cdots, k$*, and* $\nu = j+1, \cdots, \min\{n_{j,n}(\ell), n-1\}$*, where* ϱ *is given by* (3.4)*.*

Proof. It follows from Theorem 3.3 and taking into account the interlacing properties fulfilled by the eigenvalues of the matrices $J_{nk-1}, J_{nk}, \cdots, J_{nk+j-1}$. In fact, assume first $j=1$. By the interlacing property between the eigenvalues of J_{nk-1} and J_{nk}, it follows that

$$\left|z^{(1)}_{\ell,n,\nu} - z_{\ell,n,\nu}\right| \le |z_{\ell,n,\nu} - z_{\ell,n,\nu+1}| \le \frac{\rho_\ell}{n+1}$$

for all $\nu = 2,3,\cdots,n-1$, the last inequality being justified by (3.2), and so the desired result follows for $j=1$. In a similar way, for $j=2$, by the interlacing properties between the eigenvalues of J_{nk-1}, J_{nk} and J_{nk+1}, we get

$$\begin{aligned}&\left|z^{(2)}_{\ell,n,\nu} - z_{\ell,n,\nu}\right| \\ &\quad \le \left|z^{(1)}_{\ell,n,\nu} - z_{\ell,n,\nu}\right| + \left|z^{(1)}_{\ell,n,\nu+1} - z^{(1)}_{\ell,n,\nu}\right| + \left|z^{(1)}_{\ell,n,\nu+1} - z_{\ell,n,\nu+2}\right| \\ &\quad = |z_{\ell,n,\nu} - z_{\ell,n,\nu+2}| = |z_{\ell,n,\nu} - z_{\ell,n,\nu+1}| + |z_{\ell,n,\nu+1} - z_{\ell,n,\nu+2}| \le \frac{2\rho_\ell}{n+1}\end{aligned}$$

for all $\nu = 3,4,\cdots,\min\{n_{2,n}(\ell), n-1\}$. In a similar way, we get the desired result for any j. ☐

Theorem 3.7 *Under the conditions of Theorem* 3.3*, let*

$$z^{(j,\mu,\lambda)}_{\ell,n,1} < \cdots < z^{(j,\mu,\lambda)}_{\ell,n,n'_{j,n}(\ell)}, \qquad j = 0,1,\cdots,k-1,$$

be the eigenvalues of $J^{\mu,\lambda}_{kn+j-1}$ inside the interior of the interval I_ℓ ($\ell = 1,\cdots,k$). Then the following bounds

$$\left|z^{(j,\mu,\lambda)}_{\ell,n,\nu} - z_{\ell,n,\nu}\right| \le \frac{(5j+2)\varrho_\ell}{n+1} \le \frac{(5k-3)\varrho}{n+1}$$

hold for all $\nu = j+2,\cdots,\min\{n_{j,n}(\ell), n'_{j,n}(\ell), n-1\} - 1$.

Proof. From (3.2) and (3.5) and using the triangle inequality we see that the eigenvalues $z^{(j)}_{\ell,\nu}$ of the matrices J_{nk+j-1} that lies inside the interval I_ℓ satisfy

$$\begin{aligned}\left|z^{(j)}_{\ell,n,\nu} - z^{(j)}_{\ell,n,\nu+1}\right| &\le \left|z^{(j)}_{\ell,n,\nu} - z_{\ell,n,\nu}\right| + \left|z^{(j)}_{\ell,n,\nu+1} - z_{\ell,n,\nu+1}\right| + \left|z_{\ell,n,\nu+1} - z_{\ell,n,\nu}\right| \\ &\le \frac{(2j+1)\varrho_\ell}{n+1}\end{aligned}$$

for all $\nu = j+1,\cdots,\min\{n_{j,n}(\ell), n-1\}$ and $\ell = 1,\cdots,k$. On the other hand, by (ii) and (iii) in Proposition 3.1, for all the eigenvalues $z^{(j,\mu,\lambda)}_{\ell,n,\nu}$ and $z^{(j)}_{\ell,n,\nu}$ of the matrices $J^{(j,\mu,\lambda)}_{nk+j-1}$ and J_{nk+j-1} that are inside the interior of I_ℓ we can write

$$\left|z^{(j,\mu,\lambda)}_{\ell,n,\nu} - z^{(j)}_{\ell,n,\nu}\right| \le 2\max_\nu \left|z^{(j)}_{\ell,n,\nu} - z^{(j)}_{\ell,n,\nu+1}\right| \le \frac{2(2j+1)\varrho_\ell}{n+1}$$

for all $\nu = j+2,\cdots,\min\{n_{j,n}(\ell), n'_{j,n}(\ell), n-1\} - 1$ and $\ell = 1,\cdots,k$. As a consequence, for all the eigenvalues $z^{(j,\mu,\lambda)}_{\ell,n,\nu}$ and $z_{\ell,n,\nu}$ of the matrices $J^{(j,\mu,\lambda)}_{nk+j-1}$ and J_{nk-1} that are inside I_ℓ one finds

$$\left|z^{(j,\mu,\lambda)}_{\ell,n,\nu} - z_{\ell,n,\nu}\right| \le \left|z^{(j,\mu,\lambda)}_{\ell,n,\nu} - z^{(j)}_{\ell,n,\nu}\right| + \left|z^{(j)}_{\ell,n,\nu} - z_{\ell,n,\nu}\right| \le \frac{(5j+2)\varrho_\ell}{n+1}$$

for all $\nu = j+1,\ldots,\min\{n_{j,n}(\ell), n'_{j,n}(\ell), n-1\} - 1$ and $\ell = 1,\cdots,k$. ☐

Remark 3.8 *From Theorem 3.7 we see that for large n, up to a number independent of n, the eigenvalues of the matrix $J^{\mu,\lambda}_{nk+j-1}$ may be approximated by the eigenvalues of J_{nk-1} Henceforth, for large n most of the eigenvalues of $J^{\mu,\lambda}_{nk+j-1}$ are close enough to the solutions of the equation* (2.8)*.*

4 Examples and numerical experiments

4.1 Perturbation of a tridiagonal 3−Toeplitz matrix

Let us consider matrices with the following structure

$$H = \begin{pmatrix} a_1 & 0 \\ 0 & A_N \end{pmatrix}, \tag{4.1}$$

where A_N is a tridiagonal matrix. The eigenproblem

$$\begin{pmatrix} a_1 & 0 \\ 0 & A_N \end{pmatrix} \begin{pmatrix} x_1 \\ X \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ X \end{pmatrix},$$

always has the eigenvalue $\lambda = a_1$ and the eigenvector $(1, 0, \ldots, 0)^T$. To compute the remaining eigenvectors, $(0, X)^T$, and the corresponding eigenvalues, λ, we need to solve the eigenproblem

$$X \neq 0, \quad A_N X = \lambda X. \tag{4.2}$$

Here we will concentrate our attention in the case when A_N is a perturbed tridiagonal k-Toeplitz matrix. This structure is motivated by certain physical models (cf. e.g. [2]). An example of such a matrix is

$$H_N^{(3)} = \begin{pmatrix} a_1 & 0 \\ 0 & A_N \end{pmatrix}, \tag{4.3}$$

where $N = 3n + 2$ and A_N is

$$A_{3n+2} = \begin{pmatrix}
a_2 & b_2 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
b_2 & a_3 & b_3 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & b_3 & a_4 & b_4 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & b_4 & a_5 & b_2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & b_2 & a_3 & b_3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & b_3 & a_4 & b_4 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & b_4 & a_5 & b_2 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b_2 & a_3 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & b_2 & a_3 & b_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & b_3 & a_4 & b_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b_4 & a_5 & b_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & b_2 & a_2
\end{pmatrix}, \tag{4.4}$$

so that A_N is a perturbed tridiagonal 3-Toeplitz matrix. As pointed out before, $H_N^{(3)}$ has the eigenvalue $\lambda_0 = a_1$ corresponding to the eigenvector $(1, 0, \ldots, 0)^T$. To obtain the other eigenvalues we will suppose that n is large enough and we change the first entry of the diagonal of A_N by a_5 and the last entry by a_3, hence we obtain a (non-perturbed) tridiagonal 3-Toeplitz matrix, say, $\widetilde{A}_N$. According to the results in [22], or from (3.10)–(3.12) in [1], the eigenvalues λ_ℓ ($\ell = 1, 2, \ldots, 3n + 2$) of $\widetilde{A}_N \equiv \widetilde{A}_{3n+2}$ are

$$\lambda_1 = \frac{a_3 + a_5 - \sqrt{(a_3 - a_5)^2 + 4b_2^2}}{2}, \; \lambda_2 = \frac{a_3 + a_5 + \sqrt{(a_3 - a_5)^2 + 4b_2^2}}{2}$$

and the $3n$ solutions of the cubic equations

$$x^3 - (a_3 + a_4 + a_5)x^2 + (a_3 a_5 + a_4 a_5 + a_3 a_4 - b_2^2 - b_3^2 - b_4^2)x$$
$$+ a_4 b_2^2 + a_5 b_3^2 + a_3 b_4^2 - a_3 a_4 a_5 + 2 b_2 b_3 b_4 \cos \frac{k\pi}{n+1} = 0,$$

for $k = 1, 2, \ldots, n$. The corresponding eigenvectors are

$$\mathbf{v}_\ell = (0, S_0(\lambda_\ell), S_1(\lambda_\ell), \ldots, S_{3n+1}(\lambda_\ell))^T , \quad \ell = 1, 2, \ldots, 3n+2,$$

where $(S_\nu)_\nu$ is a sequence of orthonormal polynomials, defined explicitly by

$$S_{3k}(\lambda) = U_k\left(\varphi_3(\lambda)\right) + \frac{b_4}{b_2 b_3}(\lambda - a_3)\, U_{k-1}\left(\varphi_3(\lambda)\right) \ ,$$

$$S_{3k+1}(\lambda) = \frac{(\lambda - a_5)}{b_2} U_k\left(\varphi_3(\lambda)\right) + \frac{b_4}{b_3} U_{k-1}\left(\varphi_3(\lambda)\right) \ ,$$

$$S_{3k+2}(\lambda) = \frac{(\lambda - \lambda_1)(\lambda - \lambda_2)}{b_2 b_3} U_k\left(\varphi_3(\lambda)\right)$$

for all $k = 0, 1, 2, \ldots$, being

$$\varphi_3(x) := \frac{1}{2b_2b_3b_4}\left\{(x-a_3)(x-a_4)(x-a_5) - (b_2^2+b_3^2+b_4^2)(x-a_4)\right.$$
$$\left. +(a_5-a_4)b_3^2 + (a_3-a_4)b_4^2\right\},$$

and U_k is the Chebyshev polynomial of the second kind of degree k, defined in (2.7). Notice that, according to Theorem 3.7, the eigenvalues of A_N and $\widetilde{A}_N$ are close enough for N large.

4.2 Some numerical experiments

To conclude this section let us briefly discuss some numerical results. We have computed the eigenvalues corresponding to the matrices A_N and $\tilde{A}_N$ by finding an accurate agreement between the numerical and the analytical results. As expected, the numerical results confirm that the interlacing property (iv) in Proposition 3.1 holds. As an example, consider the perturbed tridiagonal $1-$Toeplitz matrix

$$A_N = \begin{pmatrix} a_2 & b_2 & 0 & \cdots & 0 & 0 \\ b_2 & a_3 & b_2 & \cdots & 0 & 0 \\ 0 & b_2 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_3 & b_2 \\ 0 & 0 & 0 & \cdots & b_2 & a_4 \end{pmatrix}. \tag{4.5}$$

In Figure 4 we show the interlacing property between the $12-$th and $15-$th eigenvalues λ_{A_N} (with stars) and $\lambda_{\tilde{A}_N}$ (open circles) of the matrices A_N and $\tilde{A}_N$ (resp.) where we have choosing $b_2 = 1/2$, $a_1 = 9/2$, $a_2 = 5/2$, $a_3 = 4$, and $N = 21$. Notice that between the $13-$th and $14-$th eigenvalues of the perturbed matrix A_N there exist two eigenvalues of the matrix $\tilde{A}_N$ as it is stated in in Proposition 3.1(iv).

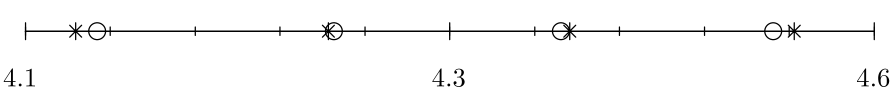

Figure 4: The interlacing property of the eigenvalues of A_N and $\tilde{A}_N$.

Next, we will include some numerical simulations for perturbed tridiagonal k-Toeplitz matrices, for several choices of k. We start with two examples with $k = 3$. In this case there are three disjoint intervals where almost all the eigenvalues lie, and only few of them are out of these intervals (see Fig. 5). The left panel is an example for the case of a matrix A_N of the form (4.4) where $a_2 = 4$, $a_3 = 2$, $a_4 = 6$, $a_5 = 1$, $b_2 = 2$, $b_3 = 3$, and $b_4 = 4$, and in the right panel we may see the eigenvalues distribution of the same matrix but now when the last diagonal element is $a_4 = 9.75$ (the first element remains the same, $a_2 = 2$). Many other examples we have simulated are in accordance with Theorem 3.7.

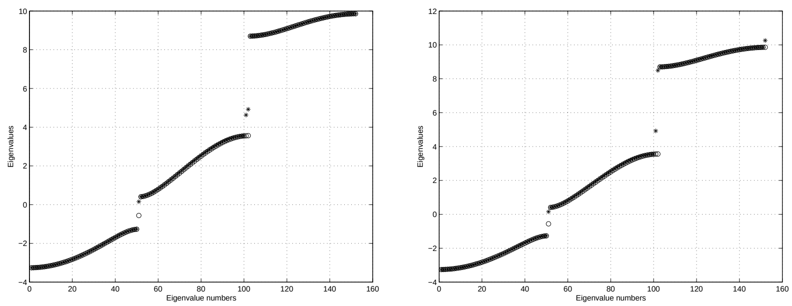

Figure 5: The eigenvalues λ_A (with stars) and $\lambda_{\tilde{A}}$ (using open circles) for tridiagonal 3−Toeplitz matrices when $n = 50$, $N = 3\,n{+}2$. The values of the remaining parameters are described in the text.

Finally, in Fig. 6 a plot of a *typical* example of 5 and 7-Toeplitz matrices is considered. In the first case the diagonal is a repetition of the elements $[1, 5, 3, 3, 2]$ and the supper and subdiagonal are $[1, 5, 4, 4, 5]$, respectively. As before, using stars we plot the eigenvalues of the perturbed matrix and by circles the eigenvalues of the unperturbed ones. As we can see in Fig. 6 (left panel) there are 5 disjoint intervals. In the second case we have a 7−Toeplitz matrix obtained by repeating the elements $[1, 5, 3, 3, 3, 2, 1]$ and $[1, 5, 4, 4, 5, 2, 1]$ in the diagonal and in the supper and subdiagonal, respectively, and the perturbation parameters are $\mu = 2$ and $\lambda = 1.5$.

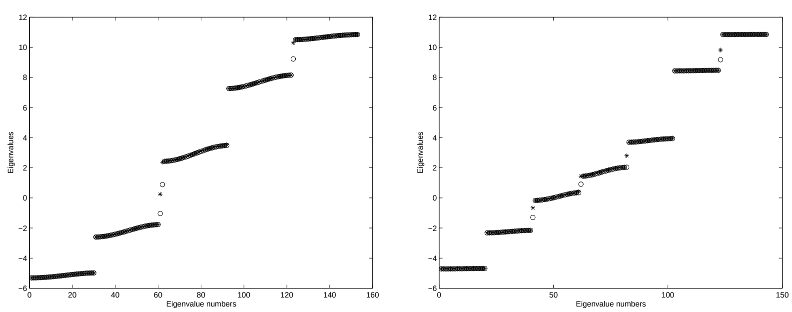

Figure 6: The eigenvalues λ_A (with stars) and $\lambda_{\tilde{A}}$ (using open circles). Parameters of the numerical simulations are: in left panel, $k = 5$, $n = 30$, $N = 5\,n + 3$; in right panel, $k = 7$, $n = 20$, $N = 7n + 3$. The parameters a_i and b_i are defined in the text.

Programs: The numerical simulations presented here have been obtained by using the commercial program MATLAB. The source code can be obtained by request via e-mail to `niurka@euler.us.es` or `ran@us.es`.

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Appendix A

Here we will deduce the solution of the eigenproblem for the matrix

$$H_N^{(1)} = \begin{pmatrix} a_1 & 0 \\ 0 & A_N \end{pmatrix},$$

where A_N is the perturbed 1−Toeplitz matrix (4.5). Using (4.2) we have that one eigenvalue of $H_N^{(1)}$ is $\lambda_0 = a_1$ and an associated eigenvector is $(1, 0, \ldots, 0)^T$. In order to obtain the other eigenvalues, put $a_2 = a_3 + \lambda$ and $a_4 = a_3 + \mu$, and consider the tridiagonal 1−Toeplitz matrix

$$\widetilde{A}_N = \begin{pmatrix} a_3 & b_2 & 0 & \cdots & 0 & 0 \\ b_2 & a_3 & b_2 & \cdots & 0 & 0 \\ 0 & b_2 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_3 & b_2 \\ 0 & 0 & 0 & \cdots & b_2 & a_3 \end{pmatrix}.$$

It is well known (see e.g. [17]) that the eigenvalues of this matrix are

$$\lambda_k = a_3 + 2b_2 \cos\left(\frac{k\pi}{N+1}\right), \quad k = 1, 2, \ldots, N,$$

and the corresponding eigenvectors are

$$\mathbf{v}_\ell = \left(0, S_0(\lambda_\ell), S_1(\lambda_\ell), \ldots, S_{N-1}(\lambda_\ell)\right)^T, \quad \ell = 1, 2, \ldots, N,$$

where

$$S_k(\lambda) = U_k\left(\frac{\lambda - a_3}{2b_2}\right), \quad k = 0, 1, 2, \ldots,$$

being U_k the Chebyshev polynomial of the second kind of degree k defined in (2.7). Hence, one sees that

$$S_k(\lambda_\ell) = \frac{\sin\frac{(k+1)\ell\pi}{N+1}}{\sin\frac{\ell\pi}{N+1}}, \quad \ell = 1, 2, \ldots, N; \quad k = 0, 1, 2, \ldots,$$

and so

$$\mathbf{v}_\ell = \frac{1}{\sin\frac{\ell\pi}{N+1}}\left(0, \sin\frac{\ell\pi}{N+1}, \sin\frac{2\ell\pi}{N+1}, \ldots, \sin\frac{N\ell\pi}{N+1}\right)^T, \quad \ell = 1, 2, \ldots, N.$$

Notice that, according to Theorem 3.7, for N large enough the eigenvalues of A_N are close enough to the eigenvalues of $\widetilde{A}_N$.