# Asymptotic properties of generalized Laguerre orthogonal polynomials

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**Abstract**

In the present paper we deal with the polynomials $L_n^{(\alpha,M,N)}(x)$ orthogonal with respect to the Sobolev inner product

$$(p,q)=\frac{1}{\Gamma(\alpha{+}1)}\int_0^{\infty}p(x)q(x)\,x^{\alpha}\,e^{-x}dx+M\,p(0)q(0)+N\,p'(0)q'(0),\quad N,M\geq 0,\ \alpha>-1,$$

firstly introduced by Koekoek and Meijer in 1993 and extensively studied in the last years. We present some new asymptotic properties of these polynomials and also a limit relation between the zeros of these polynomials and the zeros of Bessel function $J_\alpha(x)$. The results are illustrated with numerical examples. Also, some general asymptotic formulas for generalizations of these polynomials are conjectured.

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## 1 Introduction

In this paper we will deal mainly with the polynomials which are orthogonal with respect to the Sobolev-type inner product

$$(p,q)=\frac{1}{\Gamma(\alpha+1)}\int_0^{\infty}p(x)q(x)\,x^{\alpha}\,e^{-x}dx+M\,p(0)q(0)+N\,p'(0)q'(0)\,,\tag{1}$$

where $M,N\geq 0$ and $\alpha>-1$. These polynomials were introduced by Koekoek and Meijer in [10] and constitute a natural generalization of the so–called Koornwinder's generalized Laguerre

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polynomials, earlier introduced by Koornwinder in [11], which are orthogonal with respect to (1) where $N = 0$ (for details see e.g. [8, 11]).

In the following we will denote $\left(L_n^{(\alpha,M,N)}(x)\right)_n$ the sequence of orthogonal polynomials with respect to (1). In [10] the authors established different properties of the polynomials $\left(L_n^{(\alpha,M,N)}(x)\right)_n$ such as the differential equation that they satisfy, a five–term recurrence relation, a Christoffel–Darboux type formula, a representation as a hypergeometric series ${}_3F_3$ and some properties of their zeros (being one of them the fact that the zeros of these polynomials are all real and simple). Later, although it was published earlier, Koekoek in [9] considered the more general inner product

$$(p,q)_g = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty p(x)q(x)\, x^\alpha\, e^{-x} dx + \sum_{i=0}^{s} M_i\, p^{(i)}(0) q^{(i)}(0)\,, \tag{2}$$

where $M_i \geq 0$, $i = 0, \ldots, s$ and $\alpha > -1$ and studied some properties of the corresponding orthogonal polynomials with respect to the inner product (2), usually called the discrete Sobolev-type Laguerre which constitutes an instance of a larger class of orthogonal polynomials: the discrete Sobolev-type orthogonal polynomials. For more detailed description of this Sobolev-type orthogonal polynomials (including the continuous ones) we refer the readers to the recent reviews [12, 14, 15].

Our main aims here are two:

1. To fill a gap in the study of the polynomials $L_n^{(\alpha,M,N)}(x)$ orthogonal with respect to the inner product (1), that is, to obtain new asymptotic properties such as strong asymptotics, Plancherel–Rotach type asymptotics and Mehler–Heine type formulas. This will be done in Section 3, theorems 1 and 2.

2. To establish limit relations when $n \to \infty$ between the zeros of the polynomial $L_n^{(\alpha,M,N)}(x)$ and the zeros of Bessel functions $J_\alpha$, $J_{\alpha+2}$, $J_{\alpha+4}$ or their combinations according to the values of the masses $N$ and $M$. This will be done in Section 4, theorem 3.

These kind of problems have been also considered for the Jacobi Sobolev–type orthogonal polynomials (see [2]) and for the continuous Sobolev orthogonal polynomials (see, e.g., [4, 13]). Also we will show that the technique used here for the $L_n^{(\alpha,M,N)}(x)$ polynomials can be easily extended to another family of orthogonal polynomials corresponding to a *non-diagonal* case introduced later on in [5, 6]. This will be done in section 4.1.

The structure of the paper is as follows: In section 2, some preliminary results are quoted. In section 3, the asymptotics of the polynomials orthogonal with respect to the inner product (1) is deduced that allows us, in section 4, to obtain some interesting properties of the zeros of these generalized polynomials as well as to set out a conjecture about the asymptotic behavior of the polynomials orthogonal with respect to (2). In section 4.1 an example of a non-diagonal case will be discussed briefly and, finally, in section 4.2 some numerical examples illustrating the above results are presented.

## 2 Preliminaries

### 2.1 The classical Laguerre polynomials

The classical Laguerre polynomials are defined by (see e.g. [16])

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!},$$

where $\binom{\alpha}{\beta}$ are the binomial coefficients.

**Proposition 1** *Let $\left(L_n^{(\alpha)}(x)\right)_n$ be the sequence of Laguerre polynomials with leading coefficients $(-1)^n/n!$. They verifies the following properties:*

*(a) For $\alpha \in \mathbb{R}$, ([16, f. (5.1.1)])*

$$L_n^{(\alpha)}(x) - L_{n-1}^{(\alpha)}(x) = L_n^{(\alpha-1)}(x). \tag{3}$$

*(b) Strong asymptotics (Perron's formula) on $\mathbb{C} \setminus \mathbb{R}$ ([16, Th. 8.22.3]). Let $\alpha \in \mathbb{R}$. Then*

$$L_n^{(\alpha)}(x) = 2^{-1}\pi^{-1/2} e^{x/2} (-x)^{-\alpha/2-1/4} n^{\alpha/2-1/4} e^{2(-nx)^{1/2}} \left(1 + O\left(n^{-1/2}\right)\right). \tag{4}$$

*This relation holds for $x$ in the complex plane cut along the positive real semiaxis; both $(-x)^{-\alpha/2-1/4}$ and $(-x)^{1/2}$ must be taken real and positive if $x < 0$. The bound of the remainder holds uniformly in every closed domain which does not overlap the positive real semiaxis.*

*(c) It holds ([16, Section 8.22 and formula (1.71.7)])*

$$\frac{L_n^{(\alpha)}(x)}{n^{\alpha/2}} = e^{x/2} x^{-\alpha/2} J_\alpha(2\sqrt{nx}) + O(n^{-3/4}), \tag{5}$$

*uniformly on compact subsets of $(0, +\infty)$ where $J_\alpha$ is the Bessel function.*

*(d) Mehler–Heine type formula ([16, Th. 8.1.3])*

$$\lim_{n\to\infty} \frac{L_n^{(\alpha)}(x/(n+j))}{n^\alpha} = x^{-\alpha/2} J_\alpha(2\sqrt{x}), \tag{6}$$

*uniformly on compact subsets of $\mathbb{C}$ and uniformly on $j \in \mathbb{N} \cup \{0\}$.*

*(e) Scaled asymptotics on $\mathbb{C} \setminus [0,4]$. It holds ([7])*

$$\lim_{n\to\infty} 2^n \sqrt{2\pi n}\, (-1)^n L_n^{(\alpha)}(nx) \left(x - 2 + \sqrt{x^2-4x}\right)^{-n} \exp\left(\frac{-2nx}{x+\sqrt{x^2-4x}}\right) =$$
$$2^{-\alpha-1/2} x^{-\alpha} \left(x-2+\sqrt{x^2-4x}\right)^{1/2} \left(x+\sqrt{x^2-4x}\right)^{\alpha} \left(x^2-4x\right)^{-1/4}, \tag{7}$$

*uniformly on compact subsets of $\mathbb{C} \setminus [0,4]$ taking into account that the square roots in (7) are negative if $x$ is negative.*

**Remark 1** Although the Mehler–Heine type formula for Laguerre polynomials in Szegő's book is (6) with $j = 0$, it can be shown that this formula is true for $j \in \mathbb{N}$ such as it appears in (6).

On the other hand, formulas (4) and (7) allow us to obtain the ratio asymptotics for Laguerre and scaled Laguerre orthogonal polynomials ($\alpha > -1$), respectively. In fact, from (4) we deduce

$$\lim_{n\to\infty} n^{(\ell-j)/2} \frac{L_{n+k}^{(\alpha+j)}(x)}{L_{n+h}^{(\alpha+\ell)}(x)} = (-x)^{(\ell-j)/2}, \quad j,\ell \in \mathbb{R}, \quad h,k \in \mathbb{Z}. \tag{8}$$

uniformly on compact subsets of $\mathbb{C} \setminus [0,\infty)$. We will use this result with $j,\ell \in \mathbb{Z}$.

Also, from (7) we get, for $j \in \mathbb{N} \cup \{0\}$,

$$\lim_{n\to\infty} \frac{L_{n-1}^{(\alpha)}((n+j)x)}{L_n^{(\alpha)}((n+j)x)} = -\frac{1}{\varphi\left((x-2)/2\right)}, \tag{9}$$

uniformly on compact subsets of $\mathbb{C} \setminus [0,4]$ where $\varphi$ is the conformal mapping of $\mathbb{C} \setminus [-1,1]$ onto the exterior of the unit circle given by

$$\varphi(x) = x + \sqrt{x^2-1}, \quad x \in \mathbb{C} \setminus [-1,1], \tag{10}$$

with $\sqrt{x^2-1} > 0$ when $x > 1$.

### 2.2 Generalization of Laguerre polynomials

In [10] Koekoek and Meijer establish that the orthogonal polynomials, $L_n^{(\alpha,M,N)}(x)$, with respect to the inner product (1), $M, N \geq 0$, $\alpha > -1$, can be rewritten in terms of the Laguerre polynomials, $L_n^{(\alpha)}(x)$,

$$L_n^{(\alpha,M,N)}(x) = B_0(n)L_n^{(\alpha)}(x) + B_1(n)xL_{n-1}^{(\alpha+2)}(x) + B_2(n)x^2L_{n-2}^{(\alpha+4)}(x), \quad n \geq 0, \tag{11}$$

where it is assumed $L_i^{(\alpha)}(x) = 0$, for $i = -1, -2$ and

$$B_0(n) = 1 - \frac{N}{\alpha+1}\binom{n+\alpha+1}{n-2}, \;\; B_1(n) = -\frac{M}{\alpha+1}\binom{n+\alpha}{n} - \frac{(\alpha+2)N}{(\alpha+1)(\alpha+3)}\binom{n+\alpha}{n-2}, \tag{12}$$

$$B_2(n) = \frac{N}{(\alpha+1)(\alpha+2)(\alpha+3)}\binom{n+\alpha}{n-1} + \frac{MN}{(\alpha+1)^2(\alpha+2)(\alpha+3)}\binom{n+\alpha}{n}\binom{n+\alpha+1}{n-1}. \tag{13}$$

Notice that using (11) and the fact that the leading coefficients of the Laguerre polynomials are $(-1)^n/n!$ we deduce that the leading coefficients of $L_n^{(\alpha,M,N)}(x)$ are

$$\frac{(-1)^n}{n!}\left(B_0(n) - nB_1(n) + n(n-1)B_2(n)\right).$$

Following [10] we can use the above formulas (12–13) to obtain the asymptotics of the coefficients in (11):

Case $M > 0$, $N = 0$.

$$B_0(n) = 1, \quad \lim_{n\to\infty}\frac{B_1(n)}{n^\alpha} = \frac{-M}{\Gamma(\alpha+2)}, \quad B_2(n) = 0.$$

Case $M = 0$, $N > 0$.

$$\lim_{n\to\infty} \frac{B_0(n)}{n^{\alpha+3}} = \frac{-N}{(\alpha+1)\Gamma(\alpha+4)}, \quad \lim_{n\to\infty} \frac{B_1(n)}{n^{\alpha+2}} = \frac{-N(\alpha+2)}{(\alpha+1)\Gamma(\alpha+4)},$$
$$\lim_{n\to\infty} \frac{B_2(n)}{n^{\alpha+1}} = \frac{N}{(\alpha+1)\Gamma(\alpha+4)}. \tag{14}$$

Case $M > 0$, $N > 0$.

$$\lim_{n\to\infty} \frac{B_0(n)}{n^{\alpha+3}} = \frac{-N}{(\alpha+1)\Gamma(\alpha+4)}, \quad \lim_{n\to\infty} \frac{B_1(n)}{n^{\alpha+2}} = \frac{-N(\alpha+2)}{(\alpha+1)\Gamma(\alpha+4)},$$
$$\lim_{n\to\infty} \frac{B_2(n)}{n^{2\alpha+2}} = \frac{MN}{(\alpha+1)\Gamma(\alpha+3)\Gamma(\alpha+4)}. \tag{15}$$

The polynomials $L_n^{(\alpha,M,N)}(x)$ verify several interesting properties [10]: they satisfy a second–order differential equation with polynomial coefficients of degree at most three, a five–term recurrence relation, and Christoffel–Darboux type formula and they can be represented as a generalized hypergeometric series ${}_3F_3$. Of particular significance to our work is the following:

**Theorem** [Koekoek, Meijer [10]] *The polynomial $L_n^{(\alpha,M,N)}(x)$ has $n$ real simple zeros. At least $n-1$ of them lie in $(0,+\infty)$. Furthermore, when $N > 0$ and $n$ large enough they have exactly one zero in $(-\infty, 0]$.*

Before concluding let us point out that some of the above properties and results have been extended to other more general polynomials. The reader interested in these results can consult, e.g., [1, 3, 9].

## 3 Asymptotic properties of generalized Laguerre polynomials

Along this section $B_i(n)$, $i = 0, 1, 2$, take the values given by (12–13), respectively.

**Theorem 1** *The polynomials $\left(L_n^{(\alpha,M,N)}(x)\right)_n$, with $\alpha > -1$, satisfy*

*(a) Exterior asymptotics. The following limits hold uniformly on compact subsets of $\mathbb{C} \setminus [0, +\infty)$,*

- *If $M > 0$ and $N = 0$,*

$$\lim_{n\to\infty} \frac{L_n^{(\alpha,M,0)}(x)}{n^{\alpha+1} L_n^{(\alpha)}(x)} = \frac{M}{\Gamma(\alpha+2)}.$$

- *If $M = 0$ and $N > 0$,*

$$\lim_{n\to\infty} \frac{L_n^{(\alpha,0,N)}(x)}{n^{\alpha+3} L_n^{(\alpha)}(x)} = \frac{(\alpha+2)\,N}{(\alpha+1)\Gamma(\alpha+4)}.$$

- *If $M > 0$ and $N > 0$,*

$$\lim_{n\to\infty} \frac{L_n^{(\alpha,M,N)}(x)}{n^{2\alpha+4} L_n^{(\alpha)}(x)} = \frac{MN}{(\alpha+1)\Gamma(\alpha+3)\Gamma(\alpha+4)}.$$

*(b) Asymptotics on compact subsets of* $(0,+\infty)$.

- *If* $M>0$ *and* $N=0$,

$$\frac{L_n^{(\alpha,M,0)}(x)}{n^{3\alpha/2+1}} = a(n)\, e^{x/2} x\, x^{-(\alpha+2)/2} J_{\alpha+2}\left(2\sqrt{(n-1)x}\right) + O\left(n^{-\min\{\alpha+5/4,3/4\}}\right), \qquad (16)$$

*where*

$$a(n) = \frac{B_1(n)}{n^{\alpha}} \left(\frac{n-1}{n}\right)^{\alpha/2+1} \longrightarrow \frac{-M}{\Gamma(\alpha+2)} \quad \textit{when} \quad n\to\infty\,.$$

- *If* $M=0$ *and* $N>0$,

$$\frac{L_n^{(\alpha,0,N)}(x)}{n^{3\alpha/2+3}} = e^{x/2}\Big(b_0(n)\, x^{-\alpha/2} J_{\alpha}\left(2\sqrt{nx}\right) + b_1(n)\, x\, x^{-(\alpha+2)/2} J_{\alpha+2}\left(2\sqrt{(n-1)x}\right) + b_2(n)\, x^2\, x^{-(\alpha+4)/2} J_{\alpha+4}\left(2\sqrt{(n-2)x}\right)\Big) + O\left(n^{-3/4}\right), \qquad (17)$$

*where*

$$b_0(n) = \frac{B_0(n)}{n^{\alpha+3}} \longrightarrow \frac{-N}{(\alpha+1)\Gamma(\alpha+4)} \quad \textit{when} \quad n\to\infty\,,$$

$$b_1(n) = \frac{B_1(n)}{n^{\alpha+2}} \left(\frac{n-1}{n}\right)^{\alpha/2+1} \longrightarrow \frac{-(\alpha+2)N}{(\alpha+1)\Gamma(\alpha+4)} \quad \textit{when} \quad n\to\infty\,,$$

$$b_2(n) = \frac{B_2(n)}{n^{\alpha+1}} \left(\frac{n-2}{n}\right)^{\alpha/2+2} \longrightarrow \frac{N}{(\alpha+1)\Gamma(\alpha+4)} \quad \textit{when} \quad n\to\infty\,.$$

- *If* $M>0$ *and* $N>0$,

$$\frac{L_n^{(\alpha,M,N)}(x)}{n^{5\alpha/2+4}} = c(n) e^{x/2}\, x^2\, x^{-(\alpha+4)/2} J_{\alpha+4}\left(2\sqrt{(n-2)x}\right) + O\left(n^{-\min\{\alpha+5/4,3/4\}}\right), \qquad (18)$$

*where*

$$c(n) = \frac{B_2(n)}{n^{2\alpha+2}} \left(\frac{n-2}{n}\right)^{\alpha/2+2} \longrightarrow \frac{-N}{(\alpha+1)\Gamma(\alpha+4)} \quad \textit{when} \quad n\to\infty\,.$$

**Proof:** (a) We will prove here only the case when $M,N>0$. The proof of the other cases can be done in a similar way. First, we divide (11) by $n^{2\alpha+4} L_n^{(\alpha)}(x)$

$$\frac{L_n^{(\alpha,M,N)}(x)}{n^{2\alpha+4} L_n^{(\alpha)}(x)} = \frac{B_0(n)}{n^{\alpha+3}} \frac{1}{n^{\alpha+1}} + \frac{B_1(n)}{n^{\alpha+2}} \frac{1}{n^{\alpha+1}} \frac{x L_{n-1}^{(\alpha+2)}(x)}{n L_n^{(\alpha)}(x)} + \frac{B_2(n)}{n^{2\alpha+2}} \frac{x^2 L_{n-2}^{(\alpha+4)}(x)}{n^2 L_n^{(\alpha)}(x)}\,.$$

Now, (8) and (15) yield

$$\lim_{n\to\infty} \frac{L_n^{(\alpha,M,N)}(x)}{n^{2\alpha+4} L_n^{(\alpha)}(x)} = \lim_{n\to\infty} \frac{B_2(n)}{n^{2\alpha+2}} \frac{x^2 L_{n-2}^{(\alpha+4)}(x)}{n^2 L_n^{(\alpha)}(x)} = \frac{MN}{(\alpha+1)\Gamma(\alpha+3)\Gamma(\alpha+4)}\,.$$

(b) We start considering the case $M=0$ and $N>0$. In this case, if we divide (11) by $n^{3\alpha/2+3}$

$$\frac{L_n^{(\alpha,0,N)}(x)}{n^{3\alpha/2+3}} = \frac{B_0(n)}{n^{\alpha+3}} \frac{L_n^{(\alpha)}(x)}{n^{\alpha/2}} + x \frac{B_1(n)}{n^{\alpha+2}} \frac{L_{n-1}^{(\alpha+2)}(x)}{n^{(\alpha+2)/2}} + x^2 \frac{B_2(n)}{n^{\alpha+1}} \frac{L_{n-2}^{(\alpha+4)}(x)}{n^{(\alpha+4)/2}}\,,$$

and use the formulas (5) and (14), the expression (17) follows.

To prove the case when $M, N > 0$ we divide (11) by $n^{5\alpha/2+4}$ to get

$$\frac{L_n^{(\alpha,0,N)}(x)}{n^{5\alpha/2+4}} = \frac{1}{n^{\alpha+1}} \frac{B_0(n)}{n^{\alpha+3}} \frac{L_n^{(\alpha)}(x)}{n^{\alpha/2}} + \frac{1}{n^{\alpha+1}} x \frac{B_1(n)}{n^{\alpha+2}} \frac{L_{n-1}^{(\alpha+2)}(x)}{n^{(\alpha+2)/2}} + x^2 \frac{B_2(n)}{n^{2\alpha+2}} \frac{L_{n-2}^{(\alpha+4)}(x)}{n^{(\alpha+4)/2}} .$$

Therefore, taking into account the asymptotic formula (see [16, p.15])

$$J_\alpha(2\sqrt{nx}) = \left(\frac{1}{\pi\sqrt{nx}}\right)^{1/2} \cos\left(2\sqrt{nx} - \alpha\frac{\pi}{2} - \frac{\pi}{4}\right) + O(n^{-3/4}), \quad \text{when} \quad n \to \infty,$$

valid for any $x$ on compact subsets of $\mathbb{C} \setminus [0, \infty)$, and using (5) as well as (15) we find

$$\frac{L_n^{(\alpha,0,N)}(x)}{n^{5\alpha/2+4}} = O\left(\frac{1}{n^{\alpha+5/4}}\right) + c(n) e^{x/2} x^2 x^{-(\alpha+4)/2} J_{\alpha+4}\left(2\sqrt{(n-2)x}\right) + O\left(\frac{1}{n^{3/4}}\right),$$

with $c(n) = \dfrac{B_2(n)}{n^{2\alpha+2}} \left(\dfrac{n-2}{n}\right)^{\alpha/2+2}$. Thus, the application of (15) leads to (18). The case (16) is similar to this one and we omit it here. □

**Remark 2** We can deduce the strong exterior asymptotic of the polynomials $L_n^{(\alpha,M,N)}(x)$, for all $M, N \geq 0$, directly from (a) in the above theorem using Perron's formula (4). For example, for case $M, N > 0$, in the same conditions of Proposition 1 (b) we have

$$\frac{L_n^{(\alpha,M,N)}(x)}{n^{5\alpha/2+15/4} e^{2(-nx)^{1/2}}} = \frac{MN}{2\sqrt{\pi}(\alpha+1)\Gamma(\alpha+3)\Gamma(\alpha+4)} e^{x/2} (-x)^{-\alpha/2-1/4} \left(1 + O(n^{-1/2})\right) .$$

**Theorem 2** *The polynomials* $\left(L_n^{(\alpha,M,N)}(x)\right)_n$, *with* $\alpha > -1$, *satisfy*

*(a) Exterior Plancherel–Rotach type asymptotics. The following limits hold uniformly on compact subsets of* $\mathbb{C} \setminus [0, 4]$,

- *If* $M > 0$ *and* $N = 0$,

$$\lim_{n\to\infty} \frac{L_n^{(\alpha,M,0)}(nx)}{n^{\alpha+1} L_n^{(\alpha)}(nx)} = \frac{M}{\Gamma(\alpha+2)} \frac{x\,\varphi((x-2)/2)}{(\varphi((x-2)/2)+1)^2} . \tag{19}$$

- *If* $M = 0$ *and* $N > 0$,

$$\lim_{n\to\infty} \frac{L_n^{(\alpha,0,N)}(nx)}{n^{\alpha+3} L_n^{(\alpha)}(nx)} = \frac{N}{(\alpha+1)\Gamma(\alpha+4)} \times$$
$$\frac{x^2\varphi^2((x-2)/2) + (\alpha+2)x\,\varphi((x-2)/2)\left(\varphi^2((x-2)/2)+1\right)^2 - \left(\varphi^2((x-2)/2)+1\right)^4}{\left(\varphi((x-2)/2)+1\right)^4} .$$

- *If* $M > 0$ *and* $N > 0$,

$$\lim_{n\to\infty} \frac{L_n^{(\alpha,M,N)}(nx)}{n^{2\alpha+4} L_n^{(\alpha)}(nx)} = \frac{MN}{(\alpha+1)\Gamma(\alpha+3)\Gamma(\alpha+4)} \frac{x^2\varphi^2((x-2)/2)}{(\varphi((x-2)/2)+1)^4} . \tag{20}$$

*In the three cases* $\varphi(x)$ *is given by (10).*

*(b) Mehler–Heine type formulas. The following limits hold uniformly on compact subsets of* $\mathbb{C}$,

- *If* $M > 0$ *and* $N = 0$,

$$\lim_{n\to\infty} \frac{L_n^{(\alpha,M,0)}(x/n)}{n^{2\alpha+1}} = \frac{-M}{\Gamma(\alpha+2)}\, x\, x^{-(\alpha+2)/2} J_{\alpha+2}(2\sqrt{x})\,. \tag{21}$$

- *If* $M = 0$ *and* $N > 0$,

$$\begin{aligned}\lim_{n\to\infty} \frac{L_n^{(\alpha,0,N)}(x/n)}{n^{2\alpha+3}} = \frac{N}{(\alpha+1)\Gamma(\alpha+4)} & \Big(x^2\, x^{-(\alpha+4)/2} J_{\alpha+4}(2\sqrt{x}) \\ & -(\alpha+2)x\, x^{-(\alpha+2)/2} J_{\alpha+2}(2\sqrt{x}) - x^{-\alpha/2} J_{\alpha}(2\sqrt{x})\Big)\,.\end{aligned} \tag{22}$$

- *If* $M > 0$ *and* $N > 0$,

$$\lim_{n\to\infty} \frac{L_n^{(\alpha,M,N)}(x/n)}{n^{3\alpha+4}} = \frac{MN}{(\alpha+1)\Gamma(\alpha+3)\Gamma(\alpha+4)}\, x^2\, x^{-(\alpha+4)/2} J_{\alpha+4}(2\sqrt{x})\,. \tag{23}$$

**Proof:** (a) To prove (19-20) we will use a similar idea as in the proof Theorem 1 (a). We first use the identity (3) to rewrite the quotient

$$\frac{L_n^{(\alpha)}(nx)}{L_{n-1}^{(\alpha+2)}(nx)} = \frac{L_n^{(\alpha+2)}(nx) - 2L_{n-1}^{(\alpha+2)}(nx) + L_{n-2}^{(\alpha+2)}(nx)}{L_{n-1}^{(\alpha+2)}(nx)},$$

and then use the ratio asymptotics for the scaled Laguerre polynomials (9) to obtain

$$\lim_{n\to\infty} \frac{L_n^{(\alpha)}(nx)}{L_{n-1}^{(\alpha+2)}(nx)} = -\frac{\left(\varphi\left((x-2)/2\right)+1\right)^2}{\varphi\left((x-2)/2\right)}, \quad \lim_{n\to\infty} \frac{L_n^{(\alpha)}(nx)}{L_{n-2}^{(\alpha+4)}(nx)} = \frac{\left(\varphi\left((x-2)/2\right)+1\right)^4}{\varphi^2\left((x-2)/2\right)}\,.$$

Thus, dividing (11) by $L_n^{(\alpha)}(x)$, scaling $x$ as $nx$ and using the above limit relations the result follows.
(b) Since the proof of the three cases are completely analogous we will prove only the second case and will omit the other two cases.

To prove the case $M = 0$, $N > 0$, we use again the relation (11). Scaling the variable $x$ as $x/n$, in (11) yields

$$L_n^{(\alpha,0,N)}(x/n) = B_0(n) L_n^{(\alpha)}(x/n) + B_1(n)\, \frac{x}{n}\, L_{n-1}^{(\alpha+2)}(x/n) + B_2(n)\, \frac{x^2}{n^2} L_{n-2}^{(\alpha+4)}(x/n).$$

Then, dividing the above expression by $n^{2\alpha+3}$ we get

$$\frac{L_n^{(\alpha,0,N)}(x/n)}{n^{2\alpha+3}} = \frac{B_0(n)}{n^{\alpha+3}} \frac{L_n^{(\alpha)}(x/n)}{n^{\alpha}} + \frac{B_1(n)}{n^{\alpha+2}} \frac{x\, L_{n-1}^{(\alpha+2)}(x/n)}{n^{\alpha+2}} + \frac{B_2(n)}{n^{\alpha+1}} \frac{x^2\, L_{n-2}^{(\alpha+4)}(x/n)}{n^{\alpha+4}}\,.$$

Finally, we take the limit $n \to \infty$ and use (6) and (14) that lead to the result (22). □

**Remark 3** The exterior Plancherel–Rotach type asymptotics of the polynomials $L_n^{(\alpha,M,N)}(x)$ can be obtained in a straightforward way using (a) of the above theorem and (7). For example, in the simplest case $M > 0$, $N = 0$ (Koornwinder polynomials) in the same conditions of Proposition 1 (e) we have

$$\lim_{n\to\infty} \frac{(-1)^n 2^{n+1} L_n^{(\alpha,M,0)}(x)}{n^{\alpha+1/2}\varphi^n((x-2)/2)\exp\left(\frac{2nx}{x+\sqrt{x^2-4x}}\right)} = \frac{M}{2^\alpha\sqrt{\pi}\,\Gamma(\alpha+2)}\times$$
$$\frac{x^{1-\alpha}\left(x+\sqrt{x^2-4x}\right)^\alpha \left(x^2-4x\right)^{-1/4}\varphi^{3/2}((x-2)/2)}{\left(\varphi((x-2)/2)+1\right)^2}.$$

## 4 Zeros of the generalized Laguerre polynomials

In this section we will obtain the asymptotic properties of the zeros of $L_n^{(\alpha,M,N)}(x)$ that follow from the Mehler–Heine type formulas given in Theorem 2 (b). More concretely, we will establish a limit relation between the zeros of the generalized Laguerre polynomials and the zeros of Bessel function $J_\alpha$, $J_{\alpha+2}$ or $J_{\alpha+4}$ or their combinations. In fact, we will prove the following:

**Theorem 3** *Denote by $j_{\alpha,i}$ the $i$-th positive zero of the Bessel function $J_\alpha(x)$. Let $\left(x_{n,i}\right)_{i=1}^n$ be the zeros in increasing order of the polynomial $L_n^{(\alpha,M,N)}(x)$ orthogonal with respect to the inner product (1) with $\alpha > -1$. Then,*

*(a) If $M > 0$ and $N = 0$, we have*

$$\lim_{n\to\infty} n\,x_{n,1} = 0 \quad \text{and} \quad \lim_{n\to\infty} n\,x_{n,i} = \frac{j^2_{\alpha+2,i-1}}{4},\ i \geq 2. \tag{24}$$

*(b) If $M = 0$ and $N > 0$, we have*

$$\lim_{n\to\infty} n\,x_{n,i} = h_{\alpha,i},$$

*where $h_{\alpha,i}$ denotes the $i$-th real zero of function $h(x)$ defined as*

$$h(x) = \frac{N}{(\alpha+1)\Gamma(\alpha+4)}\Big(x^2\,x^{-(\alpha+4)/2}J_{\alpha+4}(2\sqrt{x}) - (\alpha+2)x\,x^{-(\alpha+2)/2}J_{\alpha+2}(2\sqrt{x})$$
$$-x^{-\alpha/2}J_\alpha(2\sqrt{x})\Big).$$

*Moreover, $h(x)$ has only one negative real zero.*

*(c) If $M, N > 0$, we have*

$$\lim_{n\to\infty} n\,x_{n,i} = 0,\ i = 1, 2 \quad \text{and} \quad \lim_{n\to\infty} n\,x_{n,i} = \frac{j^2_{\alpha+4,i-2}}{4},\ i \geq 3. \tag{25}$$

**Proof:** The results in the three cases are a consequence of Theorem 2 (b) and Hurwitz's theorem [16, Thm. 1.91.3]. Let prove now that when $M = 0$ and $N > 0$, the limit function $h(x)$ has only one negative real zero. Using the definition of Bessel function $J_\alpha(x)$ we have:

$$x^{-\alpha/2}J_\alpha(2\sqrt{x}) = \sum_{i=0}^{\infty} \frac{(-x)^i}{i!\Gamma(i+\alpha+1)}.$$

It is well known [16, §8.1] that for $\alpha > -1$ the above function only has positive real zeros. Therefore,

$$h(x) = \frac{N}{(\alpha+1)\Gamma(\alpha+4)} \sum_{i=0}^{\infty} \left( \frac{(-1)^i x^{i+2}}{i!\Gamma(i+\alpha+5)} - \frac{(\alpha+2)(-1)^i x^{i+1}}{i!\Gamma(i+\alpha+3)} - \frac{(-1)^i x^i}{i!\Gamma(i+\alpha+1)} \right)$$
$$= \frac{N}{(\alpha+1)\Gamma(\alpha+4)} \left( -\frac{1}{\Gamma(\alpha+1)} + \sum_{i=0}^{\infty} \frac{(-x)^{i+2}}{i!\Gamma(i+\alpha+3)} \frac{\alpha+2}{(i+2)(i+\alpha+4)} \right).$$

Thus,

$$\lim_{x\to-\infty} h(x) = +\infty, \quad h(0) = -\frac{N}{\Gamma(\alpha+2)\Gamma(\alpha+4)} < 0. \tag{26}$$

Moreover,

$$h'(x) = \frac{-N}{(\alpha+1)\Gamma(\alpha+4)} \sum_{i=0}^{\infty} \frac{(i+2)(-x)^{i+1}}{i!\Gamma(i+\alpha+3)} \frac{\alpha+2}{(i+2)(i+\alpha+4)} < 0, \quad \text{for all } x < 0.$$

Then, $h(x)$ is a continuous decreasing function on $(-\infty, 0)$ and gathering with (26) we obtain that $h(x)$ has one and only one zero on $(-\infty, 0)$. □

The last theorem has several important consequences. First of all, from (24) follows that in the case when $M > 0$, $N = 0$ the first scaled zero, $nx_{n,1}$, of the orthogonal polynomials (Koornwinder polynomials) $L_n^{(\alpha,M,0)}(x)$ goes to 0 when $n \to \infty$. Second, in the case $M > 0$, $N > 0$ the two first scaled zeros ($nx_{n,1}$, $nx_{n,2}$), being one of them a negative zero [10], of the corresponding orthogonal polynomials $L_n^{(\alpha,M,N)}(x)$ also tend to 0 when $n \to \infty$., i.e., in these cases the origin attracts one or two zeros of the corresponding orthogonal polynomials. This situation agrees with the results in [1]. In fact, in [1] the authors proved, in a more general framework, that if $M, N > 0$ then there are two zeros –not scaled ones– of $L_n^{(\alpha,M,N)}(x)$ that are attracted by 0 and if $M > 0$, $N = 0$ there is only one zero that is attracted by 0. We will call this *the simple* or *regular* situation. Notice that if $nx_{n,i} \to 0$ then $x_{n,i} \to 0$ but not vice versa.

Let also point out that to apply the general results of [1] we need to obtain the ratio asymptotics of the sequence $\left(L_n^{(\alpha,M,N)}(x)\right)_n$. This relation can be deduced from Theorem 1 and the relation (8), in fact we have that $\lim_{n\to\infty} L_{n+1}^{(\alpha,M,N)}(x)/L_n^{(\alpha,M,N)}(x) = 1$, uniformly on compact subsets of $\mathbb{C} \setminus [0,\infty)$ with $M, N > 0$.

The situation for the case $M = 0$, $N > 0$ is different because in this case we can not apply the general results of [1]. Furthermore, in this case, for $n$ large enough, the first scaled zero $nx_{n,1}$ is always negative and does not tends to zero when $n \to \infty$, i.e., $\lim_{n\to\infty} nx_{n,1} = h_{\alpha,1}$ is a negative real number.

Let us now consider the polynomials $L_n^{(\alpha,M_0,\ldots,M_s)}(x)$ orthogonal with respect to the inner product (2). If all masses $M_0$, …, $M_s$ are positive then, using the results of [1] it can be shown that $s+1$ zeros of $L_n^{(\alpha,M_0,\ldots,M_s)}(x)$ go to zero as $n$ tends to infinity. But now, using the fact proved here that asymptotic behavior of the smaller zeros of $L_n^{(\alpha,M,N)}(x)$ is determined by the Mehler–Heine type formulas, it is reasonable to expect for this general case a *simple* Mehler–Heine type formulas similar to the (21-23). More exactly, we pose the following:

**Conjecture 1** Let $L_n^{(\alpha,M_0,\ldots,M_s)}(x)$ be the polynomials orthogonal with respect to the inner product (2). If $M_i > 0$, $i = 1,\ldots,s$, then, for some real numbers $\beta$ and $K$

$$\lim_{n\to\infty} \frac{L_n^{(\alpha,M_0,\ldots,M_s)}(x/n)}{n^\beta} = K\, x^{s+1}\, x^{-(\alpha+2s+2)/2} J_{\alpha+2s+2}(2\sqrt{x})\,,$$

uniformly on compact subsets of $\mathbb{C}$ where $\beta \in \mathbb{R}$ and $K$ is a non-zero constant.

Moreover, in the general case these simple Mehler–Heine type formulas should not be expected when some of the constants vanish since even in the case of the polynomials $L_n^{(\alpha,M,N)}(x)$ discussed here it does not appear. What happens in this case is still an open question.

## 4.1 Another generalization of the Laguerre polynomials

In [5, 6] a different generalization of the Laguerre polynomials has been introduced. In fact in [5, 6] the authors considered the monic polynomials, $R_n^{\alpha,M_0,M_1}(x)$, orthogonal with respect to the linear functional $\mathcal{U}$ on the linear space of polynomials with real coefficients defined as

$$\langle \mathcal{U}, P\rangle \;\; = \int_0^\infty P(x)\;\; x^\alpha e^{-x} dx + M_0 P(0) + M_1 P'(0), \quad M_0, M_1 \in \mathbb{R},\; \alpha > -1.$$

Although the functional $\mathcal{U}$ is not positive definite, for $n$ large enough there exists the orthogonal polynomial $R_n^{\alpha,M_0,M_1}(x)$ for all the values of the masses $M_0$ and $M_1$. Furthermore, we have the following expression for these generalized monic Laguerre polynomials, in terms of the monic Laguerre polynomials $\hat{L}_n^\alpha(x)$

$$R_n^{\alpha,M_0,M_1}(x) = \hat{L}_n^\alpha(x) + A_1(\hat{L}_n^\alpha)'(x) + A_2(\hat{L}_n^\alpha)''(x),$$

being

$$\lim_{n\to\infty} nA_1 = 2(\alpha+2) > 0, \qquad \lim_{n\to\infty} n^2 A_2 = (\alpha+2)(\alpha+3) > 0$$

In [6] it was established for $n$ large enough and $M_0, M_1 \geq 0$, that all zeros of these polynomials are real, simple and one of them is negative. Thus, using the same ideas presented in this paper we can obtain the Mehler–Heine type formula, that is, for any $M_1 > 0$, and $M_0 \geq 0$,

$$\lim_{n\to\infty} \frac{R_n^{\alpha,M_0,M_1}(x/n)}{n^\alpha} = x^2\, x^{-(\alpha+4)/2} J_{\alpha+4}(2\sqrt{x})\,,$$

from where a similar formula to (25) follows, i.e., if $M_1 > 0$ and $M_0 \geq 0$ we have

$$\lim_{n\to\infty} n\, x_{n,i} = 0,\; i = 1,2 \quad \text{and} \quad \lim_{n\to\infty} n\, x_{n,i} = \frac{j^2_{\alpha+4,i-2}}{4}\,,\;\; i \geq 3,$$

where $x_{n,i}$ denotes, as before, the zeros of the polynomial $R_n^{\alpha,M_0,M_1}(x)$ ordered in increasing order. In other words, for this non diagonal case the origin always attracts the two first scaled zeros of $R_n^{\alpha,M_0,M_1}(x)$. In this case we expect to be true the following:

**Conjecture 2** Let $R_n^{\alpha,M_0,\ldots,M_s}(x)$ be the polynomials orthogonal with respect to linear functional $\mathcal{U}$ defined by

$$\langle \mathcal{U}, P\rangle \;\; = \int_0^\infty P(x)\;\; x^\alpha e^{-x} dx + \sum_{k=0}^{s} M_k (P(0))^{(k)}, \quad \alpha > -1.$$

If $M_s > 0$, then, for some real numbers $\beta$ and $K$

$$\lim_{n\to\infty} \frac{L_n^{(\alpha,M_0,\ldots,M_s)}(x/n)}{n^\beta} = K\, x^{s+1}\, x^{-(\alpha+2s+2)/2} J_{\alpha+2s+2}(2\sqrt{x})\,,$$

uniformly on compact subsets of $\mathbb{C}$ where $\beta \in \mathbb{R}$ and $K$ is a non-zero constant, i.e., in this case we have a simple Mehler–Heine type formula for any choice of the masses $M_0, \ldots, M_{s-1} \geq 0$.

## 4.2 Numerical examples

Finally, we illustrate with numerical examples the asymptotic behavior of the first three scaled zeros of $L_n^{(\alpha,M,N)}(x)$ in the three cases.

(a) (Koornwinder polynomials) $\alpha = 0.5$, $M = 4$ and $N = 0$.

| | $nx_{n1}$ | $nx_{n,2}$ | $nx_{n,3}$ |
|---|---|---|---|
| $n = 100$ | 0.0012226844 | 8.2437752506 | 20.530281615 |
| $n = 300$ | 0.0002383295 | 8.2838350858 | 20.628775446 |
| $n = 500$ | 0.0001110472 | 8.2920021251 | 20.649038871 |
| Limit value | 0 | $\frac{j^2_{\alpha+2,1}}{4} = 8.3043654787$ | $\frac{j^2_{\alpha+2,2}}{4} = 20.679807776$ |

(b) $\alpha = 0.5$, $M = 0$ and $N = 4$.

| | $nx_{n1}$ | $nx_{n,2}$ | $nx_{n,3}$ |
|---|---|---|---|
| $n = 100$ | $-2.9502897801$ | 5.9140361943 | 18.441785447 |
| $n = 300$ | $-2.9413138140$ | 5.9620907466 | 18.545273471 |
| $n = 500$ | $-2.9395263213$ | 5.9718231990 | 18.566504799 |
| Limit value | $h_{\alpha,1} = -2.9368500726$ | $h_{\alpha,2} = 5.9864991636$ | $h_{\alpha,3} = 18.598691651$ |

(c) When $M, N > 0$ we present three examples with the objective to compare our numerical results with the lower bound for $x_{n,1}$ obtained in [10].

(c1) $\alpha = -0.5$, $M = 2$ and $N = 30$.

| | $nx_{n1}$ | $nx_{n,2}$ | $nx_{n,3}$ |
|---|---|---|---|
| $n = 100$ | $-0.5803594677$ | 0.6641769437 | 12.232451164 |
| $n = 300$ | $-0.4596747367$ | 0.5115142683 | 12.230360187 |
| $n = 500$ | $-0.4102511939$ | 0.4511789180 | 12.227361506 |
| Limit value | 0 | 0 | $\frac{j^2_{\alpha+4,2}}{4} = 12.207798411$ |

(c2) $\alpha = -0.5$, $M = 2$ and $N = 4$.

| | $nx_{n1}$ | $nx_{n,2}$ | $nx_{n,3}$ |
|---|---|---|---|
| $n = 100$ | $-0.5803835399$ | $0.6641570515$ | $12.232438558$ |
| $n = 300$ | $-0.4596762948$ | $0.5115129187$ | $12.230359354$ |
| $n = 500$ | $-0.4102516290$ | $0.4511785349$ | $12.227361272$ |
| Limit value | 0 | 0 | $\frac{j^2_{\alpha+4,2}}{4} = 12.207798411$ |

(c3) $\alpha = -0.5$, $M = 60$ and $N = 4$.

| | $nx_{n1}$ | $nx_{n,2}$ | $nx_{n,3}$ |
|---|---|---|---|
| $n = 100$ | $-0.1176774997$ | $0.1208277520$ | $12.181114739$ |
| $n = 300$ | $-0.0903224524$ | $0.0921633932$ | $12.198979258$ |
| $n = 500$ | $-0.0797063110$ | $0.0811386546$ | $12.202659549$ |
| Limit value | 0 | 0 | $\frac{j^2_{\alpha+4,2}}{4} = 12.207798411$ |

The lower bound obtained in [10] for the smallest zero of $L_n^{(\alpha,M,N)}(x)$ when $n$ is large enough and $M, N > 0$ is

$$b(N, M) := -\frac{1}{2}\sqrt{\frac{N}{M}} \leq x_{n,1} \leq 0$$

and furthermore $\lim_{n\to\infty} x_{n,1} = 0$. We compare this lower bound with the data of the numerical examples: $\alpha = -0.5$,

- If $M = 2$, $N = 30$, then $b(2, 30) \approx -1.9364916731$,

  $$x_{100,1} \approx -0.0058035947, \quad x_{300,1} \approx -0.0015322491, \quad x_{500,1} \approx -0.0008205024.$$

- If $M = 2$, $N = 4$, then $b(2, 4) \approx -0.7071067812$

  $$x_{100,1} \approx -0.0058038354, \quad x_{300,1} \approx -0.0015322543, \quad x_{500,1} \approx -0.0008205033.$$

- If $M = 60$, $N = 4$, then $b(60, 4) \approx -0.1290994449$

  $$x_{100,1} \approx -0.0011767750, \quad x_{300,1} \approx -0.0003010748, \quad x_{500,1} \approx -0.0001594126.$$

Notice that the smallest zero, at least in these numerical examples, is very close to 0 even for values of $n$ not excessively large (it is also true for values of $n$ such as $10, 20$, etc.)

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