On a q-extension of the Hermite polynomials $H_n(x)$ with the continuous orthogonality property on $\mathbb{R}$ [1]

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Abstract

We study a polynomial sequence of q-extensions of the classical Hermite polynomials $H_n(x)$, which satisfies continuous orthogonality on the whole real line $\mathbb{R}$ with respect to the positive weight function. This sequence can be expressed either in terms of the q-Laguerre polynomials $L_n^{(\alpha)}(x;q)$, $\alpha = \pm 1/2$, or through the discrete q-Hermite polynomials $\tilde{h}_n(x;q)$ of type II.

1 Introduction

There is a well-known family of the continuous q-Hermite polynomials of Rogers [1, 2]

$$H_n(\cos\theta|q) := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q e^{i(n-2k)\theta} = e^{in\theta}\, {}_2\phi_0 \left(\begin{matrix} q^{-n}, 0 \\ \text{—} \end{matrix} \,\middle|\, q, q^n e^{-2i\theta} \right), \tag{1.1}$$

which are q-extensions of the classical Hermite polynomials $H_n(x)$ for $q \in (0,1)$. Throughout this paper we will employ the standard notations of the q-special functions theory , see [3]-[5]. In particular,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} = \begin{bmatrix} n \\ n-k \end{bmatrix}_q \tag{1.2}$$

stands for the q-binomial coefficient and $(a;q)_0 = 1$ and $(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j)$, $n = 1,2,3,\ldots$, is the q-shifted factorial. Besides, explicit forms of q-polynomials from the Askey-scheme [4] are often expressed in terms of the terminating basic hypergeometric polynomial

$$\begin{aligned} {}_r\phi_s &\left(\begin{matrix} q^{-n}, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix} \,\middle|\, q\,,\, z \right) \\ &:= \sum_{k=0}^{n} \frac{(q^{-n};q)_k (a_2;q)_k \cdots (a_r;q)_k}{(b_1;q)_k (b_2;q)_k \cdots (b_s;q)_k} \frac{z^k}{(q;q)_k} \left[(-1)^k q^{k(k-1)/2} \right]^{s-r+1} \end{aligned} \tag{1.3}$$

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of degree n in the variable z. So the continuous q-Hermite polynomials in (1) correspond to the case when $r=2$ and $s=0$.

The continuous q-Hermite polynomials $H_n(x|q)$ satisfy the three-term recurrence relation

$$2xH_n(x|q) = H_{n+1}(x|q) + (1-q^n)H_{n-1}(x|q)\,. \tag{1.4}$$

They are orthogonal on the finite interval $x \in [-1,1]$ with respect to the continuous weight function [2]

$$w(x) = \frac{1}{\sqrt{1-x^2}} \prod_{k=0}^{\infty} \left[1 + 2(1-2x^2)q^k + q^{2k}\right]. \tag{1.5}$$

In the limit as $q \to 1$, they coincide with the ordinary Hermite polynomials $H_n(x)$, *i.e.*,

$$\lim_{q\to 1} \left(\frac{1-q}{2}\right)^{-n/2} H_n\left(\sqrt{\frac{1-q}{2}}\,x \,\middle|\, q\right) = H_n(x)\,. \tag{1.6}$$

There is another family, called the discrete q-Hermite polynomials of type II , which is a q-extension of the sequence of the Hermite polynomials $H_n(x)$ [6]. Their explicit form is given (see [4, p. 119]) by

$$\tilde{h}_n(x;q) := \mathrm{i}^{-n}\, q^{-n(n-1)/2}\, {}_2\phi_0\left(\begin{matrix} q^{-n}, \mathrm{i}x \\ - \end{matrix} \,\middle|\, q, -q^n \right) = x^n\, {}_2\phi_1\left(\begin{matrix} q^{-n}, q^{1-n} \\ 0 \end{matrix} \,\middle|\, q^2, -\frac{q^2}{x^2} \right). \tag{1.7}$$

The discrete q-Hermite polynomials of type II, $\tilde{h}_n(x;q)$, satisfy the three-term recurrence relation

$$x\tilde{h}_n(x;q) = \tilde{h}_{n+1}(x;q) + q^{1-2n}(1-q^n)\tilde{h}_{n-1}(x;q)\,, \tag{1.8}$$

but, contrary to the polynomials $H_n(x|q)$ of Rogers (1.1), the sequence $\{\tilde{h}_n(x;q)\}$ is orthogonal on the infinite interval $x \in \mathbb{R}$ with respect to the discrete weight function, supported on the points $x = \pm\, cq^k\,, c>0\,, k\in\mathbb{Z}$. In the limit as $q \to 1$, the $\tilde{h}_n(x,q)$ reduce as well to the Hermite polynomials $H_n(x)$:

$$\lim_{q\to 1}(1-q^2)^{-n/2}\, \tilde{h}_n(x\sqrt{1-q^2};q) = 2^{-n} H_n(x)\,. \tag{1.9}$$

Now we are in a position to formulate the aim of this paper. We wish to find such q-extensions of the Hermite polynomials $H_n(x)$, which satisfy the following requirements: 1. They are polynomials in the variable x, which obey a three-term recurrence relation; 2. They are orthogonal on the whole real line $\mathbb{R}$ with respect to a continuous positive weight function; 3. In the limit as $q \to 1$ they coincide with the Hermite polynomials $H_n(x)$.

Such a family is of great interest from the point of view of possible applications in mathematical physics. We remind the reader that the two most fundamental problems in nonrelativistic quantum mechanics, the harmonic oscillator and the Coulomb system, are defined on $\mathbb{R}^3$ [7]. Thus our goal is equivalent to having such a q-deformed version of the linear harmonic oscillator, which is still defined on the whole real line $\mathbb{R}$ and enjoys the continuous orthogonality property on $\mathbb{R}$ with respect to a positive weight function.[2] We will not pursue this viewpoint here. Instead we now focus on the mathematical properties of the q-extensions of the Hermite polynomials $H_n(x)$ under discussion.

[2]It should be noted that there are, in fact, several publications (see [8]–[16] and references therein) devoted to the study of explicit realizations, which represent q-extensions of the Hermite functions (or the wave functions of the linear harmonic oscillator) $H_n(x)\,e^{-x^2/2}$. But none of these realizations satisfies all of the aforementioned requirements : the continuous weight functions in [8, 10, 13] are supported on the finite intervals; the continuous weight functions in [9, 15] are not positive; the q-extensions in [9], [12]–[15] are not expressed in terms of polynomials in the independent variable; and, finally, the orthogonality relations in [11]–[13], [16] are discrete.

2 Definition of the sequence $\{\mathcal{H}_n(x;q)\}$

It is known that the Hermite polynomials $H_n(x)$ can be expressed through the Laguerre polynomials $L_n^{(\alpha)}(x)$ as

$$\begin{aligned} H_{2n}(x) &= (-1)^n\, 2^{2n}\, n!\, L_n^{(-1/2)}(x^2), \\ H_{2n+1}(x) &= (-1)^n\, 2^{2n+1}\, n!\, x\, L_n^{(1/2)}(x^2)\ , \end{aligned} \tag{2.1}$$

where

$$L_n^{(\alpha)}(z) := \frac{(\alpha+1)_n}{n!}\, {}_1F_1\left(\begin{matrix} -n \\ \alpha+1 \end{matrix} \middle| z \right) = \frac{(\alpha+1)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k}{(\alpha+1)_k} \frac{z^k}{k!}, \tag{2.2}$$

and $(a)_n = \Gamma(a+n)/\Gamma(a)$, $n = 0, 1, 2, \ldots$, is the shifted factorial.

It is also known that a q-extension of the Laguerre polynomials [3] $L_n^{(\alpha)}(x;q)$, defined [17]-[19] as

$$\begin{aligned} L_n^{(\alpha)}(x;q) &= \frac{(q^{\alpha+1};q)_n}{(q;q)_n}\, {}_1\phi_1\left(\begin{matrix} q^{-n} \\ q^{\alpha+1} \end{matrix} \middle| q, -q^{n+\alpha+1}x \right) \\ &= \frac{1}{(q;q)_n}\, {}_2\phi_1\left(\begin{matrix} q^{-n}, -x \\ 0 \end{matrix} \middle| q, q^{n+\alpha+1} \right), \end{aligned} \tag{2.3}$$

satisfies two kinds of orthogonality relations, an absolutely continuous one and a discrete one. The former orthogonality relation[4] is given by

$$\int_0^\infty \frac{x^\alpha}{E_q(x)}\, L_m^{(\alpha)}(x;q)\, L_n^{(\alpha)}(x;q)\, dx = d_n^{-1}(\alpha)\delta_{mn}\ , \quad \alpha > -1\,, \tag{2.4}$$

where $E_q(x)$ is the Jackson q-exponential function,

$$E_q(z) := \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q;q)_n}\, z^n = (-z;q)_\infty\ , \tag{2.5}$$

and the normalization constant $d_n(\alpha)$ is equal to

$$d_n(\alpha) = q^n \frac{(q;q)_n}{(q^{\alpha+1};q)_n} \frac{(q;q)_\infty}{(q^{-\alpha};q)_\infty} \frac{\sin\pi(\alpha+1)}{\pi}\ . \tag{2.6}$$

It remains only to remind the reader, that in the limit as $q \to 1$ we have

$$\lim_{q\to 1} L_n^{(\alpha)}((1-q)x;q) = L_n^{(\alpha)}(x)\ . \tag{2.7}$$

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[3] There are also the continuous q-Laguerre polynomials $P_n^{(\alpha)}(x|q)$ with the orthogonality on the finite interval $x \in [-1,1]$ and the Wall (or the little q-Laguerre) polynomials $p_n(x;a|q)$ with the discrete orthogonality on the points $x = q^k$, $k = 0, 1, 2, \ldots$ (see [4], pp. 105 and 107, respectively).

[4] It is worth noting that the fact of the integrability of the weight function $x^\alpha/E_q(x)$ in (2.4) directly follows from the integral

$$\int_0^\infty \frac{t^{x-1}\, dt}{E_q((1-q)t)} = \frac{\Gamma(x)\,\Gamma(1-x)}{\Gamma_q(1-x)}\,, \qquad Re\, x > 0,$$

which is a particular case of the Ramanujan integral extension of the beta function [20].

We can now define, in complete analogy with the relationship (2.1), a family of q-extensions of the Hermite polynomials of the form

$$\begin{aligned} \mathcal{H}_{2n}(x;q) &:= (-1)^n (q;q)_n \, L_n^{(-1/2)}(x^2;q) \,, \\ \mathcal{H}_{2n+1}(x;q) &:= (-1)^n (q;q)_n \, x \, L_n^{(1/2)}(x^2;q) \,. \end{aligned} \tag{2.8}$$

With the aid of the limit relation

$$\lim_{q\to 1} \frac{(q^a;q)_n}{(1-q)^n} = (a)_n \tag{2.9}$$

it is not difficult to verify that from (2.8) and (2.3) one obtains

$$\lim_{q\to 1} (1-q)^{-n/2} \, \mathcal{H}_n(\sqrt{1-q}\, x;q) = 2^{-n} H_n(x) \,, \tag{2.10}$$

i.e., the polynomials $\mathcal{H}_n(x;q)$ so defined are indeed q-extensions of the Hermite polynomials $H_n(x)$. The next step is to establish a continuous orthogonality relation and a three-term recurrence relation for the q-polynomials $\mathcal{H}_n(x;q)$.

3 Orthogonality relation

We begin this section with the following theorem:

Theorem 1 *The sequence of the q-polynomials $\{\mathcal{H}_n(x;q)\}$, which are defined by the relations (2.8), satisfies the orthogonality relation*

$$\int_{-\infty}^{\infty} \mathcal{H}_m(x;q)\, \mathcal{H}_n(x;q) \frac{dx}{E_q(x^2)} = \pi q^{-\frac{n}{2}} \, (q^{1/2};q^{1/2})_n \, (q^{1/2};q)_{1/2} \, \delta_{mn} \tag{3.1}$$

on the whole real line $\mathbb{R}$ with respect to the continuous positive weight function $w(x) = 1/E_q(x^2)$.

Proof. Since the weight function in (3.1) is an even function of the independent variable x and $\mathcal{H}_n(-x;q) = (-1)^n \mathcal{H}_n(x;q)$ by the definition (2.8), the q-polynomials of an even degree $\mathcal{H}_{2m}(x;q)$ and of an odd degree $\mathcal{H}_{2n+1}(x;q)$, $m,n = 0,1,2,\ldots$, are evidently orthogonal to each other. Consequently, it suffices to prove only those cases in (3.1), when degrees of polynomials m and n are either simultaneously even or odd. Let us consider first the former case. From (2.8) and (2.4) it follows that

$$\begin{aligned} &\int_{-\infty}^{\infty} \mathcal{H}_{2m}(x;q)\, \mathcal{H}_{2n}(x;q) \frac{dx}{E_q(x^2)} \\ &= 2(-1)^{m+n} (q;q)_m (q;q)_n \int_0^{\infty} L_m^{(-1/2)}(x^2;q) L_n^{(-1/2)}(x^2;q) \frac{dx}{E_q(x^2)} \\ &= 2(-1)^{m+n} (q;q)_m (q;q)_n \int_0^{\infty} L_m^{(-1/2)}(x^2;q) L_n^{(-1/2)}(x^2;q) \frac{dx}{E_q(x^2)} \\ &= (-1)^{m+n} (q;q)_m (q;q)_n \int_0^{\infty} L_m^{(-1/2)}(y;q) L_n^{(-1/2)}(y;q) \frac{y^{-1/2} dy}{E_q(y)} \\ &= \frac{(q;q)_n^2}{d_n(-1/2)} \, \delta_{mn} = \pi q^{-n} (q^{1/2};q^{1/2})_{2n} (q^{1/2};q)_{1/2} \, \delta_{mn} \,, \end{aligned} \tag{3.2}$$

where we have used at the last step the equality $(q^{1/2};q)_n(q;q)_n = (q^{1/2};q^{1/2})_{2n}$ and the standard relation (for $a = q^{1/2}$ and $\beta = 1/2$)

$$\frac{(a;q)_\infty}{(aq^\beta;q)_\infty} = (a;q)_\beta\,, \tag{3.3}$$

which is valid for an arbitrary complex β.

In the same way we find

$$\begin{aligned}
&\int_{-\infty}^{\infty} \mathcal{H}_{2m+1}(x;q)\mathcal{H}_{2n+1}(x;q)\frac{dx}{E_q(x^2)} \\
&= 2(-1)^{m+n}(q;q)_m(q;q)_n \int_0^\infty L_m^{(1/2)}(x^2;q)L_n^{(1/2)}(x^2;q)\frac{x^2dx}{E_q(x^2)} \\
&= 2(-1)^{m+n}(q;q)_m(q;q)_n \int_0^\infty L_m^{(1/2)}(y;q)L_n^{(1/2)}(y;q)\frac{y^{1/2}dy}{E_q(y)} \\
&= \frac{(q;q)_n^2}{d_n(1/2)}\,\delta_{mn} = \pi q^{-(n+1/2)}(q^{1/2};q^{1/2})_{2n+1}(q^{1/2};q)_{1/2}\,\delta_{mn}\,,
\end{aligned} \tag{3.4}$$

where we have used the relation

$$(1-q^{1/2})(q^{3/2};q)_n = (1-q^{n+1/2})(q^{1/2};q)_n\,, \tag{3.5}$$

which is a straightforward consequence of the general definition of the q-shifted factorial $(a;q)_n$. Putting (3.2) and (3.4) together results in the orthogonality relation (3.1).

The positivity of Jackson's q-exponential function $E_q(x^2)$ for $x \in \mathbb{R}$ and $q \in (0,1)$ is obvious from its definition (2.5): for it is represented as an infinite sum of positive terms (or an infinite product of positive factors). This completes the proof. ■

To conclude this section, we note the obvious fact that in the limit as $q \to 1$ the orthogonality relation (3.1) reduces to the well-known orthogonality property

$$\int_{-\infty}^{\infty} H_m(x)H_n(x)e^{-x^2}dx = \sqrt{\pi}\,2^n\,n!\,\delta_{mn} \tag{3.6}$$

of the Hermite polynomials $H_n(x)$ with respect to the normal distribution e^{-x^2}. This follows immediately from the limit relations (2.9) and (2.10), upon using the fact that

$$\lim_{q\to 1} E_q((1-q)z) = e^z\,. \tag{3.7}$$

4 Recurrence relation

The easiest way to identify the sequence (2.8) (that is, to find its appropriate niche in the Askey-scheme of q-polynomial families [4]), is to derive a three-term recurrence relation for it. Since an arbitrary polynomial $p_n(x)$ satisfies a recurrence relation of the form (see [21, p. 19])

$$(a_nx+b_n)p_n(x) = p_{n+1}(x) + c_np_{n-1}(x)\,, \qquad n \geq 0\,, \tag{4.1}$$

one can try to evaluate the right-hand side of (4.1) for $p_n(x) = \mathcal{H}_n(x;q)$ and to find then such particular coefficient c_n that leads to the left-hand side of (4.1) with some a_n and b_n.

Before starting this derivation we note that in what follows it proves convenient to use the following form

$$L_n^{(\alpha)}(x;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n} \sum_{k=0}^{n} \frac{q^{k(k+\alpha)}}{(q^{\alpha+1};q)_k} \begin{bmatrix} n \\ k \end{bmatrix}_q (-x)^k \tag{4.2}$$

of the q-Laguerre polynomials $L_n^{(\alpha)}(x;q)$, which comes from the first line in definition (2.3), upon using the relation

$$\frac{(q^{-n};q)_k}{(q;q)_k} = (-1)^k q^{k(k-1)/2-nk} \begin{bmatrix} n \\ k \end{bmatrix}_q . \tag{4.3}$$

Let us first consider the case when n in (4.1) is even. Then from (2.8) and (2.3) we find that

$$\begin{aligned} &\mathcal{H}_{2n+1}(x;q) + c_{2n}(q)\mathcal{H}_{2n-1}(x;q) \\ &= (-1)^n (q^{3/2};q)_n\, x \sum_{k=0}^{n} \frac{q^{k(k+1/2)}}{(q^{3/2};q)_k} \begin{bmatrix} n \\ k \end{bmatrix}_q (-x^2)^k \\ &+ c_{2n}(q)\,(-1)^{n-1}(q^{3/2};q)_{n-1}\, x \sum_{k=0}^{n-1} \frac{q^{k(k+1/2)}}{(q^{3/2};q)_k} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q (-x^2)^k . \end{aligned} \tag{4.4}$$

The next step is to employ the relation (3.5) in order to rewrite the quotient $(q^{3/2};q)_n/(q^{3/2};q)_k$ from the first term in (4.4) as

$$\frac{(q^{3/2};q)_n}{(q^{3/2};q)_k} = \frac{1-q^{n+1/2}}{1-q^{k+1/2}} \frac{(q^{1/2};q)_n}{(q^{1/2};q)_k} . \tag{4.5}$$

In the second term in (4.4) one can use the evident relation $(q^{3/2};q)_{n-1} = (q^{1/2};q)_n/(1-q^{1/2})$ and the same formula (3.5) for the factor $(q^{3/2};q)_k$. Also, we recall the property of the q-binomial coefficient

$$\begin{bmatrix} n-1 \\ k \end{bmatrix}_q = \frac{1-q^{n-k}}{1-q^n} \begin{bmatrix} n \\ k \end{bmatrix}_q . \tag{4.6}$$

Putting this all together, we obtain

$$\begin{aligned} &\mathcal{H}_{2n+1}(x;q) + c_{2n}(q)\mathcal{H}_{2n-1}(x;q) \\ &= (-1)^n (q^{1/2};q)_n\, x \sum_{k=0}^{n} \frac{q^{k(k+1/2)}}{(q^{1/2};q)_k} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-x^2)^k}{1-q^{k+1/2}} \left\{ 1 - q^{n+1/2} - c_{2n}(q)\frac{1-q^{n-k}}{1-q^n} \right\} . \end{aligned} \tag{4.7}$$

The right-hand side of (4.7) should match with

$$\mathcal{H}_{2n}(x;q) = (-1)^n (q^{1/2};q)_n \sum_{k=0}^{n} \frac{q^{k(k-1/2)}}{(q^{1/2};q)_k} \begin{bmatrix} n \\ k \end{bmatrix}_q (-x^2)^k , \tag{4.8}$$

multiplied by $a_{2n}(q)x + b_{2n}(q)$. This means that the coefficient $c_{2n}(q)$ can be found from the equation

$$1 - q^{n+1/2} - c_{2n}(q)\frac{1-q^{n-k}}{1-q^n} = d_n(q)\, q^{-k}\,(1-q^{k+1/2}) , \tag{4.9}$$

where $d_n(q)$ is some k-independent factor. It is not difficult to verify that the only solution of the equation (4.9) is $c_{2n}(q) = 1-q^n$ and $d_n(q) = q^n$. Thus

$$\mathcal{H}_{2n+1}(x;q) + (1-q^n)\mathcal{H}_{2n-1}(x;q) = q^n\, x\, \mathcal{H}_{2n}(x;q) . \tag{4.10}$$

Similarly, in the case of an odd n from (4.8) we have

$$\begin{aligned}
&\mathcal{H}_{2n+2}(x;q) + c_{2n+1}(q)\mathcal{H}_{2n}(x;q) \\
&= (-1)^{n+1}(q^{1/2};q)_{n+1} \sum_{k=0}^{n+1} \frac{q^{k(k-1/2)}}{(q^{1/2};q)_k} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q (-x^2)^k \\
&+ c_{2n+1}(q)(-1)^n(q^{1/2};q)_n \sum_{k=0}^{n} \frac{q^{k(k-1/2)}}{(q^{1/2};q)_k} \begin{bmatrix} n \\ k \end{bmatrix}_q (-x^2)^k .
\end{aligned} \tag{4.11}$$

In this case it is even easier to find the coefficient $c_{2n+1}(q)$. Indeed, one will obtain from (4.11) an expression $[a_{2n+1}(q)\,x + b_{2n+1}(q)]\,\mathcal{H}_{2n+1}(x;q)$ only if the two constant terms in (4.11) with $k=0$ cancel each other. This means that the $c_{2n+1}(q)$ should satisfy the equation

$$(q^{1/2};q)_{n+1} - (q^{1/2};q)_n\, c_{2n+1}(q) \equiv (q^{1/2};q)_n[1 - q^{n+1/2} - c_{2n+1}(q)] = 0 . \tag{4.12}$$

Consequently, $c_{2n+1}(q) = 1 - q^{n+1/2}$ and, therefore, by (4.11) we obtain

$$\begin{aligned}
&\mathcal{H}_{2n+2}(x;q) + (1 - q^{n+1/2})\mathcal{H}_{2n}(x;q) \\
&= (-1)^{n+1}(q^{1/2};q)_{n+1} \sum_{k=0}^{n+1} \frac{q^{k(k-1/2)}}{(q^{1/2};q)_k} \left\{ \begin{bmatrix} n+1 \\ k \end{bmatrix}_q - \begin{bmatrix} n \\ k \end{bmatrix}_q \right\} (-x^2)^k \\
&= (-1)^n(q^{3/2};q)_n\, x^2 \sum_{m=0}^{n} \frac{q^{(m+1)(m+1/2)}}{(q^{3/2};q)_m} \left\{ \begin{bmatrix} n+1 \\ m+1 \end{bmatrix}_q - \begin{bmatrix} n \\ m+1 \end{bmatrix}_q \right\} (-x^2)^m \\
&= (-1)^n q^{n+1/2}\,(q^{3/2};q)_n\, x^2 \sum_{m=0}^{n} \frac{q^{m(m+1/2}}{(q^{3/2};q)_m} \begin{bmatrix} n \\ m \end{bmatrix}_q (-x^2)^m = q^{n+1/2}\, x\, \mathcal{H}_{2n+1}(x;q) ,
\end{aligned} \tag{4.13}$$

upon using the readily verified relation

$$\begin{bmatrix} n+1 \\ m+1 \end{bmatrix}_q - \begin{bmatrix} n \\ m+1 \end{bmatrix}_q = q^{n-m} \begin{bmatrix} n \\ m \end{bmatrix}_q \tag{4.14}$$

for the q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$. From (4.10) and (4.13) it thus follows that the q-polynomials $\mathcal{H}_n(x;q)$ satisfy the three-term recurrence relation of the form

$$\mathcal{H}_{n+1}(x;q) + (1 - q^{n/2})\mathcal{H}_{n-1}(x;q) = q^{n/2}\, x\, \mathcal{H}_n(x;q) . \tag{4.15}$$

An examination of the recurrence relation (4.15) reveals that the $\mathcal{H}_n(x;q)$ are related to the discrete q-Hermite polynomials $\tilde{h}(x;q)$ of type II. Indeed, change the base $q \to q^2$ in (4.15) and substitute into it

$$\mathcal{H}_n(x;q^2) = q^{n(n-1)/2}\, \tilde{h}_n(x;q) \tag{4.16}$$

to obtain the recurrence relation (1.8) for the discrete q-Hermite polynomials $\tilde{h}_n(x;q)$ of type II.

There are two immediate consequences of the formula (4.16). Firstly, combining (4.16) with (2.8), one obtains the relationship

$$\begin{aligned}
\tilde{h}_{2n}(x;q) &= (-1)^n q^{-n(2n-1)}\,(q^2;q^2)_n\, L_n^{(-1/2)}(x^2;q^2) , \\
\tilde{h}_{2n+1}(x;q) &= (-1)^n q^{-n(2n+1)}\,(q^2;q^2)_n\, x\, L_n^{(1/2)}(x^2;q^2) ,
\end{aligned} \tag{4.17}$$

between the discrete q-Hermite polynomials $\tilde{h}_n(x;q)$ of type II and the q-Laguerre polynomials (2.3) with $\alpha = \pm 1/2$. Secondly, one can represent the q-polynomials $\mathcal{H}_n(x;q)$, initially defined by (2.8), in the unified form (*cf* (1.7))

$$\mathcal{H}_n(x;q^2) = \mathrm{i}^{-n}\, {}_2\phi_0\left(\begin{matrix} q^{-n}, \mathrm{i}x \\ — \end{matrix} \,\middle|\, q, -q^n \right) = q^{n(n-1)/2} x^n\, {}_2\phi_1\left(\begin{matrix} q^{-n}, q^{1-n} \\ 0 \end{matrix} \,\middle|\, q^2, -q^2/x^2 \right). \tag{4.18}$$

As a consistency check, one may try to verify directly that $\mathcal{H}_n(x;q^2)$, defined by (4.18), does really coincide with the expressions, given by (2.8) for even and odd values of n, respectively. This will lead to the two identities

$$\begin{aligned} {}_2\phi_1\left(\begin{matrix} q^{-n}, y \\ 0 \end{matrix} \,\middle|\, q, q^{n+1/2} \right) &= q^{n(n-1)/2}\, y^n\, {}_2\phi_1\left(\begin{matrix} q^{-n}, q^{1/2-n} \\ 0 \end{matrix} \,\middle|\, q, q/y \right), \\ {}_2\phi_1\left(\begin{matrix} q^{-n}, y \\ 0 \end{matrix} \,\middle|\, q, q^{n+3/2} \right) &= q^{n(n+1)/2}\, y^n\, {}_2\phi_1\left(\begin{matrix} q^{-n}, q^{-(n+1/2)} \\ 0 \end{matrix} \,\middle|\, q, q/y \right), \end{aligned} \tag{4.19}$$

between the terminating basic hypergeometric series ${}_2\phi_1$. The relations (4.19) are straightforward to verify by using the expansion

$$(x;q)_k = \sum_{j=0}^{k} q^{j(j-1)/2} \begin{bmatrix} k \\ j \end{bmatrix}_q (-x)^j \tag{4.20}$$

of the q-shifted factorial $(x;q)_k$ in terms of powers of the variable x.

Yet another consequence of the relation (4.16) can be formulated as the following corollary of theorem (1) in section 3.

Corollary 2 *The sequence of the discrete q-Hermite polynomials $\tilde{h}_n(x;q)$ of type II, defined by (1.7), satisfies the orthogonality relation*

$$\int_{-\infty}^{\infty} \tilde{h}_m(x;q)\, \tilde{h}_n(x;q) \frac{dx}{E_{q^2}(x^2)} = \pi q^{-n^2} (q;q)_n (q;q^2)_{1/2}\, \delta_{mn} \tag{4.21}$$

on the whole real line $\mathbb{R}$ with respect to the continuous positive weight function $w(x) = 1/E_{q^2}(x^2)$.

Proof. Change the base $q \to q^2$ in (3.1) and then use the relation (4.16) to obtain (4.21). ∎

As we noted in the introduction, the discrete q-Hermite polynomials $\tilde{h}_n(x;q)$ of type II are known as polynomials with the discrete orthogonality, supported on the points $x = \pm\, cq^k, c > 0, k \in \mathbb{Z}$. The fact that they satisfy also the continuous orthogonality relation (4.21) does not contradict the general theory of orthogonal polynomials [22, 23]. The point is that the Hamburger moment problem associated with $\{\tilde{h}_n(x;q)\}$ is indeterminate, and therefore they are orthogonal with respect to an infinite class of weight functions, both continuous and discrete ones [19], [24]–[26]. C.Berg studied in [27] some families of discrete solutions to indeterminate moment problems and showed how they can be used to generate absolutely continuous solutions to the same moment problems. In particular, C.Berg derived in [27] the continuous weight function $w(x) = 1/E_{q^2}(x^2)$ for the discrete q-Hermite polynomials $\tilde{h}_n(x;q)$ of type II and evaluated its moments. It should be emphasized that in our approach the orthogonality relation (4.21) emerges as a simple consequence of the continuous orthogonality relation (2.4) for the q-Laguerre polynomials $L_n^{(\pm 1/2)}(x;q)$.

Observe that the continuous orthogonality relation (4.21) is useful for deriving some integrals, involving the q-exponential function $E_q(z)$. We do not go into this question in depth but merely present one of them as an example. One of the generating functions for the discrete q-Hermite polynomials of type II has the form (see [4, p. 119])

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q;q)_n} \tilde{h}_n(x;q) t^n = \frac{E_q(xt)}{E_{q^2}(t^2)} \,. \tag{4.22}$$

Multiply both sides of (4.21) by a constant factor $(\mathrm{i}tq^{m/2})^m(-\mathrm{i}\tau q^{n/2})^n/(q;q)_m(q;q)_n$ and sum over indices m and n from zero to infinity with the aid of the generating function (4.22) and the definition of another Jackson's q-exponential function

$$e_q(z) := \frac{1}{(z;q)_\infty} = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} \,, \qquad |z| < 1 \,. \tag{4.23}$$

This results in the following integral

$$\int_{-\infty}^{\infty} \frac{E_q(iq^{1/2}xt)E_q(-iq^{1/2}x\tau)}{E_{q^2}(x^2)}\, dx = \pi \, (q;q^2)_{1/2} \; e_q(t\tau) \, E_{q^2}(-qt^2) E_{q^2}(-q\tau^2) \,, \quad |t\tau| < 1 \,. \tag{4.24}$$

Since $\lim_{q\to 1} e_q((1-q)z) = e^z$ and the q-exponential function $E_q(z)$ has the same limit (see (3.7)), the formula (4.24) is a q-extension of the well-known integral

$$\int_{-\infty}^{\infty} e^{2i(x-y)t} e^{-t^2} dt = \sqrt{\pi} e^{-(x-y)^2} \,,$$

which reflects the important property of the normal distribution e^{-x^2} of being its own Fourier transform.

5 Concluding remarks

We have discussed in the preceding sections how one can construct the particular polynomial sequence of q-extensions of the Hermite polynomials, either in terms of the q-Laguerre polynomials $L_n^{(\alpha)}(x;q)$, $\alpha = \pm 1/2$, or in terms of the discrete q-Hermite polynomials $\tilde{h}_n(x;q)$ of type II. The sequence so defined satisfies the continuous orthogonality on $\mathbb{R}$ with respect to the positive weight function $w(x) = 1/E_q(x^2)$. It was shown that this orthogonality relation leads to an interesting integral, involving Jackson's q-exponential functions.

It seems that the same approach can be implemented for deriving a q-extension of the generalized Hermite polynomials $H_n^{(\mu)}(x)$ [28]–[30] with the continuous orthogonality property (the case of discrete orthogonality requires a different technique, see [31]). This study is under way.

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