Symmetric Functions in Noncommuting Variables

Mercedes H. Rosas, Departamento de Matemáticas,
Universidad Simón Bolívar, Apdo. Postal 89000,
Caracas, VENEZUELA,
mrosas@ma.usb.ve

and

Bruce E. Sagan, Department of Mathematics,
Michigan State University, East Lansing, MI 48824-1027, USA,
sagan@math.msu.edu

April 9, 2003

Key Words: noncommuting variables, partition lattice, Schur function, symmetric function

AMS subject classification (2000): Primary 05E05; Secondary 05E10, 05A18.

Proposed running head:

Symmetric Functions in Noncommuting Variables

Send proofs to:

Bruce E. Sagan
Department of Mathematics
Michigan State University
East Lansing, MI 48824-1027

Tel.: 517-355-8329
FAX: 517-432-1562
Email: sagan@math.msu.edu

Abstract

Consider the algebra $\mathbb{Q}\langle\langle x_1, x_2, \ldots\rangle\rangle$ of formal power series in countably many noncommuting variables over the rationals. The subalgebra $\Pi(x_1, x_2, \ldots)$ of symmetric functions in noncommuting variables consists of all elements invariant under permutation of the variables and of bounded degree. We develop a theory of such functions analogous to the ordinary theory of symmetric functions. In particular, we define analogs of the monomial, power sum, elementary, complete homogeneous, and Schur symmetric functions as will as investigating their properties.

1 Introduction

Let $\mathbb{Q}[[x_1, x_2, \ldots]] = \mathbb{Q}[[\mathbf{x}]]$ be the algebra of formal power series in a countably infinite set of commuting variables x_i with coefficients in the rational numbers. There is an action of elements g of the symmetric group $\mathfrak{S}_n$ on this algebra given by

$$gf(x_1, x_2, \ldots) = f(x_{g1}, x_{g2}, \ldots) \tag{1}$$

where $gi = i$ for $i > n$. We say that f is *symmetric* if it is invariant under the action of $\mathfrak{S}_n$ for all $n \geq 1$. The *algebra of symmetric functions*, $\Lambda = \Lambda(\mathbf{x})$, consists of all symmetric f that are also of bounded degree. This algebra has a long, venerable history and is also of interest in combinatorics, algebraic geometry, and representation theory. For more information in this regard, see the texts of Fulton [5], Macdonald [17], Sagan [24], or Stanley [30].

Now consider the algebra $\mathbb{Q}\langle\langle x_1, x_2, \ldots\rangle\rangle$ where the x_i do not commute. Define the *algebra of symmetric functions in noncommuting variables*, $\Pi = \Pi(\mathbf{x})$, to be the subalgebra consisting of all elements invariant under the action (1) and of bounded degree. (This is not to be confused with the algebra of noncommutative symmetric functions of Gelfand et. al. [9] or the partially commutative symmetric functions studied by Lascoux and Schützenberger [14] as well as by Fomin and Greene [4].) This algebra was first studied by M. C. Wolf [35] in 1936. Her goal was to provide an analogue of the fundamental theorem of symmetric functions in this context. The concept then lay dormant for over 30 years until Bergman and Cohn generalized Wolf's result [2]. Still later, Kharchenko [12] proved that if V is a positively graded vector space and G is a group of gradation-preserving automorphisms of the tensor algebra TV, then the algebra of invariants of G is also a tensor algebra. Anick [1] then showed that one could remove the hypothesis about G preserving the grading. Most recently, Gebhard and Sagan [8] revived these ideas as a tool for studying Stanley's chromatic symmetric function [27, 29].

The aim of this paper is to give the first systematic study of the properties of $\Pi(\mathbf{x})$. We define analogues of all of the standard bases for $\Lambda(\mathbf{x})$, including the Schur functions. We then study the corresponding basis change equations and inner products. For the Schur functions, which are defined combinatorially via tableaux, we derive analogues Jacobi-Trudi determinants and Robinson-Schensted-Knuth correspondence.

2 Basic definitions

If n is a positive integer, then we define $[n] = \{1, 2, \ldots, n\}$. We will let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ denote an *integer partition of* n which is just a weakly decreasing sequence of positive integers, called *parts*, summing to n. In this case we write $\lambda \vdash n$ and denote the number of parts or *length* of λ by $l = l(\lambda)$. We will also use the notation

$$\lambda = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n}) \tag{2}$$

to mean that i appears in λ with multiplicity m_i, $1 \leq i \leq n$. The bases of the symmetric function algebra $\Lambda(\mathbf{x})$ are indexed by partitions. We use the notation m_λ, p_λ, e_λ, and h_λ for the elements of the monomial, power sum, elementary, and complete homogeneous symmetric functions bases, respectively.

Analogously, define a *set partition* π *of* $[n]$ to be a family of sets, called *blocks*, whose disjoint union is $[n]$. Here we write $\pi = B_1/B_2/\ldots/B_l \vdash [n]$ where the B_i are the blocks and also define *length* $l(\pi)$ as the number of blocks. There is a natural mapping from set partitions to integer partitions given by

$$\lambda(B_1/B_2/\ldots/B_l) = (|B_1|, |B_2|, \ldots, |B_l|)$$

where we assume that the blocks of the set partition have been listed in weakly decreasing order of size. If $\lambda(\pi) = \lambda$ then we say that the integer partition λ is the *type* of the set partition π. We also write $\pi \in \lambda$ for this relation.

One of the nice features of partitions of $[n]$ is that there is an associated ranked lattice Π_n. (Do not confuse Π_n with the algebra $\Pi(\mathbf{x})$.) In Π_n the ordering is by refinement: $\pi \leq \sigma$ if each block B of π is contained in some block C of σ. The meet (greatest lower bound) and join (least upper bound) operations will be denoted $\wedge$ and $\vee$, respectively. Finally, the rank function is just

$$r(\pi) = n - l(\pi).$$

To obtain analogues of the bases of Λ in this setting, take a set partition π of $[n]$. It will be helpful to think of the elements of $[n]$ as indexing the positions in a monomial $x_{i_1}x_{i_2}\cdots x_{i_n}$. This makes sense because the variables do not commute. Now define the *monomial symmetric function in noncommuting variables*, m_π, by

$$m_\pi = \sum_{i_1,i_2,\ldots,i_n} x_{i_1}x_{i_2}\cdots x_{i_n} \text{ where } i_j = i_k \text{ iff } j,k \text{ are in the same block in } \pi.$$

For example,

$$m_{13/24} = x_1x_2x_1x_2 + x_2x_1x_2x_1 + x_1x_3x_1x_3 + x_3x_1x_3x_1 + x_2x_3x_2x_3 + x_3x_2x_3x_2 + \cdots$$

Note that these functions are exactly those gotten by symmetrizing a monomial and so are invariant under the action defined previously. It follows easily that they form a basis for $\Pi(\mathbf{x})$.

We will now define noncommuting-variable analogues of the power sum, elementary, and complete homogeneous functions. For the first of these, given a set partition π we define the *power sum function in noncommuting variables*, p_π, by

$$p_\pi = \textstyle\sum_{i_1,i_2,\ldots,i_n} x_{i_1}x_{i_2}\cdots x_{i_n} \text{ where } i_j = i_k \text{ if } j,k \text{ are in the same block in } \pi.$$

To illustrate,

$$\begin{aligned} p_{13/24} &= x_1x_2x_1x_2 + x_2x_1x_2x_1 + x_1^4 + x_2^4 + \cdots \\ &= m_{13/24} + m_{1234}. \end{aligned}$$

It is not clear why these functions form a basis for $\Pi(\mathbf{x})$ or why they deserve the power sum moniker. We will establish the former fact in the next section and the latter at the end of the current one as will also be done for the following analogues.

Define the *elementary symmetric functions in noncommuting variables* to be

$$e_\pi = \textstyle\sum_{i_1,i_2,\ldots,i_n} x_{i_1}x_{i_2}\cdots x_{i_n} \text{ where } i_j \neq i_k \text{ if } j,k \text{ are in the same block in } \pi.$$

By way of example

$$\begin{aligned} e_{13/24} &= x_1x_1x_2x_2 + x_2x_2x_1x_1 + x_1x_2x_2x_1 + x_2x_1x_2x_1 + \cdots \\ &= m_{12/34} + m_{14/23} + m_{12/3/4} + m_{14/2/3} + m_{1/23/4} + m_{1/2/34} + m_{1/2/3/4}. \end{aligned}$$

To define the analogue of the complete homogeneous symmetric functions, it will be useful to introduce another way of looking at the previous definitions. This was the method that Doubilet [3] used to define his ordinary symmetric functions associated with set partitions. Any two sets D, R and a function $f : D \to R$ determine a *kernel*, $\ker f \vdash D$, whose blocks are the nonempty sets among the $f^{-1}(r)$ for $r \in R$. If our function looks like $f : [n] \to \mathbf{x}$, then there is a corresponding *monomial*

$$M_f = f(1)f(2)\cdots f(n).$$

Directly from these definitions it follows that

$$m_\pi = \sum_{\ker f = \pi} M_f.$$

Using our running example, if $\pi = 13/24$ then the functions with $\ker f = \pi$ are exactly those of the form $f(1) = f(3) = x_i$ and $f(2) = f(4) = x_j$ where $i \neq j$. This f is the one that gives rise to the monomial $M_f = x_ix_jx_ix_j$ in the sum for $m_{13/24}$.

Now define

$$h_\pi = \sum_{(f,L)} M_f \tag{3}$$

where $f : [n] \to \mathbf{x}$ and L is a linear ordering of the elements of each block of $\ker f \wedge \pi$. Continuing with our usual example partition,

$$
\begin{aligned}
h_{13/24} = {} & m_{1/2/3/4} + m_{12/3/4} + 2m_{13/2/4} + m_{14/2/3} + m_{1/23/4} + 2m_{1/24/3} + m_{1/2/34} \\
& + m_{12/34} + 4m_{13/24} + m_{14/23} + 2m_{123/4} + 2m_{124/3} + 2m_{134/2} + 2m_{1/234} + 4m_{1234}.
\end{aligned}
$$

Now that we have defined all the symmetric functions in noncommuting variables that we will consider at this point, we would like to give some justification for their names by exhibiting their relation to the corresponding ordinary symmetric functions. To do so, consider the *forgetful* or *projection* map

$$\rho : \mathbb{Q}\langle\langle \mathbf{x} \rangle\rangle \to \mathbb{Q}[[\mathbf{x}]]$$

which merely lets the variables commute. We also need the notation

$$
\begin{aligned}
\lambda! &= \lambda_1!\lambda_2!\cdots\lambda_l! \\
\lambda^{!} &= m_1!m_2!\cdots m_n!
\end{aligned}
$$

where the m_i are the multiplicities in (2). We extend these conventions to set partitions by letting $\pi! = \lambda(\pi)!$ and similarly for the "exponential" factorial. Note that given λ,

$$\binom{n}{\lambda} := \text{number of } \pi \text{ of type } \lambda = \frac{n!}{\lambda!\lambda^{!}}. \tag{4}$$

The next proposition was proved by Doubilet for his set partition analogues of ordinary symmetric functions and a similar proof can be given in the noncommuting case. However, we will include a demonstration which tries to bring out the combinatorics behind some of Doubilet's algebraic manipulations.

Theorem 2.1 *The images of our noncommuting functions under the forgetful map are*

(i) $\rho(m_\pi) = \pi^{!} m_{\lambda(\pi)}$,

(ii) $\rho(p_\pi) = p_{\lambda(\pi)}$,

(iii) $\rho(e_\pi) = \pi! e_{\lambda(\pi)}$,

(iv) $\rho(h_\pi) = \pi! h_{\lambda(\pi)}$.

Proof For (i), pick an integer b and let $B_1, B_2, \ldots, B_k$ be the blocks of π of size b. Since m_π is constant on the positions corresponding to each B_i and they are of the same size, the variables in the B_i positions can be switched with the variables in the B_j positions for any i, j and still give the same monomial in the projection.

It follows that these blocks give rise to a factor of $k!$ in the projection, and so π will contribute $\pi^!$.

To prove (ii), note that $p_{[n]} = m_{[n]}$ and so from (i) $\lambda(p_{[n]}) = m_n = p_n$. Now $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_l}$ while $p_\pi = p_{B_1/B_2/\ldots/B_l}$ can be thought of as a shuffle of $p_{[\lambda_1]}$, $p_{[\lambda_2]}$, ..., $p_{[\lambda_l]}$ where $|B_i| = \lambda_i$ and the elements from $p_{[\lambda_i]}$ are only permitted to be in the positions indexed by B_i. The desired equality follows.

The proof of (iii) is similar. We have $\rho(e_{[n]}) = n! e_n$ since if all n positions have different variables, then they can be permuted in any of $n!$ ways and still give the same monomial in the projection. In the general case, we have the same phenomenon of multiplication corresponding to shuffling, with each block B contributing $|B|!$. So the total contribution is $\pi!$.

Finally we consider (iv). By the same argument as in (iii), it suffices to show that $\rho(h_{[n]}) = n! h_n$. Consider a monomial $M = x_{j_1}^{\lambda_1} x_{j_2}^{\lambda_2} \cdots a_{j_l}^{\lambda_l}$ in h_n. These variables can be rearranged to form $n!/\lambda!$ monomials in noncommuting variables where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$. To obtain one of these monomials in (3) we must have $\lambda(\ker f) = \lambda$ since $\ker f \wedge [n] = \ker f$. But then the number of pairs (f, L) is just $\lambda!$. So the number of monomials in h_π mapping to M under ρ is just $\lambda! \cdot n!/\lambda! = n!$, which is what we wanted. $\blacksquare$

We end this section by defining a second action of $\mathfrak{S}_n$ that is also interesting. Since our variables do not commute, we can define an action on places (rather than variables). Explicitly, consider the vector space of elements of $\Pi(\mathbf{x})$ which are homogeneous of degree n. Given a monomial of that degree, we define

$$g \circ (x_{i_1} x_{i_2} \cdots x_{i_n}) = x_{i_{g1}} x_{i_{g2}} \cdots x_{i_{gn}} \tag{5}$$

and extend linearly. It is easy to see that if b_π is a basis element for any of our four bases, then $g \circ b_\pi = b_{g\pi}$ where g acts on set partitions in the usual manner.

3 Change of basis

We will now show that all the symmetric functions in noncommuting variables defined in the previous section form bases for Λ. Since we already know this for the m_π, it suffices to find change of basis formulas expressing each function in terms of the m_π and vice-versa. Note that any other basis exchange formula can be found from these by composing. Doubilet [3] has obtained these results as well as those in the next section in a formal setting that includes ours as a special case. But we replicate his theorems and proofs here for completeness, to present them in standard notation, and to extend and simplify some of them.

Expressing each symmetric function in terms of m_π is easy to do directly from the definitions, so the following proposition is given without proof. In it, all lattice operations refer to Π_n and $\hat{0}$ is the unique minimal element $1/2/\ldots/n$.

Theorem 3.1 *We have the following change of basis formulae.*

(i) $p_\pi = \sum_{\sigma \geq \pi} m_\sigma$,

(ii) $e_\pi = \sum_{\sigma \wedge \pi = \hat{0}} m_\sigma$,

(iii) $h_\pi = \sum_{\sigma} (\pi \wedge \sigma)! m_\sigma$. ■

To express m_π in terms of the other functions, we will need the Möbius function of Π_n. The *Möbius function* of any partially ordered set P is the function $\mu : P \times P \to \mathbb{Z}$ defined inductively by

$$\mu(a,b) = \begin{cases} 1 & \text{if } a = b, \\ -\sum_{a \leq c < b} \mu(a,c) & \text{else.} \end{cases}$$

This can be rewritten in the useful and more intuitive form

$$\sum_{a \leq c \leq b} \mu(a,c) = \delta_{a,b} \tag{6}$$

where $\delta_{a,b}$ is the Kronecker delta. For more information about Möbius functions, see the seminal article of Rota [23] or the book of Stanley [28].

The Möbius function of Π_n is well known. In particular

$$\mu(\hat{0}, \hat{1}) = (-1)^{n-1}(n-1)!$$

where $\hat{1} = 12\ldots n$ is the unique maximal element of Π_n. This is enough to determine μ on any interval of this lattice. For example, for any $\pi = B_1/B_2/\ldots/B_l$ and $\lambda = \lambda(\pi)$ we have the lattice isomorphism $[\hat{0}, \pi] \cong \prod_i \Pi_{\lambda_i}$. Since the Möbius function is preserved by isomorphism and distributes over products, we have

$$\mu(\hat{0}, \pi) = \prod_i (-1)^{\lambda_i - 1}(\lambda_i - 1)!$$

Note that, up to sign, this is just the number of permutations $\alpha \in \mathfrak{S}_n$ which have disjoint cycle decomposition $\alpha = \alpha_1 \alpha_2 \cdots \alpha_l$ where the elements of α_i are just those in B_i for all i. It follows that

$$\sum_{\sigma \in \Pi_n} |\mu(\hat{0}, \sigma)| = n!$$

Or more generally, because of multiplicativity,

$$\sum_{\sigma \leq \pi} |\mu(\hat{0}, \sigma)| = \pi! \tag{7}$$

a result which will be useful shortly. We finally note that if $\sigma = C_1/C_2/\ldots/C_m$ satisfies $\sigma \leq \pi$ then we still have an isomorphism $[\sigma, \pi] \cong \prod_i \Pi_{\lambda_i(\sigma,\pi)}$ where $\lambda(\sigma, \pi)$ is the integer partition whose ith part is the the number of blocks of σ contained in the ith block of π. (We assume the blocks are listed so that the parts are in weakly decreasing order). Of course, $\lambda(\hat{0}, \pi)$ is just the type of π.

All of the rest of the proofs in this section will be based on the Möbius Inversion Theorem. This result was first proved in somewhat less generality by Weisner [33]. The reader is encouraged to consult Rota's influential article [23] for more details.

Theorem 3.2 (Möbius Inversion Theorem) *Let P be a poset, let $(G, +)$ be an abelian group, and consider two functions $f, g : P \to G$. Then*

$$f(a) = \sum_{b \geq a} g(b) \quad \textit{for all } a \in P \iff g(a) = \sum_{b \geq a} \mu(a, b) f(b) \quad \textit{for all } a \in P.$$

Dually

$$f(a) = \sum_{b \leq a} g(b) \quad \textit{for all } a \in P \iff g(a) = \sum_{b \leq a} \mu(b, a) f(b) \quad \textit{for all } a \in P. \qquad \blacksquare$$

We will also need a simple corollary of this theorem that slightly generalizes a result of Doubilet.

Corollary 3.3 *Let P be a poset, let F be a field, and consider three functions $f, g, h : P \to F$ where $g(a) \neq 0$ for all $a \in P$. Then*

$$f(a) = \sum_{b \leq a} g(b) \sum_{c \geq b} h(c) \quad \textit{for all } a \in P$$
$$\iff h(a) = \sum_{c \geq a} \frac{\mu(a, c)}{g(c)} \sum_{b \leq c} \mu(b, c) f(b) \quad \textit{for all } a \in P.$$

Proof We will prove the forward direction as the converse is obtained just by reversing the steps. Doing (dual) Möbius inversion on the outer sum for $f(a)$ gives

$$g(a) \sum_{c \geq a} h(c) = \sum_{b \leq a} \mu(b, a) f(b).$$

Since $g(a) \neq 0$, we can divide by it and then invert the sum containing $h(c)$, which gives the desired result. $\blacksquare$

We are now in a position to invert each of the sums in Theorem 3.1.

Theorem 3.4 *We have the following change of basis formulae.*

(i′) $m_\pi = \sum_{\sigma \geq \pi} \mu(\pi, \sigma) p_\sigma$,

(ii′) $m_\pi = \sum_{\sigma \geq \pi} \frac{\mu(\pi, \sigma)}{\mu(\hat{0}, \sigma)} \sum_{\tau \leq \sigma} \mu(\tau, \sigma) e_\tau$,

(iii′) $m_\pi = \sum_{\sigma \geq \pi} \frac{\mu(\pi, \sigma)}{|\mu(\hat{0}, \sigma)|} \sum_{\tau \leq \sigma} \mu(\tau, \sigma) h_\tau$.

Proof Equation (i′) follows immediately from the Möbius Inversion Theorem applied to part (i) of Theorem 3.1.

For identity (ii′), use equation (6) to write (ii) of Theorem 3.1 in the form

$$\begin{aligned} e_\pi &= \sum_{\sigma} \left(\sum_{\tau \leq \pi \wedge \sigma} \mu(\hat{0}, \tau) \right) m_\sigma \\ &= \sum_{\tau \leq \pi} \mu(\hat{0}, \tau) \sum_{\sigma \geq \tau} m_\sigma. \end{aligned}$$

Using the corollary to invert this double sum gives the desired result.

Finally consider (iii′). Applying (7) to Theorem 3.1 (iii) gives

$$\begin{aligned} h_\pi &= \sum_{\sigma} \left(\sum_{\tau \leq \pi \wedge \sigma} |\mu(\hat{0}, \tau)| \right) m_\sigma \\ &= \sum_{\tau \leq \pi} |\mu(\hat{0}, \tau)| \sum_{\sigma \geq \tau} m_\sigma. \end{aligned}$$

The corollary again provides the last step. ■

The other bases change equations are derived using similar techniques, so we will content ourselves with merely stating the result after one last bit of notation. We define the *sign of* π, $(-1)^\pi$, to be the sign of any permutation gotten by replacing each block of π by a cycle. Note that

$$\mu(\hat{0}, \pi) = (-1)^\pi |\mu(\hat{0}, \pi)|. \tag{8}$$

Theorem 3.5 *We have the following change of basis formulae.*

$$
\begin{array}{rclcrcl}
e_\pi & = & \displaystyle\sum_{\sigma\le\pi} \mu(\hat{0},\sigma) p_\sigma & \qquad & p_\pi & = & \displaystyle\frac{1}{\mu(\hat{0},\pi)} \sum_{\sigma\le\pi} \mu(\sigma,\pi) e_\sigma \\[3ex]
h_\pi & = & \displaystyle\sum_{\sigma\le\pi} |\mu(\hat{0},\sigma)| p_\sigma & & p_\pi & = & \displaystyle\frac{1}{|\mu(\hat{0},\pi)|} \sum_{\sigma\le\pi} \mu(\sigma,\pi) h_\sigma \\[3ex]
e_\pi & = & \displaystyle\sum_{\sigma\le\pi} (-1)^\sigma \lambda(\sigma,\pi)! h_\sigma & & h_\pi & = & \displaystyle\sum_{\sigma\le\pi} (-1)^\sigma \lambda(\sigma,\pi)! e_\sigma
\end{array}
\qquad \blacksquare
$$

As an application of these equations, we will derive the properties of an analogue of the involution $\omega : \Lambda(\mathbf{x}) \to \Lambda(\mathbf{x})$ defined by linearly extending

$$\omega(e_\lambda) = h_\lambda.$$

Define a map on $\Pi(\mathbf{x})$, which we will also call ω, by

$$\omega(e_\pi) = h_\pi$$

for all set partitions π and linear extension.

Theorem 3.6 *The map $\omega : \Pi(\mathbf{x}) \to \Pi(\mathbf{x})$ has the following properties.*

(i) It is an involution.

(ii) Each p_π is an eigenvector for ω with eigenvalue $(-1)^\pi$.

(iii) It commutes with ρ, where we used the standard involution in $\Lambda(\mathbf{x})$ and the noncommutative version in $\Pi(\mathbf{x})$.

Proof (i) It suffices to show that the change of basis matrix between the elementary and complete homogeneous symmetric functions equals its inverse. This follows directly from the previous theorem.

(ii) We merely compute the action of ω on the power sum basis by expressing it in terms of the elementary symmetric functions and using equation (8)

$$
\begin{aligned}
\omega(p_\pi) &= \omega\left(\frac{1}{\mu(\hat{0},\pi)} \sum_{\sigma\le\pi} \mu(\sigma,\pi) e_\sigma\right) \\
&= \frac{1}{\mu(\hat{0},\pi)} \sum_{\sigma\le\pi} \mu(\sigma,\pi) h_\sigma \\
&= \frac{(-1)^\pi}{|\mu(0,\pi)|} \sum_{\sigma\le\pi} \mu(\sigma,\pi) h_\sigma \\
&= (-1)^\pi p_\pi.
\end{aligned}
$$

(iii) It suffices to show that commutivity holds on a basis. So we compute using Theorem 2.1 (iii) and (iv)

$$\omega\rho(e_\pi) = \omega(\pi! e_{\lambda(\pi)}) = \pi! h_{\lambda(\pi)} = \rho(h_\pi) = \rho\omega(e_\pi). \qquad \blacksquare$$

4 The lifting map and inner products

We will now introduced a partial right inverse $\tilde{\rho}$ for the projection map ρ and an inner product for which $\tilde{\rho}$ is an isometry. Define the *lifting map*

$$\tilde{\rho} : \Lambda(\mathbf{x}) \to \Pi(\mathbf{x})$$

by linearly extending

$$\tilde{\rho}(m_\lambda) = \frac{\lambda!}{n!} \sum_{\pi \in \lambda} m_\pi. \tag{9}$$

Proposition 4.1 *The map $\rho\tilde{\rho}$ is the identity on $\Lambda(\mathbf{x})$.*

Proof Using equation (4) and Theorem 2.1 (i)

$$\rho\tilde{\rho}(m_\lambda) = \frac{\lambda!}{n!} \sum_{\pi \in \lambda} \lambda^! m_\lambda = m_\lambda. \qquad \blacksquare$$

The standard inner product on $\Lambda(\mathbf{x})$ is defined by

$$\langle m_\lambda, h_\mu \rangle = \delta_{\lambda,\mu}$$

and bilinear extension. Analogously, in $\Pi(\mathbf{x})$ consider the bilinear form defined by extending

$$\langle m_\pi, h_\sigma \rangle = n! \delta_{\pi,\sigma}$$

where $\pi \vdash [n]$. It follows directly from the definition that this form respects the grading of $\Pi(x)$ in the sense that if f, g are two homogeneous symmetric functions with $\langle f, g \rangle \neq 0$ then f and g have the same degree. Our next order of business will be to show that this actually defines an inner product which is invariant under the action on places (5) and that $\tilde{\rho}$ is an isometry.

Theorem 4.2 *The bilinear form $\langle \cdot, \cdot \rangle$ has the following properties.*

(i) It is symmetric and positive definite and so an inner product.

(ii) It is invariant under the action (5).

(iii) It makes the map $\tilde{\rho} : \Lambda(\mathbf{x}) \to \Pi(\mathbf{x})$ an isometry.

Proof (i) For symmetry it suffices to show, because of linearity, that $\langle h_\pi, h_\sigma \rangle = \langle h_\sigma, h_\pi \rangle$. But by Theorem 3.1 (iii)

$$\langle h_\pi, h_\sigma \rangle = \left\langle \sum_\tau (\pi \wedge \tau)! m_\tau, h_\sigma \right\rangle = n!(\pi \wedge \sigma)!$$

where we let $(\pi \wedge \sigma)! = 0$ if π and σ are partitions of different sets. Noting that $(\pi \wedge \sigma)! = (\sigma \wedge \pi)!$ completes the proof of symmetry.

As for positive definiteness, take any symmetric function f in noncommuting variables and write $f = \sum_\pi c_\pi p_\pi$ for certain coefficients c_π. Then, using the expansions for the power sums in Theorems 3.1 and 3.5, we have

$$\begin{aligned}
\langle f, f \rangle &= \left\langle \sum_\sigma c_\sigma p_\sigma, \ \sum_\tau c_\tau p_\tau \right\rangle \\
&= \left\langle \sum_\sigma c_\sigma \sum_{\pi \geq \sigma} m_\pi, \ \sum_\tau c_\tau \frac{1}{|\mu(\hat{0}, \tau)|} \sum_{\pi \leq \tau} \mu(\pi, \tau) h_\pi \right\rangle \\
&= n! \sum_\pi \left(\sum_{\sigma \leq \pi} c_\sigma \right) \left(\sum_{\tau \geq \pi} c_\tau \frac{1}{|\mu(\hat{0}, \tau)|} \mu(\pi, \tau) \right).
\end{aligned}$$

Now the coefficient of $c_\sigma c_\tau$ in this last sum is

$$\frac{n!}{|\mu(\hat{0}, \tau)|} \sum_{\sigma \leq \pi \leq \tau} \mu(\pi, \tau) = \frac{n! \delta_{\sigma,\tau}}{|\mu(\hat{0}, \tau)|}.$$

Since this is zero for $\sigma \neq \tau$ and positive otherwise, our form is positive definite.

(ii) It suffices to verify invariance on a pair of bases:

$$\langle g \circ m_\pi, g \circ h_\sigma \rangle = \langle m_{g\pi}, h_{g\sigma} \rangle = n! \delta_{g\pi, g\sigma} = n! \delta_{\pi,\sigma} = \langle m_\pi, h_\sigma \rangle.$$

(iii) It suffices to show that $\langle \tilde{\rho}(m_\lambda), \tilde{\rho}(h_\mu) \rangle = \langle m_\lambda, h_\mu \rangle$ for all λ, μ. In order to compute $\tilde{\rho}(h_\mu)$ consider

$$H_\mu = \sum_{\pi \in \mu} h_\pi.$$

Expressing H_μ in terms of the monomial symmetric function basis using Theorem 3.1 (iii), we see that the coefficient of m_σ is the sum of $(\pi \wedge \sigma)!$ over all $\pi \in \mu$.

But the usual action of the symmetric group on set partitions shows that this quantity only depends on $\lambda(\sigma)$. Thus by (9), H_μ must be in the image of $\tilde{\rho}$. Since ρ is a left-inverse for $\tilde{\rho}$, we see that H_μ is the image under $\tilde{\rho}$ of

$$\rho(H_\mu) = \sum_{\pi\in\mu} \rho(h_\pi) = \sum_{\pi\in\mu} \lambda(\pi)! h_{\lambda(\pi)} = \binom{n}{\mu} \mu! h_\mu = \frac{n!}{\mu^!} h_\mu.$$

So finally

$$\langle \tilde{\rho}(m_\lambda), \tilde{\rho}(h_\mu)\rangle = \left\langle \frac{\lambda!}{n!} \sum_{\pi\in\lambda} m_\pi, \frac{\mu^!}{n!} \sum_{\sigma\in\mu} h_\sigma \right\rangle = \frac{\lambda!\lambda^!}{n!^2} \binom{n}{\lambda} n! \delta_{\lambda,\mu} = \delta_{\lambda,\mu} = \langle m_\lambda, h_\mu\rangle. \qquad \blacksquare$$

We could define the this inner product in terms of any other pair of bases. To state what these other formulae look like, we need the *zeta function* of the lattice Π_n defined by

$$\zeta(\pi,\sigma) = \begin{cases} 1 & \text{if } \pi \le \sigma, \\ 0 & \text{else.} \end{cases}$$

Proving the equivalence of the inner products defined for each pair of bases is similar to the derivation of the formula for $\langle h_\pi, h_\sigma\rangle$ in the previous proof. So we will omit the demonstration of the following theorem.

Theorem 4.3 *The following formulae define equivalent bilinear forms.*

$$\begin{array}{rclrcl}
\langle e_\pi, e_\sigma\rangle &=& n!(\pi\wedge\sigma)! & \langle e_\pi, h_\sigma\rangle &=& n!\delta_{\pi\wedge\sigma,\hat{0}} \\
\langle e_\pi, p_\sigma\rangle &=& (-1)^\sigma n!\zeta(\sigma,\pi) & \langle e_\pi, m_\sigma\rangle &=& (-1)^\sigma n!\lambda(\sigma,\pi)!\zeta(\sigma,\pi) \\
\langle h_\pi, h_\sigma\rangle &=& n!(\pi\wedge\sigma)! & \langle h_\pi, p_\sigma\rangle &=& n!\zeta(\sigma,\pi) \\
\langle h_\pi, m_\sigma\rangle &=& n!\delta_{\pi,\sigma} & \langle p_\pi, p_\sigma\rangle &=& n!\dfrac{\delta_{\pi,\sigma}}{|\mu(\hat{0},\pi)|} \\
\langle p_\pi, m_\sigma\rangle &=& n!\dfrac{\mu(\sigma,\pi)\zeta(\sigma,\pi)}{|\mu(\hat{0},\pi)|} & \langle m_\pi, m_\sigma\rangle &=& n!\displaystyle\sum_{\tau\ge\pi\vee\sigma} \frac{\mu(\pi,\tau)\mu(\sigma,\tau)}{|\mu(\hat{0},\tau)|}
\end{array} \qquad \blacksquare$$

5 MacMahon symmetric functions

We have delayed defining Schur functions in noncommuting variables because to do so it is best to introduce another piece of machinery, namely the MacMahon symmetric functions. The connection between functions in noncommuting variables and MacMahon symmetric functions was first pointed out by Rosas [21, 22].

Consider n sets of variables

$$\begin{aligned} \dot{\mathbf{x}} &= \{\dot{x}_1, \dot{x}_2, \ldots\}, \\ \ddot{\mathbf{x}} &= \{\ddot{x}_1, \ddot{x}_2, \ldots\}, \\ &\vdots \\ \mathbf{x^{(n)}} &= \{x_1^{(n)}, x_2^{(n)}, \ldots\}. \end{aligned}$$

Let $g \in \mathfrak{S}_m$ act on $f(\dot{\mathbf{x}}, \ddot{\mathbf{x}}, \ldots, \mathbf{x^{(n)}}) \in \mathbb{Q}[[(\dot{\mathbf{x}}, \ddot{\mathbf{x}}, \ldots, \mathbf{x^{(n)}}]]$ diagonally, i.e.,

$$gf(\dot{x}_1, \ddot{x}_1, \ldots, \dot{x}_2, \ddot{x}_2, \ldots) = f(\dot{x}_{g1}, \ddot{x}_{g1}, \ldots, \dot{x}_{g2}, \ddot{x}_{g2}, \ldots).$$

We consider a function *symmetric* if it is left invariant by all $g \in \mathfrak{S}_m$ for all m.

Consider the monomial

$$M = \dot{x}_1^{a_1} \ddot{x}_1^{b_1} \cdots x_1^{(n)c_1} \dot{x}_2^{a_2} \ddot{x}_2^{b_2} \cdots x_2^{(n)c_2} \cdots.$$

Letting $\lambda^i = [a_i, b_i, \ldots, c_i]$ be the exponent sequence of the variables of subscript i, we define the *multiexponent of M* to be

$$\begin{aligned} \vec{\lambda} &= \{\lambda^1, \lambda^2, \ldots\} \\ &= \{[a_1, b_1, \ldots, c_1], [a_2, b_2, \ldots, c_2], \ldots\}. \end{aligned}$$

By summing up the vectors which make up the parts of $\vec{\lambda}$ we get the *multidegree of M*

$$\begin{aligned} \vec{m} &= [m_1, m_2, \ldots, m_n] \\ &= [a_1, b_1, \ldots, c_1] + [a_2, b_2, \ldots, c_2] + \cdots. \end{aligned}$$

In this situation we write $\vec{\lambda} \vdash \vec{m}$ and $\vec{m} \vdash m$ where $m = \sum_i m_i$.

Given $\vec{\lambda}$, there is an associated *monomial MacMahon symmetric function* defined by

$$m_{\vec{\lambda}} = \text{ sum of all the monomials with multiexponent } \vec{\lambda}.$$

By way of example

$$m_{[2,1],[3,0]} = \dot{x}_1^2 \ddot{x}_1 \dot{x}_2^3 + \dot{x}_1^3 \dot{x}_2^2 \ddot{x}_2 + \cdots$$

Note that we drop the curly brackets about $\vec{\lambda}$ for readability. Again, these functions are exactly the ones gotten by acting on a monomial with all possible permutations. We now define the *algebra of MacMahon symmetric functions*, $\mathcal{M} = \mathcal{M}(\dot{\mathbf{x}}, \ddot{\mathbf{x}}, \ldots, \mathbf{x^{(n)}})$, to be the span of the $m_{\vec{\lambda}}$ as $\vec{\lambda}$ runs over all vector partitions where each part is a vector of n nonnegative integers. As before, the $m_{\vec{\lambda}}$ are independent and so give a basis for $\mathcal{M}$.

For analogues of the other 3 bases of Λ, call a basis $b_{\vec{\lambda}}$ of $\mathcal{M}$ *multiplicative* if it is constructed by first defining $b_{[a,b,\ldots,c]}$, i.e., for a vector partition with only one part, and then letting

$$b_{\vec{\lambda}} = b_{\lambda^1} b_{\lambda^2} \cdots b_{\lambda^l}.$$

Note that it is sometimes easiest to define $b_{[a,b,\ldots,c]}$ in terms of its generating function. We now declare the *power sum, elementary, and complete homogeneous MacMahon symmetric functions* to be multiplicative with

$$\begin{aligned} p_{[a,b,\ldots,c]} &= m_{[a,b,\ldots,c]} \\ \sum_{a,b,\ldots,c} e_{[a,b,\ldots,c]} q^a r^b \cdots s^c &= \prod_{i\geq 1} \left(1 + \dot{x}_i q + \ddot{x}_i r + \cdots + x_i^{(n)} s\right) \\ \sum_{a,b,\ldots,c} h_{[a,b,\ldots,c]} q^a r^b \cdots s^c &= \prod_{i\geq 1} \frac{1}{1 - \dot{x}_i q - \ddot{x}_i r - \cdots - x_i^{(n)} s}. \end{aligned}$$

To see the connection with noncommutative symmetric functions, let $[1^n]$ denote the vector of n ones. Now consider the subspace $\mathcal{M}_{[1^n]}$ of $\mathcal{M}$ spanned by all the $m_{\vec{\lambda}}$ where $\vec{\lambda} \vdash [1^n]$. There is a map

$$\Phi : \bigoplus_{n\geq 0} \mathcal{M}_{[1^n]} \to \Pi$$

given by

$$\dot{x}_i \ddot{x}_j \cdots x_k^{(n)} \overset{\Phi}{\mapsto} x_i x_j \cdots x_k$$

and linear extension.

Theorem 5.1 ([21]) *The map Φ is an isomorphism of vector spaces. Furthermore, for each basis we have discussed*

$$b_{\lambda^1,\lambda^2,\ldots,\lambda^l} \overset{\Phi}{\mapsto} b_{B_1/B_2/\ldots/B_l}$$

where $b = m$, p, e, or h and λ^i is the characteristic vector of B_i. ∎

By way of illustration

$$b_{[1,0,1,0],[0,1,0,1]} \overset{\Phi}{\mapsto} b_{13/24}$$

for any of our bases.

6 Schur functions

We will now give a combinatorial definition of a Schur function in the setting of MacMahon symmetric functions. This will give, via the map Φ, such a function in noncommuting variables.

Let $\lambda \vdash m$ and let T be a semistandard Young tableau of shape λ, denoted $\lambda(T) = \lambda$. We will write all our shapes in English notation. Now let $\vec{m}$ be a nonnegative integer vector of n components whose entries sum to m. Define a *dotted Young tableaux* $\dot{T}$ *of multidegree* $\vec{m} = [m_1, m_2, \ldots, m_n]$ to be any array obtained from T by putting single dots on m_1 entries of T, double dots on m_2 entries of T, etc. Now define the corresponding *MacMahon Schur function* to be

$$S_\lambda^{\vec{m}} = \sum_{\lambda(\dot{T})=\lambda} M_{\dot{T}} \quad \text{where} \quad M_{\dot{T}} = \prod_{i^{(j)} \in \dot{T}} x_i^{(j)}$$

and elements $i^{(j)}$ of $\dot{T}$ appear in the product with multiplicity. For example, if $\lambda = (3,1)$ and $\vec{m} = [2,2]$, then the coefficient of $\dot{x}_1^2 \ddot{x}_1 \ddot{x}_2$ in $S_\lambda^{\vec{m}}$ is 3, corresponding to the three dotted tableaux

$$T_1 = \begin{array}{ccc} \dot{1} & \dot{1} & \ddot{1} \\ \ddot{2} & & \end{array}, \quad T_2 = \begin{array}{ccc} \dot{1} & \ddot{1} & \dot{1} \\ \ddot{2} & & \end{array}, \quad T_3 = \begin{array}{ccc} \ddot{1} & \dot{1} & \dot{1} \\ \ddot{2} & & \end{array}.$$

There is another description of these tableaux that will also be useful. Consider the alphabet

$$\begin{aligned} A &= \dot{A} \uplus \ddot{A} \uplus \cdots \uplus A^{(n)} \\ &= \{\dot{1}, \dot{2}, \ldots\} \uplus \{\ddot{1}, \ddot{2}, \ldots\} \uplus \cdots \uplus \{1^{(n)}, 2^{(n)}, \ldots\}. \end{aligned}$$

Partially order A by saying that

$$i^{(k)} < j^{(l)} \quad \text{if and only if} \quad i < j. \tag{10}$$

Then a dotted Young tableaux of shape λ and multidegree $\vec{m}$ is just a filling of the shape of λ with elements of A so that rows are nondecreasing, columns are strictly increasing, and there are m_k entries with k dots. It will also be useful to generalize the notion of having multidegree $\vec{m}$ to any multiset M of elements from A, possibly with additional structure, having m_k elements with k dots. In this case we write $\vec{m}(M) = \vec{m}$.

Note that it is not clear from these definitions that $S_\lambda^{\vec{m}}$ is a MacMahon symmetric function, so we do that now.

Theorem 6.1 *The function $S_\lambda^{\vec{m}}$ is invariant under the diagonal action of the symmetric groups.*

Proof Because any permutation is a product of adjacent transpositions, it suffices to show that $S_\lambda^{\vec{m}}$ is invariant under the transposition $(i, i+1)$ where $i \geq 1$. So it suffices to find a shape-preserving involution on dotted tableaux $\dot{T} \to \dot{T}'$ which exchanges the number of elements equal to $i^{(k)}$ with the number equal to $(i+1)^{(k)}$ for all k, $1 \leq k \leq n$. We will use a generalization of a map of Knuth [13] used to prove that the ordinary Schur functions are symmetric.

Since $\dot{T}$ is semistandard, each column contains either a pair $i^{(k)}, (i+1)^{(l)}$; exactly one of $i^{(k)}$ or $(i+1)^{(l)}$; or neither. In the first case, replace the pair by $i^{(l)}, (i+1)^{(k)}$. In the second, replace $i^{(k)}$ by $(i+1)^{(k)}$ or replace $(i+1)^{(l)}$ by $i^{(l)}$ as appropriate. And in the third case there is nothing to do. It is easy to verify that this is an involution having the desired properties. ■

If $\vec{m} = (1, 1, \ldots, 1)$ then we will write S_λ for $S_\lambda^{\vec{m}}$ and make no distinction between S_λ and its image under the map Φ. The latter will cause no problems because we will never be multiplying these functions. Note also that if $\vec{m}$ has only one component then $S_\lambda^{\vec{m}} = s_\lambda$, the ordinary Schur function.

We now wish to consider the ideas in the previous sections applied to the S_λ. Note that they do not form a basis since we only have one for every integer, rather than set, partition. We will discuss how one might associate a Schur funtion to set partitions in the open problems section at the end of the paper. But we can still express S_λ in terms of the others bases. In fact, one basis will suffice since the other transformations can be obtained by composition. We will also compute the image of S_λ under ρ and see how it behaves with respect to our inner product. However, applying the involution ω is best done after the Jacobi-Trudi determinants are discussed in Section 7. To state our current results, we will need the dominance order on integer partitions. We say λ *dominates* μ, written $\lambda \trianglerighteq \mu$, if and only if

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i \qquad \text{for all } i$$

where we let $\lambda_i = 0$ if i is greater than the number of parts in λ and similarly for μ. We will also need the *Kostka numbers*, $K_{\lambda,\mu}$, which are the number of semistandard (ordinary) Young tableaux T of shape λ and content μ (so μ_i is the number of entries equal to i in T).

Theorem 6.2 *The functions S_λ have the following properties.*

(i) In terms of the monomial symmetric functions

$$S_\lambda = \sum_{\mu \trianglelefteq \lambda} \mu! K_{\lambda,\mu} \sum_{\sigma \in \mu} m_\sigma. \tag{11}$$

(ii) The S_λ are independent.

(iii) In terms of the forgetful map $\rho(S_\lambda) = n!s_\lambda$.

(iv) The S_λ *are in the image of* $\tilde{\rho}$ *and* $\tilde{\rho}(n!s_\lambda) = S_\lambda$.

(v) In terms of the inner product $\langle S_\lambda, S_\mu \rangle = n!^2 \delta_{\lambda,\mu}$.

Proof (i) Consider a monomial $x^{\dot{T}}$ where $\dot{T}$ has shape λ and suppose that this monomial occurs in m_σ where $\sigma \in \mu$. Then the number of ordinary tableaux T with the same content, μ, as $\dot{T}$ is $K_{\lambda,\mu}$ and this is only nonzero for $\mu \trianglelefteq \lambda$. Furthermore, the number of ways to distribute the dots in T so as to give the same monomial as $x^{\dot{T}}$ is $\mu!$, so this finishes the proof.

(ii) The lexicographic order on integer partitions is a linear extension of the dominance order. So from (i), each S_λ only contains m_σ where $\lambda(\sigma)$ is lexicographically less than or equal to λ, and those with $\lambda(\sigma) = \lambda$ have nonzero coefficient. So if one orders the S_λ this way, then each Schur function will contain one or more monomial symmetric functions not found previously in the list.

(iii) Using (i) again along with Theorem 2.1 (i) and equation (4) gives

$$\rho(S_\lambda) = \sum_{\mu \trianglelefteq \lambda} \mu! K_{\lambda,\mu} \sum_{\sigma \in \mu} \rho(m_\sigma) = \sum_{\mu \trianglelefteq \lambda} \mu! K_{\lambda,\mu} \mu^! \binom{n}{\mu} m_\mu = n! \sum_{\mu \trianglelefteq \lambda} K_{\lambda,\mu} m_\mu = n! s_\lambda.$$

(iv) Clearly from (i), all m_σ with $\sigma \in \mu$ have the same coefficient in S_λ. So S_λ is in the image of $\tilde{\rho}$. The equality now follows from (iii) and the fact that ρ is a left-inverse for $\tilde{\rho}$.

(v) We compute using (iv) and the fact that $\tilde{\rho}$ is an isometry

$$\langle S_\lambda, S_\mu \rangle = \langle \tilde{\rho}(n!s_\lambda), \tilde{\rho}(n!s_\mu) \rangle = \langle n!s_\lambda, n!s_\mu \rangle = n!^2 \delta_{\lambda,\mu}. \qquad \blacksquare$$

7 Jacobi-Trudi determinants

We now prove analogs of the Jacobi-Trudi determinants [11, 32] in this setting. We will see that it is possible to do this for the $S_\lambda^{\vec{m}}$ where $\vec{m}$ is arbitrary. So we will obtain the ordinary and noncommuting variable cases as specializations. Our method will be the lattic-path approach introduced by Lindström [15] and popularized by Gessel and Viennot [10]. A more complete exposition of the method than the one which follows can be found in [24].

Consider infinite paths in the extended integer lattice $\mathbb{Z} \times (\mathbb{Z} \uplus \infty)$:

$$P = s_1, s_2, s_3, \ldots$$

where the s_t are steps of unit length either northward (N) or eastward (E). (A point of the form (i', ∞) can only be reached by ending P with an infinite number of northward steps along the line $x = i'$.) If P starts at (i, j), then we label an eastward step along the line $y = j'$ with the label

$$L(s_t) = (j' - j + 1)^{(k)}$$

for some k which can vary with the step, $1 \le k \le n$. Considering P as a multiset of labels, it has a well-defined multidegree $\vec{m}$. Then for $\vec{m} \vdash m$ we have

$$h_{\vec{m}} = \sum_P M_P \quad \text{where} \quad M_P = \prod_{\ell^{(k)}} x_\ell^{(k)},$$

the sum being over all paths of multidegree $\vec{m}$ from (i, j) to $(i + m, \infty)$ and the product being over all labels in P taken with multiplicity. Note also that if the labels in P are read off from left to right, then they correspond to a single-rowed dotted Young tableau of multidegree $\vec{m}$.

To get products of complete homogeneous symmetric functions and tableaux of shape $\lambda = (\lambda_1, \ldots, \lambda_l)$, consider initial vertices $u_1, \ldots, u_l$ and final vertices $v_1, \ldots, v_l$ with coordinates

$$u_i = (-i, 1) \quad \text{and} \quad v_i = (\lambda_i - i, \infty) \tag{12}$$

for $1 \le i \le l$. Consider a family of labeled paths $\mathcal{P} = (P_1, \ldots, P_l)$ where, for each i, P_i is a path from u_i to v_{gi} for some $g \in \mathfrak{S}_l$. We assign to $\mathcal{P}$ a *monomial* and *sign* given by

$$M_{\mathcal{P}} = \prod_{i=1}^{l} M_P \quad \text{and} \quad (-1)^{\mathcal{P}} = (-1)^g,$$

respectively. Also, $\mathcal{P}$ has *multidegree* $\vec{m}$ if the total number of labels with k dots on all paths is m_k.

Theorem 7.1 *Given a partition λ and vector $\vec{m}$ with $\lambda, \vec{m} \vdash m$, we have*

$$S_\lambda^{\vec{m}} = \det\left(\sum_{\vec{q} \vdash \lambda_i - i + j} h_{\vec{q}} \right)$$

with the convention that if the product of two monomials in the determinant does not have multidegree $\vec{m}$ then that product is zero. Also, if λ' is the conjugate of λ (whose parts are the column lengths of the shape of λ) then, with the same convention,

$$S_{\lambda'}^{\vec{m}} = \det\left(\sum_{\vec{q} \vdash \lambda_i - i + j} e_{\vec{q}} \right).$$

Proof We will only prove the first determinant as the second is obtained by a similar argument using paths labeled so as to correspond to columns of a semistandard tableau.

First consider the right-hand side which we will denote by D. Because of the convention on products

$$D = \sum_{\mathcal{P}} (-1)^{\mathcal{P}} M_{\mathcal{P}}$$

where the sum is over all path families with beginning and ending vertices given by (12) and of multidegree $\vec{m}$.

We now construct a monomial-preserving, sign-reversing involution $\imath$ on such $\mathcal{P}$ which are self-intersecting as follows. Let i be the smallest index such that P_i intersects some P_j and take j minimum. Consider the NE-most point, v_0, of $P_i \cap P_j$. Create $\mathcal{P}' = \imath\mathcal{P}$ by replacing P_i, P_j with P_i', P_j', respectively, where P_i' goes from u_i to v_0 along P_i and then continues along P_j, and similarly for P_j'.

Because $\imath$ pairs up intersecting path families of the same monomial and opposite sign, they all cancel from the determinant leaving only nonintersecting families. Furthermore, by the choice of initial and final points, a family can only be nonintersecting if its associated element of $\mathfrak{S}_l$ is the identity. So we now have

$$D = \sum_{\mathcal{P}} M_{\mathcal{P}}$$

where the sum is over all nonintersecting families. But there is a bijection between such families and tableaux. Given $\mathcal{P}$, read the elements of P_i from left to right to obtain the ith row of the associated tableau $\dot{T}$. The fact that $\mathcal{P}$ is nonintersecting is equivalent to the fact that $\dot{T}$ has increasing columns. The given initial and final vertices ensure that the shape of $\dot{T}$ is λ. And the product condition on the determinant shows that we have the right multidegree. ■

Note that if $\vec{q} = [q]$ has a single component, then $\vec{q} \vdash \lambda_i - i + j$ forces $q = \lambda_i - i + j$. So the sums in each entry of the determinants reduce to a single term and we recover the ordinary form of Jacobi-Trudi.

We now specialize to the case of noncommuting variables case so as to fill the lacuna from the last section. Note that ω exchanges the two Jacobi-Trudi determinants, even with the condition on products which is invariant under this involution. We have proved the following.

Corollary 7.2 *We have $\omega(S_\lambda) = S_{\lambda'}$.* ■

8 The Robinson-Schensted-Knuth map

The purpose of this section is to give a generalization of the famous Robinson-Schensted-Knuth bijection [13, 20, 25] to tableaux of arbitrary multidegree. First, however, we will need some definitions.

A *biword of length n over A* is a $2 \times n$ array β of elements of A such that if the dots are removed then the columns are ordered lexicographically with the top row taking precedence. The lower and upper rows of β are denoted $\check{\beta}$ and $\hat{\beta}$, respectively. Viewing $\check{\beta}$ and $\hat{\beta}$ as multisets, the multidegree of β is the pair

$$\vec{m}(\beta) = (\vec{m}(\check{\beta}), \vec{m}(\hat{\beta})).$$

We now define a map $\beta \overset{\text{R–S–K}}{\mapsto} (\dot{T}, \dot{U})$ whose image is all pairs of dotted semistandard Young Tableaux of the same shape. Peform the normal Robinson-Schensted-Knuth algorithm on β (see [24] for an exposition) by merely ignoring the dots and just having them "come along for the ride." For example, if

$$\beta = \begin{array}{cccccc} \dot{1} & \dot{2} & \ddot{2} & \dot{2} & \ddot{3} & \dot{4} \\ \dot{2} & \ddot{1} & \ddot{3} & \dot{3} & \ddot{2} & \dot{1} \end{array}$$

then the sequence of tableux built by the algorithm is

$$\begin{array}{l} \dot{2} \end{array}, \quad \begin{array}{l} \ddot{1} \\ \dot{2} \end{array}, \quad \begin{array}{ll} \ddot{1} & \ddot{3} \\ \dot{2} & \end{array}, \quad \begin{array}{lll} \ddot{1} & \ddot{3} & \dot{3} \\ \dot{2} & & \end{array}, \quad \begin{array}{lll} \ddot{1} & \ddot{2} & \dot{3} \\ \dot{2} & \ddot{3} & \end{array}, \quad \begin{array}{lll} \ddot{1} & \dot{1} & \dot{3} \\ \dot{2} & \ddot{2} & \\ \ddot{3} & & \end{array} \quad = \dot{T},$$

$$\begin{array}{l} \dot{1} \end{array}, \quad \begin{array}{l} \dot{1} \\ \dot{2} \end{array}, \quad \begin{array}{ll} \dot{1} & \ddot{2} \\ \dot{2} & \end{array}, \quad \begin{array}{lll} \dot{1} & \ddot{2} & \dot{2} \\ \dot{2} & & \end{array}, \quad \begin{array}{lll} \dot{1} & \ddot{2} & \dot{2} \\ \dot{2} & \ddot{3} & \end{array}, \quad \begin{array}{lll} \dot{1} & \ddot{2} & \dot{2} \\ \dot{2} & \ddot{3} & \\ \ddot{4} & & \end{array} \quad = \dot{U}.$$

The next theorem follows directly from the definitions and the analogous result for the ordinary Robinson-Schensted-Knuth map.

Theorem 8.1 *The map*

$$\beta \overset{\text{R–S–K}}{\mapsto} (\dot{T}, \dot{U})$$

is a bijection between biwords and pairs of dotted semistandard Young tableaux of the same shape such that

$$\vec{m}(\beta) = (\vec{m}(\dot{T}), \vec{m}(\dot{U})). \qquad \blacksquare$$

Because this analogue is so like the original, most of the properties of the ordinary Robinson-Schensted-Knuth correspondence carry over into this setting with virtually no change. By way of illustration, here is the corresponding Cauchy identity [16] which follows directly by turning each side of the previous bijection into a generating function. Note that for $\hat{\beta}$ and $\dot{U}$ we are using a second set of variables $\dot{\mathbf{y}}, \ddot{\mathbf{y}}, \ldots, \mathbf{y}^{(\mathbf{n})}$.

Theorem 8.2 *We have*

$$\sum_{m \geq 0} \sum_{\lambda, \vec{m}, \vec{p} \ \vdash m} S_\lambda^{\vec{m}}(\dot{\mathbf{x}}, \ldots, \mathbf{x}^{(\mathbf{n})}) S_\lambda^{\vec{p}}(\dot{\mathbf{y}}, \ldots, \mathbf{y}^{(\mathbf{n})}) = \prod_{i,j \geq 1} \frac{1}{1 - \sum_{k,l=1}^{n} x_i^{(k)} y_j^{(l)}}. \qquad \blacksquare$$

9 Comments and questions

We now list various open problems raised by this research in the hopes that the reader will be tempted to tackle some of them.

(I) One can compute specializations for symmetric functions in noncommuting variables or, more generally, for MacMahon symmetric functions. This has been done by Rosas [22].

(II) Jacobi [11] also showed that the Schur functions could be expressed as a quotient of two determinants called alternants. In fact, this was the expression Schur himself used [26] when defining the functions which bear his name. This definition has the advantage that it becomes immediately apparent that s_λ is symmetric since it is the quotient of two skew-symmetric determinants. However, we have not been able to find an analogue of this result in the case of noncommuting variables, much less for general MacMahon Schur functions.

(III) One of the other important properties of s_λ is that, when expanded in terms of the power sum basis, the coefficients are essentially the character values of the irreducible representation of the symmetric group corresponding to λ. The function itself is the character of the polynomial representation of the general linear group corresponding to λ. It would be very interesting to have some connection between the S_λ, or even the $S_\lambda^{\vec{m}}$, and representation theory.

(IV) The reader will have noticed that we only defined Schur functions in noncommuting variables for integer, rather than set, partitions. This is because we have not been able to come up with a completely satisfactory definition in the set case.

One possible approach would be to define functions S_π such that

$$S_\lambda = \sum_{\pi \in \lambda} S_\pi \tag{13}$$

as follows. For each m_σ in the expansion (11) we have two possibilities. If $\sigma \le \pi$ for some $\pi \in \lambda$ then we uniformly distribute the coefficient of m_σ among all S_π with $\pi \ge \sigma$. If $\sigma \not\le \pi$ for all $\pi \in \lambda$, then we uniformly distribute the coefficient of m_σ among all S_π with $\pi \in \lambda$. More explicitly, if $\lambda(\pi) = \lambda$ then define

$$S_\pi = \sum_{\mu \trianglelefteq \lambda} \mu! K_{\lambda,\mu} \sum_{\sigma \in \mu} c_\sigma m_\sigma$$

where

$$c_\sigma = \begin{cases} 1/|\{\tau \ge \sigma \ : \ \tau \in \lambda\}| & \text{if } \sigma \le \pi, \\ 1/|\{\tau : \ \tau \in \lambda\}| & \text{if } \sigma \not\le \tau \text{ for all } \tau \in \lambda, \\ 0 & \text{else.} \end{cases}$$

It is easy to see directly from the definition that we have (13). It is also straightforward to verify that the S_π are a basis for $\Pi(\mathbf{x})$. However, there are difficulties with many of the other desired properties for such a function. For example, the S_π are not orthogonal, although one can show that $\langle S_\pi, S_\lambda \rangle = 0$ if $\pi \notin \lambda$. And we have no idea what the Jacobi-Trudi determinants or Robinson-Schensted-Knuth map look like for these functions. Perhaps there is even another definition of the S_π that will be better for developing such analogues. We should also note that Doubilet [3] also posed the problem of finding an analogue of the Schur basis indexed by set partitions in his setting.

(V) As mentioned in the introduction, one of the motivations for introducing noncommuting variables is to study Stanley's chromatic symmetric function [8, 27, 29]. Let $G = (V, E)$ be a graph and let $P_G(n)$ denote the chromatic polynomial of G, i.e., the number of proper colorings of G from a set with n colors. This object was first studied by Whitney [34] who proved, among other things, that it is always a polynomial in n. It has many interesting properties as well as connections with Möbius functions of partially ordered sets, hyperplane arrangements, etc.

Stanley introduced a related chromatic symmetric function. Let the variables $\mathbf{x}$ commute and suppose G has vertices $V = \{v_1, v_2, \ldots, v_n\}$. Then define

$$X_G = X_G(\mathbf{x}) = \sum_c x_{c(v_1)} x_{c(v_2)} \cdots x_{c(v_n)} \tag{14}$$

where the sum is over all proper colorings of G using the positive integers as colors. It is easy to see that X_G is a symmetric function and that if we set the first n variables equal to 1 and the rest equal to 0 then we recover $P_G(n)$. Stanley was able to prove various results for X_G generalizing those known for the chromatic polynomial as well as theorems that have no analogue in the earlier setting.

One thing that was missing from Stanley's development was a version of the Deletion-Contraction Law which is a useful tool for proving theorems about $P_G(n)$

by induction. Gebhard and Sagan showed that if one does not allow the variables to commute then one can use (14) to define an analogue of $P_G(n)$ in noncommuting variables, $Y_G(\mathbf{x})$. Of course, $X_G = \rho(Y_G)$. But more importantly, Y_G satisfies a Deletion-Contraction Law. (It should be noted that Noble and Welsh [19] also have a version of this Law for X_G itself in the category of vertex-weighted graphs.) This permits the derivation of a number of Stanley's results by straightforward induction.

It is also possible to use this inductive approach to make progress on the $(\mathbf{3}{+}\mathbf{1})$-free Conjecture of Stanley and Stembridge [31]. Call a poset (partially ordered set) $(\mathbf{a}{+}\mathbf{b})$-*free* if it has no induced subposet isomorphic to the dijoint union of an a-element chain and a b-element chain. Any poset P has a corresponding *incomparability graph*, $G(P)$, whose vertices are the elements of P with an edge between two vertices if they are not comparable in P.

Conjecture 9.1 (Stanley-Stembridge) *Let P be a $(\mathbf{3}{+}\mathbf{1})$-free poset and suppose*

$$X_{G(P)} = \sum_{\lambda} c_\lambda e_\lambda$$

for certain coefficients c_λ. Then $c_\lambda \geq 0$ for all λ.

This conclusion is summarized by saying that $X_{G(P)}$ is *e-positive*.

There is quite a bit of evidence for this conjecture. Stembridge has verified it by computer for small values of $|V|$. Gasharov [6] has shown that the analogous, but weaker, result that $X_{G(P)}$ is s-positive (for the Schur function basis). Gebhard and Sagan focused on the case where one keeps the elementary symmetric functions, but specializes to the case where P is both $(\mathbf{3}{+}\mathbf{1})$-free and $(\mathbf{2}{+}\mathbf{2})$-free. These $G(P)$ are called *indifference graphs* and have another characterization which is suitable for induction. Specifically, the indifference graphs are precisely those obtained by using $V = \{1, 2, \ldots, n\}$ together with a collection of intervals $I_1, I_2, \ldots, I_l \subseteq V$ and putting a complete graph on the vertices of each I_j.

Theorem 9.2 (Gebhard-Sagan) *If G is an indifference graph with $|I_j \cap I_k| \leq 1$ for all j, k then X_G is e-positive.* ■

It would be wonderful to use the theory of symmetric functions in noncommuting variables to extend this result to all indifference graphs.

Acknowledgments. Part of the research for this paper was done while both authors were resident at the Isaac Newton Institute for Mathematical Sciences in Cambridge, England. We would like to thank the Institute for support during this period. We are also indebted to Timothy Chow, Ira Gessel, and Larry Smith for helpful discussions.

References

[1] David J. Anick, On the homogeneous invariants of a tensor algebra, in "Algebraic Topology: Proceedings of the International Conference held March 21–24, 1988," Mark Mahowald and Stewart Priddy eds., Contemporary Mathematics, Vol. 96, American Math. Society, Providence, RI, 1989, 15–17.

[2] G. M. Bergman and P. M. Cohn, Symmetric elements in free powers of rings, *J. London Math. Soc. (2)* **1** (1969) 525–534.

[3] P. Doubilet, On the foundations of combinatorial theory. VII: Symmetric functions through the theory of distribution and occupancy, *Studies in Applied Math.* **51** (1972), 377–396.

[4] S. Fomin and C. Greene, Noncommutative Schur functions and their applications, *Discrete Math.* **193** (1998), 179–200.

[5] W. Fulton, "Young Tableaux," London Mathematical Society Student Texts 35, Cambridge University Press, Cambridge, 1999.

[6] V. Gasharov, Incomparability graphs of (**3**+**1**)-free posets are s-positive, *Discrete Math.* **157** (1996), 193–197.

[7] D. Gebhard and B. Sagan, Sinks in acyclic orientations of graphs, *J. Combin. Theory, Ser. B* **80** (2000), 130–146.

[8] D. Gebhard and B. Sagan, A chromatic symmetric function in noncommuting variables, *J. Algebraic Combin.* **13** (2001), 227–255.

[9] I. M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. Retakh, J.-I. Thibon, Noncommutative symmetric functions, *Adv. in Math.* **112** (1995) 218–348.

[10] I. Gessel and G. Viennot, Binomial determinants, paths, and hook length formulae, *Adv. in Math.* **58** (1985), 300–321.

[11] C. Jacobi, De functionibus alternantibus earumque divisione per productum e differentiis elementorum conflatum, *J. Reine Angew. Math. (Crelle)* **22** (1841), 360–371. Also in "Mathematische Werke," Vol. 3, Chelsea, New York, NY, 1969, 439–452.

[12] V. K. Kharchenko, Algebras of invariants of free algebras, *Algebra i Logika* **17** (1978) 478–487 (Russian); *Algebra and Logic* **17** (1978), 316–321 (English translation).

[13] D. E. Knuth, Permutations, matrices and generalized Young tableaux, *Pacific J. Math.* **34** (1970), 709–727.

[14] A. Lascoux and M.-P. Schützenberger, Le monoid plaxique, in "Noncommutative Structures in Algebra and Geometric Combinatorics, (Naples, 1978)," Quad. Ricerca Sci., Vol. 109, CNR, Rome, 1981, 129–156.

[15] B. Lindström, On the vector representation of induced matroids, *Bull. London Math. Soc.* **5** (1973), 85–90.

[16] D. E. Littlewood, "The Theory of Group Characters," Oxford University Press, Oxford, 1950.

[17] I. G. Macdonald, "Symmetric functions and Hall polynomials," 2nd edition, Oxford University Press, Oxford, 1995.

[18] P. A. MacMahon, "Combinatorial Analysis," Vols. 1 and 2, Cambridge University Press, Cambridge, 1915, 1916; reprinted by Chelsea, New York, NY, 1960.

[19] S. D. Noble and D. J. A. Welsh, A weighted graph polynomial from chromatic invariants of knots, Symposium à la Mémoire de François Jaeger (Grenoble, 1998) *Annales l'Institut Fourier* **49** (1999), 1057–1087.

[20] G. de B. Robinson, On representations of the symmetric group, *Amer. J. Math.* **60** (1934), 745–760.

[21] M. H. Rosas, MacMahon symmetric functions, the partition lattice, and Young subgroups, *J. Combin. Theory Ser. A* **96** (2001), 326–340.

[22] M. H. Rosas, Specializations of MacMahon symmetric functions and the polynomial algebra, *Discrete Math.* **246** (2002), 285–293.

[23] G.-C. Rota, On the foundations of combinatorial theory I. Theory of Möbius functions, *Z. Wahrscheinlichkeitstheorie* **2** (1964), 340–368.

[24] B. Sagan, "The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions," 2nd edition, Springer-Verlag, New York, 2001.

[25] C. Schensted, Longest increasing and decreasing subsequences, *Canad. J. Math.* **13** (1961), 179–191.

[26] I. Schur, "Über eine Klasse von Matrizen die sich einer gegebenen Matrix zuordnen lassen," Inaugural-Dissertation, Berlin, 1901.

[27] R. P. Stanley, A symmetric function generalization of the chromatic polynomial of a graph, *Advances in Math.* **111** (1995), 166–194.

[28] R. P. Stanley, "Enumerative Combinatorics, Volume 1," Cambridge University Press, Cambridge, 1997.

[29] R. P. Stanley, **G**raph Colorings **a**nd **r**elated **s**ymmetric functions: **i**deas and **a**pplications: A description of results, interesting applications, & notable open problems, Selected papers in honor of Adriano Garsia (Taormina, 1994) *Discrete Math.* **193** (1998), 267–286.

[30] R. P. Stanley, "Enumerative Combinatorics, Volume 2," Cambridge University Press, Cambridge, 1999.

[31] R. P. Stanley and J. Stembridge, On immanants of Jacobi-Trudi matrices and permutations with restricted position, *J. Combin. Theory Ser. A* **62** (1993), 261–279.

[32] N. Trudi, Intorno un determinante piu generale di quello che suol dirsi determinante delle radici di una equazione, ed alle funzioni simmetriche complete di queste radici, *Rend. Accad. Sci. Fis. Mat. Napoli* **3** (1864), 121–134. Also in *Giornale di Mat.* **2** (1864), 152–158 and 180–186.

[33] L. Weisner, Abstract theory of inversion of finite series, *Trans. Amer. Math. Soc.* **38** (1935) 474–484.

[34] H. Whitney, A logical expansion in mathematics, *Bull. Amer. Math. Soc.* **38** (1932), 572–579.

[35] M. C. Wolf, Symmetric functions of noncommuting elements, *Duke Math. J.* **2** (1936) 626–637.