The Computation of Abelian Subalgebras in Low-Dimensional Solvable Lie Algebras

MANUEL CEBALLOS, JUAN NÚÑEZ
University of Seville
Geometry and Topology
Aptdo. 1160. 41080-Seville.
SPAIN
mceballos@us.es, jnvaldes@us.es

ÁNGEL F. TENORIO
Pablo de Olavide University
Economics, Quantitative Methods and Ec. History
Ctra. Utrera km. 1. 41013-Seville
SPAIN
aftenorio@upo.es

Abstract: The main goal of this paper is to compute the *maximal abelian dimension* of each solvable non-decomposable Lie algebra of dimension less than 7. To do it, we apply an algorithmic method which goes ruling out non-valid maximal abelian dimensions until obtaining its exact value. Based on Mubarakzyanov and Turkowsky's classical classifications of solvable Lie algebras (see [13] and [19]) and the classification of 6-dimensional nilpotent Lie algebras by Goze and Khakimdjanov [7], we have explicitly computed the maximal abelian dimension for the algebras given in those classifications.

Key–Words: Solvable Lie algebra, maximal abelian dimension.

1 Introduction

The main topic of this paper is the *maximal abelian dimension* of a given finite-dimensional Lie algebra g. Let us recall that it is the maximum among the dimensions of the abelian subalgebras of g. Most papers about this topic (like [4, 18]) work with abelian ideals instead of abelian subalgebras, which implies more restrictive hypotheses. In our work, such assumptions and restrictions are not used because we are considering every abelian subalgebra of the Lie algebra g. In this sense, we are working in the line of [3, 6].

Previously, we have already studied this concept. The reader can consult [17] for some properties of Lie algebras using the maximal abelian dimension. The computation of the maximal abelian dimension was done for the Lie algebra g_n, of $n \times n$ strictly upper-triangular matrices in [1, 2] and for the Lie algebra h_n, of $n \times n$ upper-triangular ones in [5]. These papers proved properties of these algebras determining the maximal abelian dimension depending on the order n.

To continue our research, the maximal abelian dimension is computed for non-decomposable solvable Lie algebras of dimension up to 7, applying and adjusting the algorithmic method of [2], which uses the main and non-main vectors already introduced in those papers.

Before we get down to work with the topic of this paper, we would like to comment, as examples, some applications of Lie Theory to several fields of research. First of all, Lie groups and algebras have been profusely used as tools in Theoretical and Mathematical Physics. In fact, a very classical use corresponds to the study of symmetries in problems involving differential equations as can be seen in Senashov et al. [15] or the classical reference by Olver [14]. In this way, Lie Theory can be applied to dynamical systems and, more specifically, control problems. Examples in this last field are given in Kirillova et al. [10] or Lantos [11]. Let us note that there exist applications of Lie Theory, by means of optimal control problems, related to fields like Medicine as can be seen in [12]. Finally, we would like to indicate that Social Sciences are also applying Lie Theory for dealing with their subjects. In this sense, we would like to cite the papers by Sumedrea and Sangeorzan [16] and Hernández et al. [9] in order to exemplify this assertion.

2 Preliminaries

This section is devoted to recall some preliminary concepts and results on Lie algebras. For a general overview, the reader can consult [20]. From here on, only finite-dimensional Lie algebras over the complex number field are considered.

The *upper central series* of a given Lie algebra g is defined by

$$\mathcal{C}_1(g) = g,\ \mathcal{C}_2(g) = [g, g],\ \mathcal{C}_3(g) = [\mathcal{C}_2(g), \mathcal{C}_2(g)],$$
$$\ldots, \mathcal{C}_k(g) = [\mathcal{C}_{k-1}(g), \mathcal{C}_{k-1}(g)],\ \ldots \quad (1)$$

When there exists $m \in N$ such that $\mathcal{C}_m(g) \equiv 0$, the Lie algebra g is called *solvable*.

A special class of solvable Lie algebras is formed by abelian algebras. A Lie algebra h is *abelian* if $[\mathbf{v}, \mathbf{w}] = 0$, for all $\mathbf{v}, \mathbf{w} \in h$.

A very important abelian subalgebra of a Lie algebra g is its *center*, which is defined as: $\mathrm{cen}(g) = \{X \in g \mid [X, Y] = 0,\ \forall\, Y \in g\}$. Another useful ideal of g is its nilradical, which corresponds to the sum of all the nilpotent ideals of the algebra g.

Finally, the *maximal abelian dimension* of g, which will be denoted by $\mathcal{M}(g)$, is the maximum among the dimensions of its abelian subalgebras.

The invariant $\mathcal{M}(g)$ is monotone and additive: for a subalgebra h of g we have $\mathcal{M}(h) \leq \mathcal{M}(g)$, and for two Lie algebras a and b we have $\mathcal{M}(a \oplus b) = \mathcal{M}(a) + \mathcal{M}(b)$. By applying these properties, the maximal abelian dimension can be computed for all decomposable solvable Lie algebras of dimension less than 7.

Now we explain the method used in this paper to compute the maximal abelian dimension of a given Lie algebra g of dimension d. Let $\mathcal{B}_d = \{X_i\}_{i=1}^d$ be a basis of g and let $\mathcal{B} = \{v_h\}_{h=1}^r$ be a basis of an arbitrary r-dimensional (abelian) subalgebra h (with $r \leq d$). Each vector $v_h \in \mathcal{B}$ is expressible as a linear combination $v_h = \sum_{i=1}^d a_{h,i} X_i$ of the vectors in $\mathcal{B}_d$. Hence, $\mathcal{B}$ can be translated to a matrix in which the h^{th} row saves the coordinates of v_h with respect to $\mathcal{B}_d$

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,d} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r,1} & a_{r,2} & \cdots & a_{r,d} \end{pmatrix} \tag{2}$$

Since the rank of the previous matrix is equal to r, its echelon form, obtained by using elementary row and column transformations, is the following

$$\begin{pmatrix} b_{1,1} & 0 & \cdots & 0 & b_{1,r+1} & \cdots & b_{1,d} \\ 0 & b_{2,2} & \cdots & 0 & b_{2,r+1} & \cdots & b_{2,d} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{r,r} & b_{r,r+1} & \cdots & b_{r,d} \end{pmatrix} \tag{3}$$

Note that the vector $X_i \in \mathcal{B}_d$ can be associated with a different row depending on Expressions (2) or (3).

Consequently, we can suppose that every basis $\mathcal{B}$ of h is expressible by a matrix similar to (3) and each vector in $\mathcal{B}$ is a linear combination of two different types of vectors X_i: the ones coming from the pivots and the remaining ones. The vectors X_i corresponding to pivots are called *main vectors* of $\mathcal{B}$ with respect to $\mathcal{B}_d$, whereas the rest are called *non-main vectors*.

3 Solvable Lie algebras of dimension less than 5

In this section, the maximal abelian dimension is studied for solvable non-decomposable Lie algebras of dimension less than 5, following Mubarakzyanov's classification [13]. Obviously, the unique 1-dimensional Lie algebra is the abelian Lie algebra of dimension 1 and its maximal abelian dimension is equal to its dimension. Since the maximal abelian dimension of abelian algebras is exactly their dimension, abelian algebras are not considered in this paper.

3.1 Solvable Lie algebras of dimension 2

We compute the maximal abelian dimension of the Lie algebra $g_{2,1}$.

Proposition 1 *It is verified that* $\mathcal{M}(g_{2,1}) = 1$.

Proof: The Lie algebra $g_{2,1}$ is generated by the vectors $\{Z_1, Z_2\}$ and there is a unique nonzero bracket: $[Z_1, Z_2] = Z_1$. Hence, $g_{2,1}$ is non-abelian and $\mathcal{M}(g_{2,1}) < 2$. Since 1-dimensional Lie algebras are abelian, both $\langle Z_1 \rangle$ and $\langle Z_2 \rangle$ are abelian subalgebras of $g_{2,1}$ and $\mathcal{M}(g_{2,1}) = 1$. □

3.2 Solvable Lie algebras of dimension 3

Now, let us compute the maximal abelian dimension of 3-dimensional solvable Lie algebras.

Proposition 2 *The maximal abelian dimension of the Lie algebras $g_{3,i}$, where $i \in \{1, \ldots, 5\}$, is given by* $\mathcal{M}(g_{3,i}) = 2$.

Proof: Fixed and given $i \in \{1, 2, 3, 4, 5\}$, let us prove that $\mathcal{M}(g_{3,i}) = 2$. We have to find a 2-dimensional abelian subalgebra in $g_{3,i}$, in addition to determine the nonexistence of 3-dimensional abelian subalgebras.

First, $\mathcal{M}(g_{3,i}) \leq 2$ because the Lie algebra $g_{3,i}$ is non-abelian. Hence, the problem is now reduced to prove the existence of 2-dimensional abelian subalgebras of $g_{3,i}$. Obviously, the subalgebra $\langle Z_1, Z_2 \rangle$ of $g_{3,i}$ is abelian for $i \in \{1, 2, 3, 4, 5\}$. So, $\mathcal{M}(g_{3,i}) = 2$. □

3.3 Solvable Lie algebras of dimension 4

Next, we study the maximal abelian dimension of solvable Lie algebra of dimension 4.

Proposition 3 *It is verified that*

$$\mathcal{M}(g_{4,j}) = \begin{cases} 3, & \text{if } j \in \{1, 4, 5, 6, 7, 8\}, \\ 2, & \text{if } j \in \{2, 3, 9, 10, 11\}. \end{cases} \tag{4}$$

Proof: Fixed and given $i \in \{1,4,5,6,7,8\}$, let us prove that $\mathcal{M}(g_{4,i}) = 3$. So, we have to find a 3-dimensional abelian subalgebra in $g_{4,i}$, in addition to determine the nonexistence of 4-dimensional abelian subalgebras. Since the Lie algebra $g_{4,i}$ is non-abelian, we can set that $\mathcal{M}(g_{4,i}) \leq 3$. In this way, we only have to prove the existence of 3-dimensional abelian subalgebras. Let us consider the subalgebras $h_1 = \langle Z_1, Z_2, Z_3\rangle$, $h_2 = \langle Z_1, Z_2, Z_4\rangle$ and $h_3 = \langle Z_2, Z_3, Z_4\rangle$. It is easy to prove that h_1, h_2 and h_3 are abelian subalgebras of $g_{4,5}$, $g_{4,6}$ and $g_{4,j}$ (for $j \in \{1,4,7,8\}$), respectively. Consequently, $\mathcal{M}(g_{4,i}) = 3$, for $i \in \{1,4,5,6,7,8\}$.

Now, let us prove that 2 is the maximal abelian dimension of the Lie algebras $g_{4,k}$, for $k \in \{2,3,9,10,11\}$. Since the Lie algebra $g_{4,k}$ is non-abelian, its maximal abelian dimension is less than 4. We define the subalgebras

$$a_l = \langle\{Z_i + \lambda_i Z_l \mid 1 \leq i \leq 4 \wedge i \neq l\}\rangle, \quad (5)$$

where Z_l is the non-main vector. It can be proved that the subalgebras a_l are not abelian for $l = 1,2,3,4$ as subalgebras of $g_{4,k}$. Hence, the maximal abelian dimension of $g_{4,k}$ is less than 3. Since the subalgebra $\langle Z_3, Z_4\rangle$ of $g_{4,k}$ is abelian, we can assert that $\mathcal{M}(h_{4,k}) = 2$. □

4 Solvable Lie algebras of dimension 5

Due to the complexity of the problem, we will study solvable non-decomposable Lie algebras of dimension 5 by considering Mubarakzyanov's classification [13] and distinguishing the following cases:

4.1 Nilpotent Lie algebras and solvable non-nilpotent ones containing a 4-dimensional abelian subalgebra

We deal now with the maximal abelian dimension of 5-dimensional nilpotent Lie algebras and solvable ones containing an abelian subalgebra of dimension 4.

Proposition 4 *Given $i \in \{1,\dots,4\}$, it is verified that $\mathcal{M}(g_{5,i}) = 3$.*

Proof: For $i \in \{1,\dots,4\}$, we have to compute a 3-dimensional abelian subalgebra of $g_{5,i}$ and to prove the nonexistence of abelian subalgebras of dimension greater than 3. Obviously, the maximal abelian dimension of $g_{5,i}$ is less than 5, due to not being abelian. Moreover, the subalgebra $\langle Z_1, Z_2, Z_3\rangle$ is abelian.

Consequently, it is sufficient to prove that it is not possible to obtain 4-dimensional abelian subalgebras of $g_{5,i}$, for $i \in \{1,\dots,4\}$. For reason of length, only one of these algebras is studied explicitly. The same reasoning can be applied for the rest of these algebras.

Let us consider the Lie algebra $g_{5,3}$ whose law with respect to a certain basis $\{Z_i\}_{i=1}^5$ is given by:

$$[Z_2, Z_4] = Z_3, [Z_2, Z_5] = Z_1, [Z_4, Z_5] = Z_2. \quad (6)$$

Let us suppose the existence of a 4-dimensional abelian subalgebra. We can suppose that each vector in a basis of such subalgebra is expressed as a linear combination of a main vector and the non-main one. So 4-dimensional subalgebras would be expressed as

$$a_k = \langle\{Z_i + \lambda_i Z_k \mid 1 \leq i \leq 5 \wedge i \neq k\}\rangle, \quad (7)$$

where Z_k is the non-main vector. Proving that a_k is non-abelian is equivalent to finding a nonzero bracket in its law.

- For $k \in \{1,3,5\}$: $[Z_2 + \lambda_2 Z_k, Z_4 + \lambda_4 Z_k] = Z_3 + v$ is nonzero, because $v \in \langle Z_1, Z_2, Z_4, Z_5\rangle$.
- For $k = 2$: $[Z_4 + \lambda_4 Z_2, Z_5 + \lambda_5 Z_2] = Z_2 + v$ is nonzero, because $v \in \langle Z_1, Z_3, Z_4, Z_5\rangle$.
- For $k = 4$: $[Z_2 + \lambda_2 Z_4, Z_5 + \lambda_5 Z_4] = Z_1 + v$ is nonzero, because $v \in \langle\{Z_i\}_{i=2}^{i=5}\rangle$.

Hence, there do not exist 4-dimensional abelian subalgebras of $g_{5,3}$ and $\mathcal{M}(g_{5,3}) = 3$. □

Proposition 5 *Fixed and given $j \in \{5,\dots,18\}$ the maximal abelian dimension of $g_{5,j}$ is $\mathcal{M}(g_{5,j}) = 4$.*

Proof: Let us note that the 4-dimensional subalgebra of $g_{5,j}$ $\langle Z_1, Z_2, Z_3, Z_4\rangle$ is abelian for $j \in \{5,\dots,18\}$. Since $g_{5,j}$ is not abelian, its maximal abelian dimension is exactly 4 and the abelian subalgebra previously expounded is a maximal abelian subalgebra. □

4.2 Solvable, non-nilpotent Lie algebras of dimension 5 containing a 3-dimensional abelian subalgebra

Next, we study the maximal abelian dimension of the solvable Lie algebras of dimension 5 that contain a 3-dimensional abelian subalgebra.

Proposition 6 *For $j \in \{19,\dots,38\}$, the maximal abelian dimension of $g_{5,j}$ is $\mathcal{M}(g_{5,j}) = 3$.*

Proof: The subalgebras $\langle Z_1, Z_3, Z_4\rangle$ and $\langle Z_1, Z_2, Z_3\rangle$ are abelian for $j \in \{19,\dots,29\}$ and for $j \in \{30,\dots,38\}$, respectively. So, we can set that $\mathcal{M}(g_{5,j}) \geq 3$ for $j \in \{19,\dots 38\}$.

Consequently, it is sufficient to prove the nonexistence of 4-dimensional abelian subalgebras of $g_{5,j}$. Once more, the reasoning is analogous for all these algebras. So we only study explicitly the algebra $g_{5,22}$, whose law is with respect to a certain basis $\{Z_i\}_{i=1}^5$:

$$[Z_2, Z_3] = Z_1, [Z_2, Z_5] = Z_3, [Z_4, Z_5] = Z_4. \quad (8)$$

By applying the reasonings and notations used in Proposition 4, let us suppose the existence of a 4-dimensional abelian subalgebra and find a nonzero bracket in its law.

- For $k \in \{1, 4, 5\}$: $[Z_2 + \lambda_2 Z_k, Z_3 + \lambda_3 Z_k] = Z_1 + v$ is nonzero, because $v \in \langle \{Z_i\}_{i=2}^5 \rangle$.
- For $k \in \{2, 3\}$:$[Z_4+\lambda_4 Z_k, Z_5+\lambda_5 Z_k] = Z_4+w$ is nonzero, because $w \in \langle Z_1, Z_2, Z_3, Z_5\rangle$.

Hence, there do not exist 4-dimensional abelian subalgebras of $g_{5,22}$ and $\mathcal{M}(g_{5,22}) = 3$. □

4.3 Solvable, non-nilpotent Lie algebras of dimension 5 containing a 2-dimensional abelian subalgebra.

Finally, we compute the maximal abelian dimension of 5-dimensional Lie algebras containing a 2-dimensional abelian subalgebra.

Proposition 7 *It is verified that* $\mathcal{M}(g_{5,39}) = 2$.

Proof: The subalgebra $\langle Z_1, Z_2\rangle$ of $g_{5,39}$ is abelian. Hence, $\mathcal{M}(g_{5,39}) \geq 2$ and we only have to prove the nonexistence of abelian subalgebras of dimension 3.

Let us suppose the existence of a 3-dimensional abelian subalgebra. We can suppose that each vector in a basis of such subalgebra is expressed as a linear combination of a main vector and two non-main ones. So 3-dimensional subalgebras would be expressed as

$$a_{j,k} = \langle \{Z_i + \lambda_i Z_j + \mu_i Z_k \mid 1 \leq i \leq 5 \wedge j, k \neq i\}\rangle, \quad (9)$$

where Z_j and Z_k are the two non-main vectors. Proving that a_k is non-abelian is equivalent to finding a nonzero bracket in its law.

- For $(j, k) \in \{(1, 2), (1, 4)\}$: the bracket $[Z_3+\lambda_3 Z_j+\mu_3 Z_k, Z_5+\lambda_5 Z_j+\mu_5 Z_k] = Z_2+v$ is nonzero, because $v \in \langle Z_1, Z_3, Z_4, Z_5\rangle$.
- For $(j, k) \in \{(1, 3), (1, 5)\}$: the bracket $[Z_2+\lambda_2 Z_j+\mu_2 Z_k, Z_4+\lambda_4 Z_j+\mu_4 Z_k] = Z_2+v$ is nonzero, because $v \in \langle Z_1, Z_3, Z_4, Z_5\rangle$.
- For $(j, k) \in \{(2, 5), (3, 5)\}$: the bracket $[Z_1+\lambda_1 Z_j+\mu_1 Z_k, Z_4+\lambda_4 Z_j+\mu_4 Z_k] = 2Z_1+w$ is nonzero, because $w \in \langle Z_i\rangle_{i=2}^5$.
- For $(j, k) = (4, 5)$: the bracket $[Z_2+\lambda_2 Z_j+\mu_2 Z_k, Z_3+\lambda_3 Z_j+\mu_3 Z_k] = Z_1+w$ is nonzero, because $w \in \langle Z_i\rangle_{i=2}^5$.
- For $(j, k) = (2, 3)$: we consider the bracket $[Z_1 + \lambda_1 Z_2 + \mu_1 Z_3, Z_4 + \lambda_4 Z_2 + \mu_4 Z_3]= (2 + \lambda_1\mu_4 - \mu_1\lambda_4)Z_1 + \lambda_1 Z_2 + \mu_1 Z_3$. If the subalgebra $a_{2,3}$ is abelian, this bracket would have to be zero, obtaining the system $\{2 + \lambda_1\mu_4 - \mu_1\lambda_4 = 0,\ \lambda_1 = 0,\ \mu_1 = 0\}$, which has no solution.
- For $(j, k) = (2, 4)$: we have the brackets

$$[Z_3 + \lambda_3 Z_2 + \mu_3 Z_4, Z_5 + \lambda_5 Z_2 + \mu_5 Z_4] = -\lambda_5 Z_1 +(1 + \lambda_3\mu_5 - \mu_3\lambda_5)Z_2 + (\mu_5 - \lambda_3)Z_3, \quad (10)$$

$$[Z_1 + \lambda_1 Z_2 + \mu_1 Z_4, Z_5 + \lambda_5 Z_2 + \mu_5 Z_4] = 2\mu_5 Z_1 - \lambda_1 Z_3 + (\lambda_1\mu_5 - \mu_1\lambda_5)Z_2. \quad (11)$$

If the subalgebra $a_{2,4}$ is abelian, these brackets would have to be zero and the following system without solutions would be obtained: $\{1 + \lambda_3\mu_5 - \mu_3\lambda_5 = 0,\ \lambda_5 = 0,\ \mu_5 - \lambda_3 = 0,\ 2\mu_5 = 0,\ \lambda_1 = 0,\ \lambda_1\mu_5 - \mu_1\lambda_5 = 0\}$.

- For $(j, k) = (3, 4)$: we consider the brackets

$$[Z_2 + \lambda_2 Z_3 + \mu_2 Z_4, Z_5 + \lambda_5 Z_3 + \mu_5 Z_4] = \lambda_5 Z_1 +(\lambda_2\mu_5 - \mu_2\lambda_5 - 1)Z_3 + (\mu_5 + \lambda_2)Z_2, \quad (12)$$

$$[Z_1 + \lambda_1 Z_3 + \mu_1 Z_4, Z_2 + \lambda_2 Z_3 + \mu_2 Z_4] = -\mu_1 Z_2 +(2\mu_2 - \lambda_1)Z_1 + (\lambda_1\mu_2 - \mu_1\lambda_2)Z_3, \quad (13)$$

$$[Z_1 + \lambda_1 Z_3 + \mu_1 Z_4, Z_5 + \lambda_5 Z_3 + \mu_5 Z_4] = 2\mu_5 Z_1 + \lambda_1 Z_2 + (\lambda_1\mu_5 - \mu_1\lambda_5)Z_3, \quad (14)$$

If the subalgebra $a_{3,4}$ is abelian, these brackets would have to be zero, obtaining the following system without solutions $\{\lambda_2\mu_5 - \mu_2\lambda_5 - 1 = 0,\ \lambda_5 = 0,\ \mu_5 + \lambda_2 = 0,\ 2\mu_2 - \lambda_1 = 0,\ \mu_1 = 0,\ \lambda_1\mu_2 - \mu_1\lambda_2 = 0,\ \mu_5 = 0,\ \lambda_1 = 0,\ \lambda_1\mu_5 - \mu_1\lambda_5 = 0\}$.

Hence, there do not exist 3-dimensional abelian subalgebras of $g_{5,39}$ and $\mathcal{M}(g_{5,39}) = 2$. □

5 Solvable Lie algebras of dimension 6

This section is devoted to compute the maximal abelian dimension of each 6-dimensional solvable, non-decomposable Lie algebra g.

5.1 Nilpotent Lie algebras of dimension 6.

We deal now with the maximal abelian dimension of 6-dimensional nilpotent Lie algebras. The classification of these algebras is the one given by Goze and Khakimdjanov [7], but including the corrections given later by Goze and Remm [8].

Proposition 8 *It is verified that*

$$\mathcal{M}(g_{6,j}) = \begin{cases} 5, & \text{if } j \in \{1,2\}, \\ 4, & \text{if } j \in \{3,4,\dots,18\}, \\ 3, & \text{if } j \in \{19,20\}. \end{cases} \quad (15)$$

Proof: Fixed and given $i \in \{1,2\}$, let us prove that $\mathcal{M}(g_{6,i}) = 5$. We have to find a 5-dimensional abelian subalgebra in $g_{6,i}$, and determine the nonexistence of 6-dimensional abelian subalgebras. Since the Lie algebra $g_{6,i}$ is non-abelian, we can set that $\mathcal{M}(g_{6,i}) \leq 5$. In this way, we only have to prove the existence of 5-dimensional abelian subalgebras. Since the Lie algebra $\langle Z_2, Z_3, Z_4, Z_5, Z_6 \rangle$ is a 5-dimensional abelian subalgebra of $g_{6,i}$, we can affirm that $\mathcal{M}(g_{6,i}) = 5$, for $i \in \{1,2\}$.

For $i \in \{3,\dots,18\}$, we have to compute a 4-dimensional abelian subalgebra of $g_{6,i}$ and to prove the nonexistence of abelian subalgebras of dimension greater than 4. Obviously, the maximal abelian dimension of $g_{6,i}$ is less than 6, due to not being abelian. For reason of length, only one of these algebras is studied explicitly. The same reasoning can be applied for the rest of these algebras. We consider the Lie algebra $g_{6,5}$ whose law with respect to a certain basis $\{Z_i\}_{i=1}^5$ is given by:

$$[Z_1, Z_2] = Z_3, [Z_1, Z_4] = Z_5, [Z_2, Z_4] = -Z_6. \quad (16)$$

Let us note that $\langle Z_3, Z_4, Z_5, Z_6 \rangle$ is a 4-dimensional abelian subalgebra of $g_{6,5}$. Let us suppose the existence of a 5-dimensional abelian subalgebra. We can suppose that each vector in a basis of such subalgebra is expressed as a linear combination of a main vector and the non-main one. So 5-dimensional subalgebras would be expressed as

$$a_k = \langle \{Z_i + \lambda_i Z_k \mid 1 \leq i \leq 6 \wedge i \neq k\} \rangle, \quad (17)$$

where Z_k is the non-main vector. Proving that a_k is non-abelian is equivalent to finding a nonzero bracket in its law.

- For $k \in \{3,4,5,6\}$: $[Z_1 + \lambda_1 Z_k, Z_2 + \lambda_2 Z_k] = Z_3 + v$ is nonzero, because $v \in \langle Z_1, Z_2, Z_4, Z_5, Z_6 \rangle$.
- For $k = 2$: $[Z_1 + \lambda_1 Z_2, Z_4 + \lambda_4 Z_2] = Z_5 + v$ is nonzero, because $v \in \langle Z_3, Z_6 \rangle$.
- For $k = 1$: $[Z_2 + \lambda_2 Z_1, Z_4 + \lambda_4 Z_1] = -Z_6 + v$ is nonzero, because $v \in \langle Z_3, Z_5 \rangle$.

Hence, there do not exist 5-dimensional abelian subalgebras of $g_{6,5}$ and $\mathcal{M}(g_{6,5}) = 4$.

Now, let us prove that 3 is the maximal abelian dimension of the Lie algebras $g_{6,19}$ and $g_{6,20}$. Since these Lie algebra are non-abelian, their maximal abelian dimension is less than 6. For both Lie algebras, $\langle Z_4, Z_5, Z_6 \rangle$ is a 3-dimensional abelian subalgebra. We have to prove that it is not possible to find a 4-dimensional abelian subalgebra of $g_{6,19}$ or $g_{6,20}$. To do it, we express a generic 4-dimensional subalgebra in $g_{6,19}$ (an analogous reasoning can be applied to $g_{6,20}$) as

$$a_{j,k} = \langle \{Z_i + \lambda_i Z_j + \mu_i Z_k \mid 1 \leq i \leq 6 \wedge i \neq j,k\} \rangle \quad (18)$$

- For $(j,k) \in \{(3,4),(3,5),\dots,(4,6),(5,6)\}$, $[Z_1 + \lambda_1 Z_j + \mu_1 Z_k, Z_2 + \lambda_2 Z_j + \mu_2 Z_k] = Z_3 + v$ is a nonzero bracket, because $v \in \{Z_h\}_{h \neq 3}$.
- For $(j,k) \in \{(2,4),(2,5),(2,6)\}$: the bracket $[Z_1 + \lambda_1 Z_j + \mu_1 Z_k, Z_3 + \lambda_3 Z_j + \mu_3 Z_k] = Z_4 + v$ is nonzero, because $v \in \langle Z_1, Z_3, Z_4, Z_5 \rangle$.
- For $(j,k) \in \{(1,2),(1,5),(1,6)\}$, the bracket $[Z_3 + \lambda_3 Z_j + \mu_3 Z_k, Z_4 + \lambda_4 Z_j + \mu_4 Z_k] = -Z_6 + v$ is nonzero, because $v \in \{Z_h\}_{h \leq 5}$.
- For $(j,k) \in \{(1,3),(1,4)\}$, the bracket $[Z_2 + \lambda_2 Z_j + \mu_2 Z_k, Z_5 + \lambda_5 Z_j + \mu_5 Z_k] = Z_6 + v$ is nonzero, because $v \in \{Z_h\}_{h \leq 5}$.
- For $(j,k) = (2,3)$, the following bracket: $[Z_1 + \lambda_1 Z_j + \mu_1 Z_k, Z_4 + \lambda_4 Z_j + \mu_4 Z_k] = Z_5 + v$ is nonzero, because $v \in \langle Z_1, Z_2, Z_3, Z_4, Z_6 \rangle$.

Therefore, there do not exist 4-dimensional abelian subalgebras of $g_{6,19}$ and $\mathcal{M}(g_{6,19}) = 3$. Analogously, the same conclusion can be obtained for the Lie algebra $g_{6,20}$. □

5.2 Solvable, non-nilpotent Lie algebras of dimension 6.

We show the propositions where the invariant $\mathcal{M}(g)$ has been computed for solvable, non-nilpotent Lie algebras of dimension 6. Let us note that the proof of these results will be done in the same way as in the previous subsection. Here we have considered Mubarakzyanov and Turkowski's classification (see [13] and [19]).

Proposition 9 *For $j \in \{21, 22 \dots, 47\}$, it is satisfied that $\mathcal{M}(g_{6,j}) = 4$.*

Proof: To prove this proposition, we have to find a 4-dimensional abelian subalgebra of the Lie algebra $g_{6,j}$ (for $j \in \{21, 22 \ldots, 47\}$) and prove that there do not exist any abelian subalgebras of dimension 5. Let us note that no 6-dimensional abelian subalgebra can be found in these algebras, because they are not abelian.

Indeed, the subalgebra $\langle Z_3, Z_4, Z_5, Z_6\rangle$ of $g_{6,j}$ is abelian for $j = 21, \ldots, 47$. So, the maximal abelian dimension of $g_{6,j}$ is, at least, 4. In this way, we only need to see if there exists some abelian subalgebra of dimension 5. Due to reasons of length, we only study one of the Lie algebras shown in the statement of the proposition. The rest can be studied using an analogous reasoning.

We consider the algebra $g_{6,39}$ whose law with respect to a certain basis $\{Z_i\}_{i=1}^6$ is given by the following nonzero brackets:

$$[Z_1, Z_3] = Z_4 + Z_5, [Z_1, Z_4] = -Z_3 + Z_6,$$
$$[Z_1, Z_5] = Z_6, [Z_1, Z_6] = -Z_5, [Z_2, Z_3] = Z_3, \quad (19)$$
$$[Z_2, Z_4] = Z_4, [Z_2, Z_5] = Z_5, [Z_2, Z_6] = Z_6.$$

Let us suppose that there exists a 5-dimensional abelian subalgebra of $g_{6,19}$. Then, we can assume that each vector in the basis of such a subalgebra are expressed as a linear combination of one main vector and the non-main one. So any 5-dimensional abelian subalgebra is expressed by

$$a_k = \langle\{Z_i + \lambda_i Z_k \mid 1 \leq i \leq 6 \wedge i \neq k\}\rangle \quad (20)$$

where Z_k is the non- main vector. Now, we are proving that we can find a nonzero bracket in its law.

- For $k \in \{1, 3, 4, 5\}$: a nonzero bracket is $[Z_2 + \lambda_2 Z_k, Z_6 + \lambda_6 Z_k] = Z_6 + v$, because $v \in \langle Z_1, Z_2, Z_3, Z_4, Z_5\rangle$.
- For $k \in \{2, 6\}$, a nonzero bracket is $[Z_1 + \lambda_2 Z_k, Z_5 + \lambda_6 Z_k] = Z_6 + v$, because $v \in \langle Z_1, Z_2, Z_3, Z_4, Z_5\rangle$.

Hence, there do not exist any 5-dimensional abelian subalgebras of $g_{6,39}$ and $\mathcal{M}(g_{6,39}) = 4$. □

Proposition 10 *For* $j \in \{48, 49, \ldots, 60\}$*, it is verified that* $\mathcal{M}(g_{6,j}) = 3$.

Proof: Let us note that $\langle Z_3, Z_4, Z_5\rangle$ and $\langle Z_3, Z_4, Z_6\rangle$ are abelian subalgebras of $g_{6,48}$ and $g_{6,p}$, for $p = \{49, \ldots, 60\}$, respectively. Hence, the maximal abelian dimension of these algbras is greater than or equal to 3. So we are now going to prove that no 4-dimensional subalgebra of $g_{6,p}$ is abelian for $p = 48, \ldots, 60$. We are going to prove this fact for one of the algebras and an analogous reasoning can be applied to the rest.

Let us consider the algebra $g_{6,51}$, whose law with respect to a certain basis $\{Z_i\}_{i=1}^6$ is given by the following nonzero brackets

$$[Z_1, Z_4] = Z_4, [Z_1, Z_5] = -Z_5,$$
$$[Z_2, Z_3] = Z_3, [Z_2, Z_5] = Z_5, \quad (21)$$
$$[Z_2, Z_6] = Z_1 + Z_6, [Z_4, Z_5] = Z_1.$$

Now, we prove that it is not possible to obtain a 4-dimensional abelian subalgebra of $g_{6,51}$ by arguing analogously to Proposition 7. In this way, we are assuming that the subalgebras are expressed as follows

$$a_{j,k} = \langle\{Z_i + \lambda_i Z_j + \mu_i Z_k \mid 1 \leq i \leq 6 \wedge j, k \neq i\}\rangle \quad (22)$$

where Z_j and Z_k are the two non-main vectors. To prove the nonexistence of abelian subalgebras of dimension 4, we only need to find a nonzero bracket in each of these subalgebras:

- For $(j, k) \in \{(1, 2), (1, 3), (1, 6), (2, 3)\}$, we consider the nonzero bracket given by $[Z_4 + \lambda_4 Z_j + \mu_4 Z_k, Z_5 + \lambda_5 Z_j + \mu_5 Z_k] = Z_1 + v$, because $v \in \langle Z_3, Z_4, Z_5\rangle$.
- For $(j, k) \in \{(1, 4), (1, 5), (3, 5)\}$, this bracket $[Z_2 + \lambda_2 Z_j + \mu_2 Z_k, Z_6 + \lambda_6 Z_j + \mu_6 Z_k] = Z_1 + v$ is nonzero, because $v \in Z_3, Z_4, Z_5\rangle$.
- For $(j, k) \in \{(2, 5), (2, 6)\}$, the bracket $[Z_1 + \lambda_1 Z_j + \mu_1 Z_k, Z_4 + \lambda_4 Z_j + \mu_4 Z_k] = Z_4 + v$ is nonzero, because $v \in \langle Z_1, Z_3, Z_5, Z_6\rangle$.
- For $(j, k) \in \{(3, 4), (3, 6)\}$, the bracket $[Z_2 + \lambda_2 Z_j + \mu_2 Z_k, Z_5 + \lambda_5 Z_j + \mu_5 Z_k] = Z_5 + v$ is nonzero, because $v \in \langle Z_1, Z_3, Z_4, Z_6\rangle$.
- For $(j, k) \in \{(4, 5), (4, 6), (5, 6)\}$, the bracket $[Z_2 + \lambda_2 Z_j + \mu_2 Z_k, Z_3 + \lambda_3 Z_j + \mu_3 Z_k] = Z_3 + v$ is nonzero, because $v \in \langle Z_1, Z_4, Z_5, Z_6\rangle$.
- For $(j, k) = (2, 4)$, we consider the brackets

  $$[Z_1 + \lambda_1 Z_2 + \mu_1 Z_4, Z_3 + \lambda_3 Z_2 + \mu_3 Z_4] = \mu_3 Z_4 + \lambda_1 Z_3$$
  $$[Z_1 + \lambda_1 Z_2 + \mu_1 Z_4, Z_5 + \lambda_5 Z_2 + \mu_5 Z_4] = \mu_1 Z_1 \quad (23)$$
  $$+ \mu_5 Z_4 + (\lambda_1 - 1) Z_5$$

  If the subalgebra $a_{2,4}$ is abelian, these brackets would have to be zero, obtaining the following system without solutions $\{\lambda_1 = 0, \lambda_1 - 1 = 0\}$.

Consequently, there are not abelian subalgebras of dimension 4 in $g_{6,51}$ and $\mathcal{M}(g_{6,51}) = 3$. □

6 Conclusions

In this paper a new method for computing abelian subalgebras and, in particular, the maximum among the dimension of all the abelian subalgebras of a Lie algebra has been proposed. We hope to continue with this research in the future in order to provide the classification of nilpotent and solvable Lie algebras.

References:

[1] J.C. Benjumea, F.J. Echarte, J. Núñez and A.F. Tenorio: An obstruction to represent abelian Lie algebras by unipotent matrices. *Extracta Math.* 19, 2004, pp. 269–277.

[2] J.C. Benjumea, J. Núñez and A.F. Tenorio: The maximal abelian dimension of linear algebras formed by strictly-upper triangular matrices. *Theor. Math. Phys.* 152, 2007, pp. 1225–1233.

[3] R. Campoamor-Stursberg: Number of missing label operators and upper bounds for dimensions of maximal Lie subalgebras. *Acta Phys. Polon. B* 37, 2006, pp. 2745–2760.

[4] C. Caranti, S. Mattarei and F. Newman: Graded Lie algebras of maximal class. *Trans. Amer. Math. Soc.* 349, 1997, pp. 4021–4051.

[5] M. Ceballos, J. Núñez and A.F. Tenorio: The computation of abelian subalgebras in the Lie algebra of upper-triangular matrices. *An. Stiint. Univ. "Ovidius" Constanta Ser. Mat.* 16:1, 2008, pp. 59–66.

[6] D. Gorenstein, R. Lyons and R. Solomon: The classification of the finite simple groups, vol. 3. *Mathematical surveys and monographs*, American Mathematical Society, Providence, 1997.

[7] M. Goze and Y. Khakimdjanov: Nilpotent and solvable Lie algebras. In M. Hazewinkel (ed.): *Handbook of Algebra* Vol 2. Elsevier, Amsterdam, 2000, pp. 615–664

[8] M. Goze and E. Remm: *Classifications of complex nilpotent Lie algebras*. Available from World Wide Web (http://www.math.uha.fr/ algebre/goze/LIE/Lie.html), 2004.

[9] I. Hernández, C. Mateos, J. Núñez and A.F. Tenorio: Lie Theory: Applications to problems in Mathematical Finance and Economics. *Applied Mathematics and Computation* 208:2, 2009, pp. 446–452.

[10] E. Kirillova, T. Hoch and K. Spindler: Optimal control of a spin system acting on a single quantum bit. *WSEAS Transactions on Mathematics* 7:12, 2008, pp. 687–697.

[11] B. Lantos: Path design and control for collision avoidance system of cars. In M.I. García-Planas and C. González-Concepción (eds.): *Proceedings of 5th WSEAS International Conference on System Science and Simulation in Engineering (ICOSSE'06)*, WSEAS, Tenerife, 2006, pp. 337–345.

[12] U. Ledzewicz and H. Schättler: Singular controls in systems describing tumor anti-angiogenesis. In M.I. García-Planas and C. González-Concepción (eds.): *Proceedings of 5th WSEAS International Conference on System Science and Simulation in Engineering (ICOSSE'06)*, WSEAS, Tenerife, 2006, pp. 156–161.

[13] G.M. Mubarakzyanov: Classification of real structures of Lie algebras of fifth order. *Izv. Vysš. Učebn. Zaved. Matematika* 3:34, 1963, pp. 99–106.

[14] P.J. Olver: *Applications of Lie groups to differential equations*. Springer, New York, 1986.

[15] S.I. Senashov, A. Yakhno and L. Yakhno: Lie-Backlund symmetries of homogeneus system of bi-dimensional equations. *WSEAS Transactions on Mathematics* 6:1, 2007, pp. 16–21.

[16] A.G. Sumedrea and L. Sangeorzan: A Mathematical Theory of Psychological Dynamics. *WSEAS Transactions on Mathematics* 8:10, 2009, pp. 604–613.

[17] A.F. Tenorio: Solvable Lie algebras and maximal abelian dimensions. *Acta Math. Univ. Comenian. (N.S.)* 77, 2008, pp. 141–145.

[18] J.-L. Thiffeault and P.J. Morrison: Classification and Casimir invariants of Lie-Poisson brackets. *Phys. D* 136, 2000, pp. 205–244.

[19] P. Turkowski: Solvable Lie algebras of dimension six. *J. Math. Phys.* 31, 1990, pp. 1344–1350.

[20] V.S. Varadarajan: *Lie groups, Lie algebras and their representations*. Springer, New York, 1984.

A Tables

Table 1: Non-decomposable Solvable Lie algebras of dimension less than 5.

Algebra	Nonzero brackets	$\mathcal{M}(g)$
$g_{2,1}$	$[Z_1, Z_2] = Z_1$	1
$g_{3,1}$	$[Z_1, Z_3] = Z_2$	2
$g_{3,2}$	$[Z_1, Z_3] = Z_1, [Z_2, Z_3] = Z_2$	2
$g_{3,3}$	$[Z_1, Z_3] = Z_2, [Z_2, Z_3] = -Z_1$	2
$g_{3,4}$	$[Z_1, Z_3] = -Z_1, [Z_2, Z_3] = -Z_1 - Z_2$	2
$g_{3,5}$	$[Z_1, Z_3] = -Z_1$	2
$g_{4,1}$	$[Z_1, Z_3] = Z_2, [Z_1, Z_4] = Z_3$	3
$g_{4,2}$	$[Z_1, Z_3] = Z_3, [Z_1, Z_4] = Z_4,$ $[Z_2, Z_3] = Z_4$	2
$g_{4,3}$	$[Z_1, Z_3] = Z_3, [Z_1, Z_4] = Z_4,$ $[Z_2, Z_3] = -Z_4, [Z_2, Z_4] = Z_3$	2
$g_{4,4}$	$[Z_1, Z_3] = Z_3, [Z_1, Z_2] = Z_4$	3
$g_{4,5}^{\alpha,\beta}$	$[Z_4, Z_1] = Z_1, [Z_4, Z_2] = \alpha Z_2,$ $[Z_4, Z_3] = \beta Z_3$	3
$g_{4,6}^{\alpha}$	$[Z_3, Z_1] = \alpha Z_1, [Z_3, Z_2] = Z_2,$ $[Z_3, Z_4] = Z_2 + Z_4$	3
$g_{4,7}$	$[Z_1, Z_2] = Z_2 + Z_3,$ $[Z_1, Z_3] = Z_3 + Z_4, [Z_1, Z_4] = Z_4$	3
$g_{4,8}^{\alpha,\beta}$	$[Z_1, Z_4] = \alpha Z_4, [Z_1, Z_2] = \beta Z_2 - Z_3,$ $[Z_1, Z_3] = Z_2 + \beta Z_3$	3
$g_{4,9}^{\alpha}$	$[Z_2, Z_3] = Z_4, [Z_1, Z_2] = (\alpha - 1)Z_2,$ $[Z_1, Z_3] = Z_3, [Z_1, Z_4] = \alpha Z_4$	2
$g_{4,10}$	$[Z_2, Z_3] = Z_4, [Z_1, Z_2] = Z_2 + Z_3,$ $[Z_1, Z_3] = Z_3, [Z_1, Z_4] = 2Z_4$	2
$g_{4,11}^{\alpha}$	$[Z_2, Z_3] = Z_4, [Z_1, Z_2] = \alpha Z_2 - Z_3,$ $[Z_1, Z_4] = 2\alpha Z_4, [Z_1, Z_3] = Z_2 + \alpha Z_3$	2

Table 2: Non-decomposable Solvable Lie algebras of dimension 5 (I).

Algebra	Nonzero brackets	$\mathcal{M}(g)$
$g_{5,1}$	$[Z_2, Z_4] = Z_3, [Z_2, Z_5] = Z_1,$ $[Z_4, Z_5] = Z_2$	3
$g_{5,2}$	$[Z_2, Z_4] = Z_1, [Z_3, Z_5] = Z_1$	3
$g_{5,3}$	$[Z_3, Z_4] = Z_1, [Z_2, Z_5] = Z_1,$ $[Z_3, Z_5] = Z_2$	3
$g_{5,4}$	$[Z_3, Z_4] = Z_1, [Z_2, Z_5] = Z_1,$ $[Z_3, Z_5] = Z_2, [Z_4, Z_5] = Z_3$	3
$g_{5,5}$	$[Z_3, Z_5] = Z_1, [Z_4, Z_5] = Z_2$	4

Table 3: Non-decomposable Solvable Lie algebras of dimension 5 (I).

Algebra	Nonzero brackets	$\mathcal{M}(g)$
$g_{5,6}$	$[Z_2, Z_5] = Z_1, [Z_3, Z_5] = Z_2,$ $[Z_4, Z_5] = Z_3$	4
$g_{5,7}^{\alpha,\beta,\gamma}$	$[Z_1, Z_5] = Z_1, [Z_2, Z_5] = \alpha Z_2,$ $[Z_3, Z_5] = \beta Z_3, [Z_4, Z_5] = \gamma Z_4$	4
$g_{5,8}^{\gamma}$	$[Z_2, Z_5] = Z_1, [Z_3, Z_5] = Z_3,$ $[Z_4, Z_5] = \gamma Z_4,$	4
$g_{5,9}^{\beta,\gamma}$	$[Z_1, Z_5] = Z_1, [Z_2, Z_5] = Z_1 + Z_3,$ $[Z_3, Z_5] = \beta Z_3, [Z_4, Z_5] = \gamma Z_4$	4
$g_{5,10}$	$[Z_2, Z_5] = Z_1, [Z_3, Z_5] = Z_2,$ $[Z_4, Z_5] = Z_4$	4
$g_{5,11}^{\gamma}$	$[Z_1, Z_5] = Z_1, [Z_2, Z_5] = Z_1 + Z_2,$ $[Z_3, Z_5] = Z_2 + Z_3, [Z_4, Z_5] = \gamma Z_4$	4
$g_{5,12}$	$[Z_1, Z_5] = Z_1, [Z_2, Z_5] = Z_1 + Z_2,$ $[Z_3, Z_5] = Z_2 + Z_3, [Z_4, Z_5] = Z_3 + Z_4$	4
$g_{5,13}^{p,s,\gamma}$	$[Z_1, Z_5] = Z_1, [Z_2, Z_5] = \gamma Z_2,$ $[Z_3, Z_5] = pZ_3 - sZ_4,$ $[Z_4, Z_5] = sZ_3 + pZ_4$	4
$g_{5,14}$	$[Z_2, Z_5] = Z_1, [Z_3, Z_5] = pZ_3 - Z_4,$ $[Z_4, Z_5] = Z_3 + pZ_4$	4
$g_{5,15}^{\gamma}$	$[Z_1, Z_5] = Z_1, [Z_3, Z_5] = \gamma Z_3,$ $[Z_2, Z_5] = Z_1 + Z_2,$ $[Z_4, Z_5] = Z_3 + \gamma Z_4$	4
$g_{5,16}^{p,s}$	$[Z_1, Z_5] = Z_1, [Z_2, Z_5] = Z_1 + Z_2,$ $[Z_3, Z_5] = pZ_3 - sZ_4,$ $[Z_4, Z_5] = sZ_3 + pZ_4$	4
$g_{5,17}^{p,q,s}$	$[Z_1, Z_5] = pZ_1 - Z_2,$ $[Z_2, Z_5] = Z_1 + pZ_2,$ $[Z_3, Z_5] = qZ_3 - sZ_4,$ $[Z_4, Z_5] = sZ_3 + qZ_4$	4
$g_{5,18}^{p}$	$[Z_3, Z_5] = Z_1 + pZ_3 - Z_4,$ $[Z_2, Z_5] = Z_1 + pZ_2,$ $[Z_1, Z_5] = pZ_1 - Z_2,$ $[Z_4, Z_5] = Z_2 + Z_3 - pZ_4$	4
$g_{5,19}^{\alpha,\beta}$	$[Z_2, Z_3] = Z_1, [Z_1, Z_5] = (1 + \alpha)Z_1,$ $[Z_2, Z_5] = Z_2, [Z_3, Z_5] = \alpha Z_3,$ $[Z_4, Z_5] = \beta Z_4$	3
$g_{5,20}^{\alpha}$	$[Z_2, Z_3] = Z_1, [Z_1, Z_5] = (1 + \alpha)Z_2,$ $[Z_2, Z_5] = Z_2, [Z_3, Z_5] = \alpha Z_3,$ $[Z_4, Z_5] = Z_1 + (1 + \alpha)Z_4$	3
$g_{5,21}$	$[Z_2, Z_3] = Z_1, [Z_1, Z_5] = 2Z_1,$ $[Z_2, Z_5] = Z_2 + Z_3, [Z_4, Z_5] = Z_4,$ $[Z_3, Z_5] = Z_3 + Z_4$	3
$g_{5,22}$	$[Z_2, Z_3] = Z_1, [Z_2, Z_5] = Z_3,$ $[Z_4, Z_5] = Z_4$	3
$g_{5,23}^{\beta}$	$[Z_2, Z_3] = Z_1, [Z_1, Z_5] = 2Z_1,$ $[Z_2, Z_5] = Z_2 + Z_3,$ $[Z_3, Z_5] = Z_3, [Z_4, Z_5] = \beta Z_4$	3
$g_{5,24}^{\epsilon}$	$[Z_2, Z_3] = Z_1, [Z_1, Z_5] = 2Z_1,$ $[Z_2, Z_5] = Z_2 + Z_3, [Z_3, Z_5] = Z_3,$ $[Z_4, Z_5] = \epsilon Z_1 + 2Z_4$	3
$g_{5,25}^{\beta,\epsilon}$	$[Z_2, Z_3] = Z_1, [Z_1, Z_5] = 2Z_1,$ $[Z_2, Z_5] = Z_2 + Z_3, [Z_4, Z_5] = Z_4,$ $[Z_3, Z_5] = Z_3 + Z_4$	3
$g_{5,26}^{p,\epsilon}$	$[Z_2, Z_5] = pZ_2 + Z_3, [Z_1, Z_5] = 2pZ_1,$ $[Z_2, Z_3] = Z_1, [Z_3, Z_5] = -Z_2 + pZ_3,$ $[Z_4, Z_5] = \epsilon Z_1 + 2pZ_4$	3
$g_{5,27}$	$[Z_2, Z_3] = Z_1, [Z_3, Z_5] = Z_3 + Z_4,$ $[Z_1, Z_5] = Z_1, [Z_4, Z_5] = Z_1 + Z_4$	3

Table 4: Non-decomposable Solvable Lie algebras of dimension 5 (II).

Algebra	Nonzero brackets	$\mathcal{M}(g)$
$g_{5,28}^{\alpha}$	$[Z_2,Z_3]=Z_1,[Z_2,Z_5]=\alpha Z_2,$ $[Z_1,Z_5]=(1+\alpha)Z_1,[Z_4,Z_5]=Z_4,$ $[Z_3,Z_5]=Z_3+Z_4$	3
$g_{5,29}$	$[Z_2,Z_3]=Z_1,[Z_1,Z_5]=Z_1,$ $[Z_2,Z_5]=Z_2,[Z_3,Z_5]=Z_4$	3
$g_{5,30}^{h}$	$[Z_2,Z_4]=Z_1,[Z_3,Z_4]=Z_2,$ $[Z_1,Z_5]=(2+h)Z_1,[Z_4,Z_5]=Z_4,$ $[Z_2,Z_5]=(1+h)Z_2,[Z_3,Z_5]=hZ_3$	3
$g_{5,31}$	$[Z_2,Z_4]=Z_1,[Z_3,Z_4]=Z_2,$ $[Z_1,Z_5]=3Z_1,[Z_3,Z_5]=Z_3,$ $[Z_2,Z_5]=2Z_2,[Z_4,Z_5]=Z_3+Z_4$	3
$g_{5,32}^{h}$	$[Z_2,Z_4]=Z_1,[Z_3,Z_4]=Z_2,$ $[Z_1,Z_5]=Z_1,[Z_2,Z_5]=Z_2,$ $[Z_3,Z_5]=hZ_1+Z_3$	3
$g_{5,33}^{\beta,\gamma}$	$[Z_1,Z_4]=Z_1,[Z_3,Z_4]=\beta Z_3,$ $[Z_2,Z_5]=Z_2,[Z_3,Z_5]=\gamma Z_3$	3
$g_{5,34}^{\alpha}$	$[Z_1,Z_4]=\alpha Z_1,[Z_2,Z_4]=Z_2,$ $[Z_3,Z_4]=Z_3,[Z_1,Z_5]=Z_1,$ $[Z_3,Z_5]=Z_2$	3
$g_{5,35}^{h,\alpha}$	$[Z_1,Z_4]=hZ_1,[Z_2,Z_4]=Z_2,$ $[Z_3,Z_4]=Z_3,[Z_2,Z_5]=-Z_3,$ $[Z_1,Z_5]=\alpha Z_1,[Z_3,Z_5]=Z_2$	3
$g_{5,36}$	$[Z_2,Z_3]=Z_1,[Z_1,Z_4]=Z_1,$ $[Z_2,Z_4]=Z_2,[Z_3,Z_5]=Z_3,$ $[Z_2,Z_5]=-Z_2$	3
$g_{5,37}$	$[Z_1,Z_4]=Z_1,[Z_2,Z_4]=Z_2,$ $[Z_1,Z_5]=-Z_2,[Z_2,Z_5]=Z_1,$ $[Z_4,Z_5]=Z_3$	3
$g_{5,38}$	$[Z_1,Z_4]=Z_1,[Z_2,Z_5]=Z_2,$ $[Z_4,Z_5]=Z_3$	3
$g_{5,39}$	$[Z_2,Z_3]=Z_1,[Z_1,Z_4]=2Z_1,$ $[Z_2,Z_4]=Z_2,[Z_3,Z_4]=Z_3,$ $[Z_2,Z_5]=-Z_3,[Z_3,Z_5]=Z_2$	2

Table 5: Non-decomposable Solvable Lie algebras of dimension 6 (I).

Algebra	Nonzero brackets	$\mathcal{M}(g)$
$g_{6,1}$	$[Z_1,Z_2]=Z_3,[Z_1,Z_3]=Z_4,$ $[Z_1,Z_4]=Z_5,[Z_1,Z_5]=Z_6.$	5
$g_{6,2}$	$[Z_1,Z_2]=Z_3,[Z_1,Z_3]=Z_4,$ $[Z_1,Z_5]=Z_6.$	5
$g_{6,3}$	$[Z_1,Z_2]=Z_3,[Z_1,Z_4]=Z_5,$ $[Z_2,Z_6]=-Z_5.$	4
$g_{6,4}$	$[Z_1,Z_2]=Z_3,[Z_1,Z_3]=Z_4,$ $[Z_1,Z_4]=Z_5,[Z_1,Z_5]=Z_6,$ $[Z_2,Z_3]=Z_5,[Z_2,Z_4]=Z_6.$	4
$g_{6,5}$	$[Z_1,Z_2]=Z_3,[Z_1,Z_4]=Z_5,$ $[Z_2,Z_4]=-Z_6.$	4
$g_{6,6}$	$[Z_1,Z_2]=Z_3,[Z_1,Z_3]=Z_4,$ $[Z_1,Z_4]=Z_5,[Z_2,Z_6]=Z_4,$ $[Z_3,Z_6]=Z_5.$	4
$g_{6,7}$	$[Z_1,Z_2]=Z_3,[Z_1,Z_3]=Z_4,$ $[Z_1,Z_4]=Z_5,[Z_2,Z_6]=Z_5,$ $[Z_2,Z_3]=Z_5.$	4
$g_{6,8}$	$[Z_1,Z_2]=Z_3,[Z_1,Z_3]=Z_4,$ $[Z_1,Z_4]=Z_5,[Z_2,Z_6]=Z_5.$	4
$g_{6,9}$	$[Z_1,Z_2]=Z_3,[Z_1,Z_3]=Z_4,$ $[Z_1,Z_4]=Z_5,[Z_2,Z_3]=Z_6.$	4

Table 6: Non-decomposable Solvable Lie algebras of dimension 6 (II).

Algebra	Nonzero brackets	$\mathcal{M}(g)$
$g_{6,10}$	$[Z_1,Z_2]=Z_3,[Z_1,Z_3]=Z_4,$ $[Z_1,Z_4]=Z_5,[Z_2,Z_3]=Z_6,$ $[Z_2,Z_6]=Z_5.$	4
$g_{6,11}$	$[Z_1,Z_2]=Z_3,[Z_1,Z_3]=Z_4,$ $[Z_1,Z_5]=Z_6,[Z_3,Z_5]=-Z_4,$ $[Z_2,Z_6]=Z_4.$	4
$g_{6,12}$	$[Z_1,Z_2]=Z_3,[Z_1,Z_3]=Z_4,$ $[Z_1,Z_5]=Z_6,[Z_5,Z_6]=Z_4.$	4
$g_{6,13}$	$[Z_1,Z_2]=Z_3,[Z_1,Z_3]=Z_4,$ $[Z_1,Z_5]=Z_6,[Z_2,Z_3]=Z_4.$	4
$g_{6,14}$	$[Z_1,Z_2]=Z_3,[Z_1,Z_3]=Z_4,$ $[Z_1,Z_5]=Z_6,[Z_2,Z_3]=Z_6.$	4
$g_{6,15}$	$[Z_1,Z_2]=Z_3,[Z_1,Z_3]=Z_4,$ $[Z_1,Z_5]=Z_6,[Z_2,Z_3]=Z_6,$ $[Z_2,Z_5]=Z_6.$	4
$g_{6,16}$	$[Z_1,Z_2]=Z_3,[Z_1,Z_3]=Z_4,$ $[Z_1,Z_5]=Z_6,[Z_2,Z_5]=Z_4.$	4
$g_{6,17}$	$[Z_1,Z_2]=Z_3,[Z_1,Z_3]=Z_4,$ $[Z_1,Z_4]=Z_5,[Z_1,Z_5]=Z_6,$ $[Z_2,Z_3]=Z_6.$	4
$g_{6,18}$	$[Z_1,Z_2]=Z_3,[Z_1,Z_3]=Z_4,$ $[Z_5,Z_6]=-Z_4.$	4
$g_{6,19}$	$[Z_1,Z_2]=Z_3,[Z_1,Z_3]=Z_4,$ $[Z_1,Z_4]=Z_5,[Z_1,Z_5]=Z_6,$ $[Z_2,Z_5]=Z_6,[Z_3,Z_4]=-Z_6.$	3
$g_{6,20}$	$[Z_1,Z_2]=Z_3,[Z_1,Z_3]=Z_4,$ $[Z_1,Z_4]=Z_5,[Z_1,Z_5]=Z_6,$ $[Z_2,Z_3]=Z_5-Z_6,[Z_2,Z_4]=Z_6,$ $[Z_2,Z_5]=Z_6,[Z_3,Z_4]=-Z_6.$	3
$g_{6,21}^{\alpha,\beta,\gamma,\delta}$	$[Z_1,Z_3]=\alpha Z_3,[Z_1,Z_4]=\gamma Z_4,$ $[Z_1,Z_6]=Z_6,[Z_2,Z_3]=\beta Z_3,$ $[Z_2,Z_4]=\delta Z_4,[Z_2,Z_5]=Z_5$	4
$g_{6,22}^{\alpha,\beta,\gamma}$	$[Z_1,Z_3]=\alpha Z_3,[Z_1,Z_4]=Z_4,$ $[Z_1,Z_5]=Z_6,[Z_2,Z_6]=Z_6,$ $[Z_2,Z_3]=\beta Z_3,[Z_2,Z_4]=\gamma Z_4,$ $[Z_2,Z_5]=Z_5$	4
$g_{6,23}^{\alpha}$	$[Z_1,Z_3]=Z_3,[Z_2,Z_5]=Z_5,$ $[Z_1,Z_4]=Z_4,[Z_1,Z_5]=Z_6,$ $[Z_2,Z_4]=\alpha Z_4,[Z_2,Z_6]=Z_6,$ $[Z_2,Z_3]=\alpha Z_3+Z_4$	4
$g_{6,24}^{\alpha,\beta}$	$[Z_1,Z_3]=Z_3,[Z_1,Z_4]=Z_4,$ $[Z_1,Z_5]=Z_6,[Z_2,Z_6]=\alpha Z_6,$ $[Z_2,Z_3]=Z_4,[Z_2,Z_4]=-Z_3,$ $[Z_2,Z_5]=\alpha Z_5+\beta Z_6$	4
$g_{6,25}^{\alpha,\beta}$	$[Z_1,Z_3]=\alpha Z_3,[Z_1,Z_6]=Z_6,$ $[Z_2,Z_3]=\beta Z_3,[Z_2,Z_4]=Z_4,$ $[Z_1,Z_5]=Z_5+Z_6$	4
$g_{6,26}^{\alpha,\beta}$	$[Z_1,Z_3]=\alpha Z_3,[Z_1,Z_4]=\alpha Z_4,$ $[Z_1,Z_5]=Z_5+Z_6,[Z_1,Z_6]=Z_6,$ $[Z_2,Z_3]=Z_3+Z_4,[Z_2,Z_4]=Z_4$	4
$g_{6,27}^{\alpha,\beta,\gamma}$	$[Z_1,Z_3]=\alpha Z_3,[Z_1,Z_4]=\alpha Z_4,$ $[Z_2,Z_5]=\beta Z_6,[Z_2,Z_3]=\gamma Z_3+Z_4,$ $[Z_1,Z_6]=Z_6,[Z_2,Z_4]=-Z_3+\gamma Z_4,$ $[Z_1,Z_5]=Z_5+Z_6$	4
$g_{6,28}$	$[Z_1,Z_3]=Z_1,[Z_1,Z_4]=Z_6,$ $[Z_2,Z_4]=Z_2,$ $[Z_2,Z_5]=Z_5+Z_6,[Z_2,Z_6]=Z_6$	4
$g_{6,29}$	$[Z_1,Z_3]=Z_3,[Z_1,Z_4]=Z_6,$ $[Z_2,Z_6]=Z_6,[Z_2,Z_4]=Z_4+Z_5,$ $[Z_2,Z_5]=Z_5+\alpha Z_6$	4
$g_{6,30}^{\alpha,\beta}$	$[Z_1,Z_3]=\alpha Z_3,[Z_1,Z_4]=Z_3+\beta Z_6,$ $[Z_1,Z_5]=Z_5,[Z_2,Z_5]=Z_6,$ $[Z_1,Z_6]=Z_6,[Z_2,Z_3]=Z_3,$ $[Z_2,Z_4]=Z_5$	4

Table 7: Non-decomposable Solvable Lie algebras of dimension 6 (III).

Algebra	**Nonzero brackets**	$\mathcal{M}(g)$
$g_{6,31}^{\alpha}$	$[Z_1,Z_3]=Z_4,[Z_2,Z_3]=Z_3,$ $[Z_2,Z_4]=Z_4,[Z_1,Z_6]=Z_6,$ $[Z_2,Z_5]=\alpha Z_5,[Z_1,Z_5]=Z_5+Z_6,$ $[Z_2,Z_6]=\alpha Z_6$	4
$g_{6,32}^{\alpha,\beta}$	$[Z_1,Z_3]=Z_3+Z_4,[Z_1,Z_4]=Z_4,$ $[Z_1,Z_5]=Z_5+Z_6,[Z_1,Z_6]=Z_6,$ $[Z_2,Z_4]=Z_6,[Z_2,Z_6]=-Z_4,$ $[Z_2,Z_3]=\alpha Z_4+Z_5-\beta Z_6,$ $[Z_2,Z_5]=-Z_3+\beta Z_4+\alpha Z_6$	4
$g_{6,33}^{\alpha,\beta,\gamma,\delta}$	$[Z_1,Z_3]=\alpha Z_3,[Z_1,Z_4]=\gamma Z_4,$ $[Z_1,Z_5]=Z_6,[Z_2,Z_5]=Z_5,$ $[Z_1,Z_6]=-Z_5,[Z_2,Z_3]=\beta Z_3,$ $[Z_2,Z_4]=\delta Z_4,[Z_2,Z_6]=Z_6$	4
$g_{6,34}^{\alpha,\beta,\gamma}$	$[Z_2,Z_4]=Z_4,[Z_1,Z_6]=-Z_5+\gamma Z_6,$ $[Z_1,Z_3]=\alpha Z_3,[Z_2,Z_3]=\beta Z_3,$ $[Z_1,Z_5]=\gamma Z_5+Z_6$	4
$g_{6,35}^{\alpha,\beta,\gamma,\delta}$	$[Z_1,Z_3]=Z_3,[Z_1,Z_4]=Z_4,$ $[Z_1,Z_5]=\alpha Z_5+\beta Z_6,$ $[Z_2,Z_6]=\delta Z_6,[Z_1,Z_6]=-\beta Z_5+\alpha Z_6,$ $[Z_2,Z_3]=\gamma Z_3+Z_4,[Z_2,Z_5]=\delta Z_5,$ $[Z_2,Z_4]=-Z_3+\gamma Z_4$	4
$g_{6,36}^{\alpha,\beta}$	$[Z_1,Z_3]=Z_4,[Z_1,Z_5]=\alpha Z_5+Z_6,$ $[Z_1,Z_6]=-Z_5+\alpha Z_6,[Z_2,Z_3]=Z_3,$ $[Z_2,Z_4]=Z_4,[Z_2,Z_5]=\beta Z_5,$ $[Z_2,Z_6]=\beta Z_6$	4
$g_{6,37}^{\alpha}$	$[Z_1,Z_3]=\alpha Z_3+Z_4,[Z_1,Z_4]=\alpha Z_4,$ $[Z_1,Z_5]=Z_6,[Z_1,Z_6]=-Z_5,$ $[Z_2,Z_5]=Z_5,[Z_2,Z_6]=Z_6$	4
$g_{6,38}^{\alpha,\beta,\gamma}$	$[Z_1,Z_3]=Z_4,[Z_1,Z_4]=-Z_3,$ $[Z_1,Z_5]=\alpha Z_5+\beta Z_6,[Z_2,Z_6]=\delta Z_6,$ $[Z_2,Z_3]=Z_3,[Z_2,Z_4]=Z_4,$ $[Z_1,Z_6]=-\beta Z_5+\alpha Z_6,[Z_2,Z_5]=\gamma Z_5$	4
$g_{6,39}$	$[Z_1,Z_3]=Z_4+Z_5,[Z_1,Z_6]=-Z_5,$ $[Z_1,Z_5]=Z_6,[Z_2,Z_5]=Z_5,$ $[Z_1,Z_4]=-Z_3+Z_6,[Z_2,Z_3]=Z_3,$ $[Z_2,Z_4]=Z_4,[Z_2,Z_6]=Z_6$	4
$g_{6,40}^{\alpha,\beta}$	$[Z_1,Z_4]=\alpha Z_4,[Z_1,Z_6]=Z_6,$ $[Z_2,Z_4]=\beta Z_4,[Z_2,Z_5]=Z_5,$ $[Z_1,Z_2]=Z_3$	4
$g_{6,41}^{\alpha}$	$[Z_1,Z_4]=Z_4,[Z_1,Z_5]=Z_6,$ $[Z_2,Z_4]=\alpha Z_4,[Z_2,Z_5]=Z_5,$ $[Z_2,Z_6]=Z_6,[Z_1,Z_2]=Z_3$	4
$g_{6,42}^{\alpha,\epsilon}$	$[Z_1,Z_3]=Z_3,[Z_1,Z_5]=Z_6,$ $[Z_2,Z_3]=\alpha Z_3,[Z_2,Z_4]=Z_4,$ $[Z_1,Z_2]=\epsilon Z_5$	4
$g_{6,43}^{\alpha,\epsilon}$	$[Z_1,Z_3]=Z_3,[Z_1,Z_4]=Z_4,$ $[Z_1,Z_5]=Z_6,[Z_2,Z_3]=Z_4,$ $[Z_2,Z_4]=-Z_3,[Z_2,Z_5]=\alpha Z_6,$ $[Z_1,Z_2]=\epsilon Z_5$	4
$g_{6,44}$	$[Z_1,Z_5]=Z_5+Z_6,[Z_1,Z_6]=Z_6,$ $[Z_2,Z_4]=Z_4,[Z_1,Z_2]=Z_1$	4
$g_{6,45}^{\alpha,\beta}$	$[Z_1,Z_4]=\alpha Z_4,[Z_1,Z_5]=Z_6,$ $[Z_1,Z_6]=-Z_5,[Z_2,Z_5]=Z_5,$ $[Z_2,Z_6]=Z_6,[Z_1,Z_2]=Z_3,$ $[Z_2,Z_3]=\beta Z_4$	4
$g_{6,46}^{\alpha}$	$[Z_1,Z_5]=\alpha Z_5+Z_6,[Z_2,Z_4]=Z_4,$ $[Z_1,Z_6]=-Z_5+\alpha Z_6,[Z_1,Z_2]=Z_3$	4
$g_{6,47}^{\epsilon}$	$[Z_1,Z_3]=Z_4,[Z_1,Z_5]=Z_6,$ $[Z_1,Z_6]=-Z_5+\alpha Z_6[Z_1,Z_6]=-Z_5,$ $[Z_2,Z_5]=Z_5,[Z_2,Z_6]=Z_6,$ $[Z_1,Z_2]=\epsilon Z_3$	4

Table 8: Non-decomposable Solvable Lie algebras of dimension 6 (IV).

Algebra	**Nonzero brackets**	$\mathcal{M}(g)$
$g_{6,48}$	$[Z_1,Z_3]=Z_3,[Z_4,Z_6]=Z_3,$ $[Z_1,Z_5]=-Z_5,[Z_1,Z_6]=Z_6,$ $[Z_2,Z_4]=Z_4,[Z_2,Z_5]=2Z_5,$ $[Z_2,Z_6]=-Z_6,[Z_5,Z_6]=Z_4$	3
$g_{6,49}^{\alpha,\beta}$	$[Z_1,Z_3]=Z_3,[Z_4,Z_5]=Z_3,$ $[Z_1,Z_4]=Z_4,[Z_1,Z_6]=\alpha Z_6,$ $[Z_2,Z_3]=Z_3,[Z_2,Z_5]=Z_5,$ $[Z_2,Z_6]=\beta Z_6$	3
$g_{6,50}^{\alpha}$	$[Z_1,Z_3]=2Z_1,[Z_1,Z_5]=Z_5,$ $[Z_2,Z_4]=Z_5,[Z_2,Z_6]=Z_6,$ $[Z_1,Z_6]=\alpha Z_6,[Z_4,Z_5]=Z_1,$ $[Z_1,Z_4]=Z_4$	3
$g_{6,51}$	$[Z_1,Z_4]=Z_4,[Z_1,Z_5]=-Z_5,$ $[Z_2,Z_3]=Z_3,[Z_2,Z_5]=Z_5,$ $[Z_2,Z_6]=Z_1+Z_6,[Z_4,Z_5]=Z_1$	3
$g_{6,52}^{\alpha}$	$[Z_1,Z_4]=Z_4,[Z_1,Z_5]=-Z_5,$ $[Z_1,Z_6]=Z_3,[Z_2,Z_6]=Z_6,$ $[Z_2,Z_3]=Z_3,[Z_2,Z_4]=\alpha Z_4,$ $[Z_2,Z_5]=(1-\alpha)Z_5,[Z_4,Z_5]=Z_3$	3
$g_{6,53}$	$[Z_1,Z_3]=Z_3,[Z_1,Z_4]=Z_4,$ $[Z_2,Z_3]=Z_3,[Z_2,Z_5]=Z_5+Z_6,$ $[Z_2,Z_6]=Z_6,[Z_4,Z_5]=Z_3$	3
$g_{6,54}^{\alpha}$	$[Z_1,Z_3]=Z_3,[Z_1,Z_4]=Z_4,$ $[Z_1,Z_5]=Z_6,[Z_2,Z_6]=Z_6,$ $[Z_2,Z_3]=(1+\alpha)Z_3,[Z_2,Z_4]=\alpha Z_4,$ $[Z_2,Z_5]=Z_5,[Z_4,Z_5]=Z_3$	3
$g_{6,55}^{\alpha,\beta}$	$[Z_1,Z_4]=Z_5,[Z_1,Z_6]=\alpha Z_6,$ $[Z_2,Z_3]=2Z_3,[Z_2,Z_4]=Z_4,$ $[Z_1,Z_5]=-Z_4,[Z_2,Z_5]=Z_5,$ $[Z_2,Z_6]=\beta Z_6,[Z_4,Z_5]=Z_3$	3
$g_{6,56}$	$[Z_1,Z_4]=Z_5,[Z_1,Z_5]=-Z_4,$ $[Z_2,Z_3]=2Z_3,[Z_4,Z_5]=Z_3,$ $[Z_2,Z_4]=Z_4,[Z_2,Z_5]=Z_5,$ $[Z_2,Z_6]=Z_3+2Z_6$	3
$g_{6,57}^{\alpha}$	$[Z_2,Z_4]=Z_4+\alpha Z_5,[Z_1,Z_4]=Z_5,$ $[Z_1,Z_5]=-Z_4,[Z_1,Z_6]=Z_3,$ $[Z_2,Z_3]=2Z_3,[Z_2,Z_5]=-\alpha Z_4+Z_5,$ $[Z_2,Z_6]=2Z_6,[Z_4,Z_5]=Z_3$	3
$g_{6,58}$	$[Z_1,Z_3]=Z_3,[Z_1,Z_4]=Z_4,$ $[Z_2,Z_3]=Z_3,[Z_4,Z_5]=Z_1,$ $[Z_2,Z_5]=Z_5,[Z_1,Z_2]=Z_6$	3
$g_{6,59}$	$[Z_1,Z_2]=Z_6,[Z_2,Z_5]=Z_5,$ $[Z_1,Z_4]=Z_5,[Z_1,Z_5]=-Z_4,$ $[Z_2,Z_3]=2Z_3,[Z_2,Z_4]=Z_4,$ $[Z_4,Z_5]=Z_3$	3
$g_{6,60}$	$[Z_1,Z_2]=Z_3,[Z_1,Z_4]=Z_5,$ $[Z_1,Z_5]=-Z_4,[Z_2,Z_6]=Z_6,$ $[Z_4,Z_5]=Z_3$	3