ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 42, Number 4, 2012

SURVEY ARTICLE: CONSEQUENCES OF SOME OUTERPLANARITY EXTENSIONS

L. BOZA, E.M. FEDRIANI AND J. NÚÑEZ

ABSTRACT. In this expository paper we revise some extensions of Kuratowski planarity criterion, providing a link between the embeddings of infinite graphs without accumulation points and the embeddings of finite graphs with some distinguished vertices in only one face. This link is valid for any surface and for some pseudosurfaces.

On the one hand, we present some key ideas that are not easily accessible. On the other hand, we state the relevance of infinite, locally finite graphs in practice and suggest some ideas for future research.

1. Introduction. The problem of extending Kuratowski's planarity criterion [**27**] to other surfaces different from the sphere, S^2, looks very difficult. This statement is shown by the fact that there has been little progress on this research from 1930 until 1979, when Archdeacon [**1**] and Glover, Huneke and Wang [**23**] determined the class of finite graphs which cannot be drawn in the projective plane, P_2, and which are minimal with this property under topological containment. These graphs were denoted by $T(P_2)$, and they found that $T(P_2)$ had 103 elements, in front of the only two in $T(S^2)$. However, the problem still remains open when dealing with any other compact surface different from S^2 and P_2.

More or less at the same time, infinite graphs constituted a relevant generalization of classic, finite graphs. The handling of these graphs presents substantial differences, but it is also possible to find common aspects. In fact, in this paper we establish a link between infinite graphs and finite graphs which verify some properties. Besides, we revise some characterizations of embeddings of infinite graphs, paying

2010 AMS *Mathematics subject classification.* Primary 05C10 (planar graphs), 05C83 (graph minors), 05C63 (infinite graphs).
Keywords and phrases. Graph embeddings, infinite graphs, tubular surfaces, Halin's theorem, increasing systems.
The third author is the corresponding author.
Received by the editors on April 25, 2012.

DOI:10.1216/RMJ-2012-42-4-1073

special attention to those in tubular surfaces, without accumulation and with all their vertices in one face.

This paper comprises three sections besides this introduction. The first one is devoted to motivating the topic, showing the preliminary attempts to face the kinds of problems we are dealing here. The next section provides the reader with some useful, preliminary concepts and results, linking Halin's theorem and Oubiña and Zucchello's theorem for any surface, and trying to do the same with pseudosurfaces. Afterwards, we deal with the problem of outer-embeddings without accumulation points in different tubular surfaces, including some reflections about the topic and ideas for future research.

2. The interest of infinite graphs. Regarding infinite graphs, uncountable graphs are of scientific significance, but its interest is mainly theoretical (see, for example, [**11, 38**], where planar embeddings and outer-embeddings are, respectively, characterized). However, contrary to popular belief, countable (locally finite) graphs involve an intrinsic, practical interest, since they are useful to model increasing systems, especially those which are periodic.

About the planarity of infinite graphs, Dirac and Schuster [**19**] proved that a countable graph is planar if and only if each finite subgraph is planar, and Wagner characterized in [**38**] all the planar graphs. But, as many authors have pointed out (see, for example, [**24, 29, 30, 37**]), it is advisable to add some supplementary properties to planarity in the case of infinite graphs. In particular, from a practical point of view, accumulation points must be avoided. Hence, Halin gave in [**24**] the characterization of (locally finite) graphs with a planar embedding and without vertex accumulation points (VAP-free planarity) in terms of forbidden subgraphs. And Thomassen introduced EAP-free planarity, showing that all connected VAP-free planar graphs admit locally finite planar representations such that the edge set has no accumulation point in the plane ([**36**]). Later, the complete set of EAP-free planar graphs was characterized in [**6**].

Other non-compact surfaces, different from the plane, also arouse some interest. In fact, the characterization of graphs admitting embeddings with no vertex accumulation point on tubular surfaces $S(n)$ of finite genus was already found in [**33**]. This fact gives special relevance to two research lines. On the one hand, VAP-free embeddability and EAP-free embeddability are closely related for every non-compact surface (see [**14**], to verify the relationship between VAP-free-$S(n)$ and EAP-free-$S(n)$ graphs). On the other hand, we should pay some extra attention to the graphs admitting embeddings with all their vertices in one face. These outerplanar embeddings have many practical and useful properties, above all, for the modeling of increasing systems (something also pointed out for general infinite graphs). Therefore, three typical fields of interest are architecture [**35**], printed circuit boards [**28**] or communication networks and routing [**21**], respectively. Simultaneously, outerplanarity has developed into a study of other compact surfaces and pseudosurfaces, as the Bananas surface [**10**] or others [**9**].

Sometimes, only a few vertices are of interest to the future growth of the modeled system. Therefore, Oubiña and Zucchello introduced a concept very related to our aims. They defined and characterized in [**32**] the W-outerplanar graphs, where W is any non-empty set of vertices of a graph G, G is planar and each vertex of W is on the boundary of the outer face. In this sense, we say that a graph is *W-S-embeddable* if it has an embedding in S such that all the vertices of W are in the same face.

Another step ahead for our infinite graphs is the analysis of infinite outerplanar graphs, and the first attempts to generalize outerplanar graphs to the case of infinite graphs were [**8, 16**]. They characterized some specific families of graphs with planar embeddings and without accumulation points. Since then, more papers about infinite graphs have been produced (see, for example, [**11**] with respect to uncountable graphs and [**12**] regarding countable graphs).

As another consequence of the previously mentioned relations and of the important results about graph minors obtained by Robertson and Seymour in [**34**] (for any compact surface and for the spindle surface), it is proved that the minimal set of forbidden minors for graph embeddings with no vertex accumulation point in non-compact surfaces is finite, something that we are about to recall for infinite graphs.

3. Some basic concepts and results. All graphs in this paper will be considered undirected and without loops or multiple edges. We will use the standard graph-theoretical terminology, as it is presented

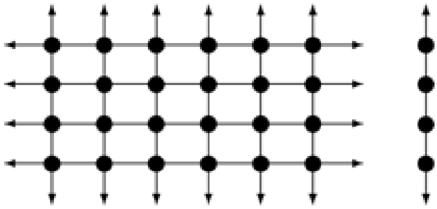

FIGURE 1. The grid (left) has one unstable end, while the Euclidean line (right) has two stable ends.

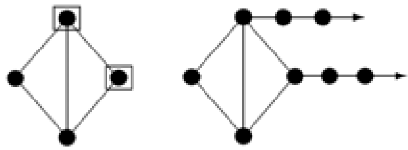

FIGURE 2. G_W (right) is the strongly stable graph built from G (left); W is the set of both distinguished vertices.

in [**25**], except *vertex* instead of *point* and *edge* instead of *line*. When infinite graphs are considered, we use the terminology in [**26, 27, 31**]. When we deal with infinite graphs in this paper, we mean locally finite graphs with a countable vertex set, i.e., countable graphs such that the degree of any vertex is finite. The formal definition of an embedding for this kind of graphs in tubular surfaces can be consulted in [**29, 30**].

These tubular surfaces are built from a compact surface S, of a finite genus, where n open discs are replaced by n open cylinders. For tubular surfaces of finite genus, we will use an invariant of non-compact spaces, namely *Freudenthal end* [**22**]. So, $S(n)$ represents a non-compact surface of finite genus with n Freudental ends. For example, if S^2 is the sphere and P_2 is the projective plane, then $S^2(1)$ is homeomorphic to the plane, $S^2(2)$ is the open cylinder and $P_2(1)$ is homeomorphic to the Möbius band.

In addition, when G is a graph we can use the following countable sequence $G_1 \subseteq G_2 \subseteq \cdots$ of finite subgraphs to define the ends of G. An infinite ray in a graph G is a morphism $\psi : P_w \to G$ inducing an injection on both the vertex set and the edge set, where P_w represents a graph such that its underlying topological space is homeomorphic to the positive half-line $\mathbf{R}^+$. Two infinite rays in G define the same Freudenthal end if vertices exist in $G-H$ for any subgraph H of G. For example, the Euclidean half-line $\mathbf{R}^+ = [0, +\infty)$ has one Freudenthal end and the Euclidean line has two. All Euclidean spaces $\mathbf{R}^n$, with $n \geq 2$, have, exactly, one Freudenthal end.

An end of a graph defined by P_w is said to be *stable* if any $G - K$ (for any K compact in G) defined by P_w is a tree. Otherwise, the end is said to be *unstable* (see Figure 1). An interesting theorem about unstable ends can be found in [**18**]. We say that an end of a graph G is *strongly stable* if a finite subgraph H exists such that every component of $G - H$ is an infinite ray. If G is a finite graph and W is a set of vertices of G, we denote by G_W to the strongly stable graph built from G with one infinite ray starting from every vertex of W (see Figure 2). Therefore, a graph G' is strongly stable if and only if G and a subset of its vertices, W, exist such that G' is isomorphic to G_W. However, we have other methods to obtain infinite graphs:

In short, the way to characterize VAP-free-$S(n)$ graphs is based on removing some points to make the graph non-compact (i.e., replacing an open disc from S by an open cylinder and replacing an edge from the graph by an infinite ray). From now on, we denote this process by *decompactification* (see Figure 3). In general, one can apply a sequence of decompactifications (or a decompactification by n points), but some extra difficulties emerge, as can be checked in [**12**]. If G is a countable graph with all its n ends strongly stable and admitting an embedding without accumulation points in tubular surface $S(n)$, then it is possible to obtain some graph G^* from which G is the decompactification of G^* by n points. In general, such a graph G^* is not unique, since it depends upon the embedding chosen in G (moreover, it depends upon the "remaining" vertices of degree two after contracting each end). We define a *main n-compactification* when the rays are replaced by n vertices and one vertex of each ray remains.

In 1966 Halin [**24**] already characterized VAP-free-$S^2(1)$ graphs in terms of forbidden subgraphs:

Theorem 3.1 [**24**]. *A planar graph is* VAP*-free if and only if it has no subgraph homeomorphic to* K_5^∞, $L_{3,3}^\infty$, $K_{3,3}^\infty$ *or* L_5^∞ (*see Figure* 4).

Clearly, if G is a minor of G' and G' is VAP-free-$S(n)$, then G is VAP-free-$S(n)$. In this way, the characterization of the VAP-free-$S(n)$ graphs

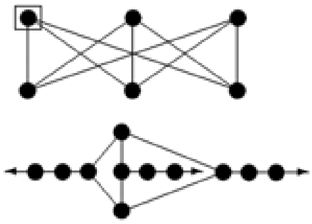

FIGURE 3. Decompactification of $K_{3,3}$ in a vertex (left) and in an inner point of an edge (right).

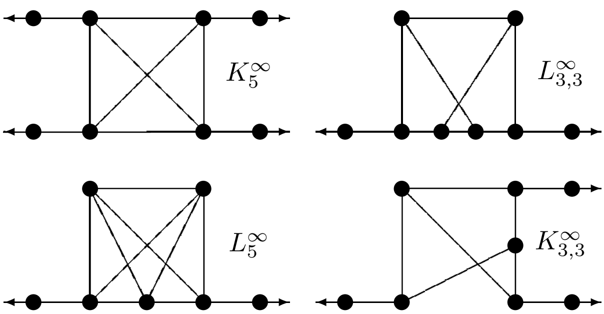

FIGURE 4. Halin's graphs.

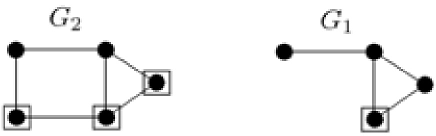

FIGURE 5. $(G_2, W_2)\mathcal{R}_1(G_1, W_1)$: subgraph.

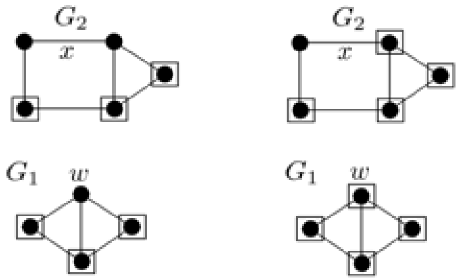

FIGURE 6. $(G_2, W_2)\mathcal{R}_2(G_1, W_1)$: contracting edge x in w.

can be given in terms of forbidden minors. We denote by $\mathcal{K}_{\text{VAP}}(S(n))$ the set of forbidden VAP-free-$S(n)$ minors. A graph G is in $\mathcal{K}_{\text{VAP}}(S(n))$ if it is not VAP-free-$S(n)$, and it verifies that if H is a minor of G and G is not a minor of H then H is VAP-free-$S(n)$.

The explicit characterization of graph embeddings with no vertex accumulation point in the Möbius band was independently obtained by Revuelta [**33**] and Archdeacon, et al. [**3**]; they gave the list of forbidden minors for VAP-free-$P_2(1)$-embeddability.

In the following, we are going to prove that there exists an equivalence between one infinite-type problem and one finite-type problem. In this way, we will allow the characterization of VAP-free-S embeddings (a generalization of Theorem 3.1) for any compact surface S. But we need some previous results related to W-S-embeddable graphs. First, in order to enunciate Oubiña and Zucchello's theorem (see [**32**] for more details), we define some elementary relationships on the set $\mathcal{L}$ of pairs (G, W), where G is a graph and W is a set of vertices of G (in the corresponding figures, the vertices in this set W will be marked).

1. $(G_2, W_2)\mathcal{R}_1(G_1, W_1)$ if G_1 is a subgraph of G_2, W_1 is a subset of W_2 and $(G_1, W_1) \neq (G_2, W_2)$ (see Figure 5).

2. $(G_2, W_2)\mathcal{R}_2(G_1, W_1)$ if G_1 is obtained from G_2 by contracting an edge $x = \{u, v\}$ in a new vertex w and

- if $u, v \notin W_2$, then W_1 is W_2, and
- if $\{u, v\} \cap W_2 \neq \varnothing$, then $W_1 = (W_2 \cap V(G_1)) \cup \{w\}$ (Figure 6).

3. $(G_2, W_2)\mathcal{R}_3(G_1, W_1)$ if $v \in W_2$ exists such that $G_1 = G_2 - v$ and W_1 is the union of W_2 and the set of adjacent vertices of v (Figure 7).

FIGURE 7. $(G_2, W_2)\mathcal{R}_3(G_1, W_1)$: deleting vertex $v \in W_2$.

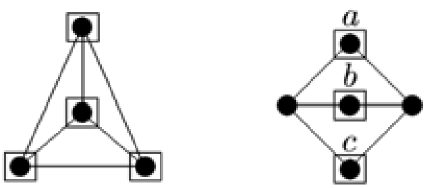

FIGURE 8. $(K_4, V(K_4))$ (left) and $(K_{2,3}, \{a, b, c\})$ (right).

Now we are able to define the relation $>_i$, for $i = 1, 2$, in the following way: let (G, W) and (G', W') be two arbitrary elements of $\mathcal{L}$. We will say that $(G, W) >_i (G', W')$ if a sequence of elements of $\mathcal{L}$ exists, $(G_1, W_1), (G_2, W_2), \ldots, (G_n, W_n)$, with $(G_1, W_1) = (G, W)$ and $(G_n, W_n) = (G', W')$ such that $(G_k, W_k)\mathcal{R}_{h_k}(G_{k+1}, W_{k+1})$, with $h_k \in \{1, 2, \ldots, i+1\}$, for each $k = 1, 2, \ldots, n-1$. We will also denote by $(G, W) \geq_i (G', W')$ if $(G, W) = (G', W')$ or $(G, W) >_i (G', W')$.

Obviously, if $(G, W) >_1 (G', W')$, then $(G, W) >_2 (G', W')$. Besides, $\geq_1$ is closely related to the *minor ordering* and $\geq_2$ is also related to the $Y\Delta$ *ordering* (see [**5**], for example, for a detailed description of these orderings). The above introduced relation is interesting to us because Oubiña and Zucchello's theorem can be re-formulated in the following way:

Theorem 3.2 [**32**]. *A graph G is not W-outerplanar if and only if $(G, W) \geq_2 (K_4, V(K_4))$ or $(G, W) \geq_2 (K_{2,3}, \{a, b, c\})$, where a, b and c are vertices of $K_{2,3}$ with degree* 2 (*Figure* 8).

By using this result, it is easy to check the following:

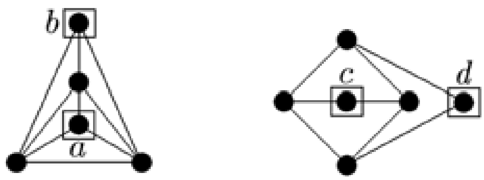

FIGURE 9. $(K_5 - K_2, \{a, b\})$ (left) and $(K_{3,3} - K_2, \{c, d\})$ (right).

Corollary 3.3 *A graph G is not W-outerplanar if and only if $(G, W) \geq_1 (K_4, V(K_4))$, $(G, W) \geq_1 (K_{2,3}, \{a, b, c\})$, where a, b and c are the vertices of $K_{2,3}$ with degree 2, $(G, W) \geq_1 (K_5 - K_2, \{a, b\})$, where a and b are the vertices of $K_5 - K_2$ with degree 3, or $(G, W) \geq_1 (K_{3,3} - K_2, \{a, b\})$, where a and b are the vertices of $K_{3,3} - K_2$ with degree 2 (Figures 8 and 9).*

Oubiña and Zucchello's theorem presents the characterization of the W-S^2-embeddable graphs and Cáceres gave in [**15**] the characterization of the W-P_2-embeddable graphs. Now we can provide the reader with some properties for the general compact surface S. Later, we will consider the case of pseudosurfaces (at the end of this section).

It is easy to check that, if $(G, W)\mathcal{R}_k(G', W')$ (with $k = 1, 2, 3$) and G is W-S-embeddable, then G' is W'-S-embeddable. Thus, if $(G, W) >_i (G', W')$ (with $i = 1, 2$) and G is W-S-embeddable, then G' is W'-S-embeddable and the characterization of the W-S-embeddable graphs can be given in terms of minimal elements of $\mathcal{L}$ in the order $>_i$, in the sense that (G, W) is minimal if G is non-W-S-embeddable and if $(G, W) >_i (G', W')$, then G' is W'-S-embeddable.

We denote by $\mathcal{L}_i(S)$, with $i = 1, 2$, the set of minimal elements in the order $>_i$. Hence, $\mathcal{L}_2(S^2)$ is $\{(K_4, V(K_4)), (K_{2,3}, \{a, b, c\})\}$, where a, b and c are the vertices of $K_{2,3}$ with degree 2 (Figure 8), and $\mathcal{L}_1(S^2)$ is $\mathcal{L}_2(S^2) \cup \{(K_5 - K_2, \{a, b\}), (K_{3,3} - K_2, \{c, d\})\}$, where a and b are the vertices of $K_5 - K_2$ with degree 3, and c and d are the vertices of $K_{3,3} - K_2$ with degree 2 (Figure 9). In this sense, we can state that Cáceres found in [**15**] the set of minimal elements of $\mathcal{L}_1(P_2)$ and $\mathcal{L}_2(P_2)$.

It is obvious that a relationship exists between the infinite graphs that have a planar embedding with no vertex accumulation point, characterized by Halin, and the finite graphs with a planar embedding such that some distinguished vertices are in the same face, characterized by Oubiña and Zucchello. Moreover, as a consequence of Corollary 3.3, we can see that, if we add an infinite ray in each vertex, the forbidden graphs for W-outerplanarity (by Oubiña and Zucchello) are the same as the forbidden graphs for VAP-free-planarity (by Halin). So the four graphs of Halin's theorem are the graphs G_W, where the (G, W) are the elements of $\mathcal{L}_1(S^2(1))$. By the following results, this relationship is generalized, showing that, for any compact surface S, the characterization of VAP-free-$S(1)$ graphs is equivalent to the one of W-S-embeddable graphs. In other words, it is sufficient to determine one of the sets $\mathcal{K}_{\mathrm{VAP}}(S(1))$ or $\mathcal{L}_1(S(1))$ to obtain the other. This relation is expressed by the following theorem:

Theorem 3.4 [**33**]. *$\mathcal{K}_{\mathrm{VAP}}(S(1))$ is the set $\{G_W : (G, W) \in \mathcal{L}_1(S(1))\}$, for any compact surface S.*

The proof of Theorem 3.4 (the same applies to Lemmas 3.5 and 3.6) can also be found in [**33**], although in Spanish and with minor typos; we include it in the following to allow subsequent generalizations. In order to prove this result, we firstly need the two following lemmas. In fact, there is a proof of Lemma 3.5 in [**7**], but we propose this as an alternative.

The following results are also useful to our purposes.

Lemma 3.5 [**33**]. *Every graph in $\mathcal{K}_{\mathrm{VAP}}(S(1))$ is strongly stable.*

Proof. We suppose that there are infinitely many components of $G - H$ different from an infinite ray for any finite subgraph H of G. We consider a sequence $\{G_k\}$ of finite graphs such that, for any $k \in \mathbf{N}$, G_k is a subgraph of G_{k+1} and $\lim_{k\to+\infty} G_k = G$.

For any $k \in \mathbf{N}$, we consider W_k as the set of vertices of G_k such that for any infinite component of $G - G_k$ incident with G_k, there is a vertex of W_k. Then, the graph $(G_k)_{W_k}$ is strongly stable and it is a subgraph of G, since it is not a minor of G. So, $(G_k)_{W_k}$ is VAP-free-$S(1)$, since $G \in \mathcal{K}_{\mathrm{VAP}}(S(1))$.

Every VAP-free embedding of $(G_k)_{W_k}$ in $S(1)$ induces a VAP-free embedding of $(G_1)_{W_1}$ in $S(1)$. As there is a finite number of different embeddings of $(G_1)_{W_1}$ in $S(1)$, there is a subsequence of $\{(G_k)_{W_k}\}$ such that all of its elements induce the same VAP-free embedding of $(G_1)_{W_1}$ in $S(1)$. We call $\{(G_k)_{W_k}\}$ to this subsequence. Then, by using the same argument with $G_2, G_3 \ldots$, we build a VAP-free embedding of G in $S(1)$, which is contradictory.

We suppose now that a finite subgraph H exists such that every component of $G - H$ is an infinite ray and that some rays start at the same vertex. We consider the graph G' obtained from G by deleting an infinite ray such that there is another infinite ray starting at the same vertex. G' is a subgraph of G and G is not a subgraph of G'. Then, G' is VAP-free-$S(1)$, since $G \in \mathcal{K}_{\text{VAP}}(S(1))$. However, a VAP-free embedding of G' in $S(1)$ induces a VAP-free embedding of G, which is also contradictory. □

Lemma 3.6 [**33**]. *Let (G, W) be an element of $\mathcal{L}$. Then, G is W-S-embeddable if and only if G_W is VAP-free-$S(1)$.*

Proof. We consider an embedding of G in S having all the vertices of W in one same face. In this face, we replace an open disc with an open cylinder, and we draw an infinite ray starting at each vertex of W in the cylinder. Thus, we obtain a VAP-free embedding of G in $S(1)$.

We now consider a VAP-free embedding of G_W in $S(1)$. The infinite rays are in the cylinder of $S(1)$. If we cut them and we replace the open cylinder with an open disc, we obtain an embedding of G in S such that the vertices of W are in the same face. □

Proof of Theorem 3.4. Let (G, W) be an element of $\mathcal{L}_1(S(1))$. According to Lemma 3.6, G_W is not a VAP-free-$S(1)$ graph. Let H be a minor of G_W such that G_W is not a minor of H. We can suppose that H is obtained from G_W by contracting or by deleting an edge of G.

If H is obtained from G_W by contracting an edge $x = \{u, v\}$ of G, we can denote by G/x the graph obtained from G by contracting x. Let w be the new vertex in G/x. We consider three subcases:

Case 1. If no vertices of x are in W, then H is $(G/x)_W$ and, since $(G,W)\mathcal{R}_2\ (G/x,W)$, G/x is embeddable, and thus H is VAP-free-$S(1)$, according to Lemma 3.6.

Case 2. If one vertex of x, for instance u, is in W, then H is $(G/x)_{(W\setminus\{u\})\cup\{w\}}$ and, as $(G,W)\mathcal{R}_2(G/x,(W\setminus\{u\})\cup\{v\})$, then H is VAP-free-$S(1)$.

Case 3. If u and v are in W, $(G/x)_{(W\setminus\{u,v\})\cup\{w\}}$ is VAP-free-$S(1)$, because $(G,W)\mathcal{R}_2(G/x,(W\setminus\{u,v\})\cup\{v\})$. As H is the union of $(G/x,(W\setminus\{u,v\})\cup\{w\})$ with a second ray starting at w, the VAP-free embedding of $(G/x,(W\setminus\{u\})\cup\{v\}))$ in $S(1)$ induces a VAP-free embedding of H, and thus H is VAP-free-$S(1)$.

If H is obtained from G_W by deleting an edge x of G, then $H=(G-x)_W$ and $(G,W)\mathcal{R}_1(G-x,W)$. So, $G-x$ is S-embeddable and thus, H is VAP-free-$S(1)$, according to Lemma 3.6.

The same thing occurs if H is obtained from G_W by deleting an edge x of an infinite ray starting at a vertex $w\in W$, because as $(G,W)\mathcal{R}_1(G,W\setminus\{w\})$, $G_W-\{w\}$ is VAP-free-$S(1)$. Then, as H is the union of an infinite ray and $(G/x)_{W\setminus w}$ with a finite chain starting at w, the VAP-free embedding of G_W-w in $S(1)$ induces a VAP-free embedding of H, and thus H is VAP-free-$S(1)$, according to Lemma 3.6. Therefore, G_W is in $\mathcal{K}_{\text{VAP}}(S(1))$.

Now, let H be a graph of $\mathcal{K}_{\text{VAP}}(S(1))$. According to Lemma 3.5, H is strongly stable. Therefore, G and W exist such that H is G_W and G is non-W-S-embeddable, according to Lemma 3.6. Let (G',W') be an element of $\mathcal{L}$ such that $(G,W)\mathcal{R}_i(G',W')$, with $i=1,2$. In any case, $G'_{W'}$ is a minor of H and H is not a minor of $G'_{W'}$. So, $G'_{W'}$ is VAP-free-$S(1)$. Lemma 3.6 implies that G' is W'-S-embeddable and $(G,W)\in\mathcal{L}_2(S(1))$. $\square$

These results (Theorem 3.4, Lemma 3.5 and Lemma 3.6) can be generalized in several ways for the general tubular surface $S(n)$; some of these generalizations are easy to prove and will be applied in Section 4.

Another natural generalization of the previous results could be the consideration of pseudosurfaces instead of compact surfaces, but W-S-embeddability is not a hereditary property for minors. In fact, in some pseudosurfaces, S-embeddability is not even a hereditary property for minors. Next we propose an original example of a pseudosurface which shows both problems:

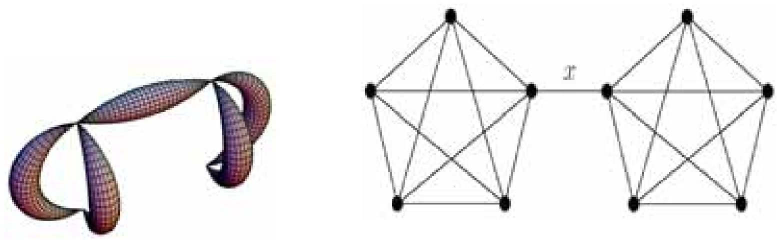

FIGURE 10. Pseudosurface S_1 (left) and a non-planar, S_1-embeddable graph.

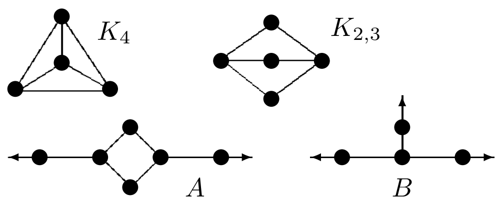

FIGURE 11. Forbidden p-outerplanar minors.

Let us consider two copies of the bananas-surface, choose one of the singular points of each copy and connect both points by a sphere. The resulting pseudosurface will be S_1 (see Figure 10, left). Now, let G be the graph obtained when taking two copies of K_5 and connecting one vertex of each copy by an edge, which we call x (as in Figure 10, right). G is obviously S_1-embeddable, but if one contracts x, a non-S_1-embeddable minor appears.

Similar examples can be designed when dealing with W-S-embeddability. If more specific cases are needed, the reader may consult Section 3 of [**9**], where some related ideas are presented.

4. Outer-embeddings in tubular surfaces and open problems. The concepts and results previously presented inspire the corresponding ones for the following situation. In the case of outer-embeddings, which can be seen as a particular case of W-embeddability, the first generalization leads us to the consideration of p-outerplanar (i.e., VAP-free-outerplanar) graphs. These graphs admit embeddings

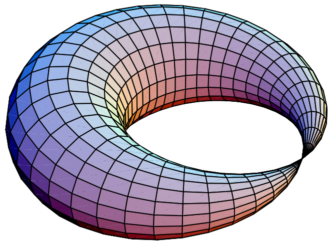

FIGURE 12. Spindle surface; its pinch point is P_1.

in the plane without any vertex accumulation point and with all their vertices in the same face; its characterization can be found in [**8**]. Analogously, we can define the set of forbidden p-outer-$S(n)$ (i.e., VAP-free-outer-$S(n)$) minors, $\mathcal{K}_p(S(n))$. So, $\mathcal{K}_p(S^2(1))$ is represented in Figure 11; this list is obtained in [**8**] from the list of two forbidden outerplanar minors (K_4 and $K_{2,3}$) given for the very first time in [**17**].

As was pointed out before for the VAP-free-$S(n)$ embeddings, every graph which does not admit a p-outer-$S(n)$ embedding (a non-p-outer-$S(n)$ graph) is the decompactification of a non-outer-S graph by n points (see [**12**] if more details are needed). If $n = 1$, this fact provides us with a method to obtain lists of forbidden minors (and even lists of forbidden subgraphs). In fact, forbidden p-outer-$S(1)$ minors can be immediately generated when forbidden outer-S subgraphs are known. The difference with respect to the process of obtaining forbidden VAP-free graphs from the obstruction lists for finite graphs is that spare graphs may appear, which have to be removed from the final list (see [**20**] if more details are needed).

Two particular cases of p-outer-$S(n)$-embeddability are presented in [**13**]: the open cylinder ($S = S^2$ and $n = 2$) and the Möbius band ($S = P_2$ and $n = 1$). In the first case, the list of forbidden minors consists of 11 graphs, and it is obtained from the set of forbidden p-outerplanar graphs. In the second, a list of 92 forbidden minors as well as a list of 182 forbidden subgraphs exist; both come from the characterization of outer-projective graphs (see [**5**]). Nowadays, other lists of forbidden minors (or subgraph) do not exist that allow the study of other particular cases.

To conclude, Kuratowski's theorem has unquestionable importance. In fact, it was one of the most cited papers during the twentieth century, allowing a number of practical applications and generalizations. Some examples of its relevance are the frequent attempts to characterize embeddings in different surfaces and pseudosurfaces, as well as the definition of outerplanar graphs. Regarding this last concept, several generalizations have emerged, too. For instance, we have commented upon some ideas about S-embeddable and W-S-embeddable graphs, since S is a surface or even a pseudosurface. Both extensions seem to possess enough relevance to deserve our future attention; above all, because they allow the study of indefinitely increasing systems.

In the first sections of this paper we have introduced the nature of topological problem and presented a link between an infinite-type problem (the characterization of VAP-free embeddable graphs) and the "finite" problem of W-S-embeddability. We think that this relationship may be useful when modeling increasing systems, but its main interest for us is the possibility of obtaining forbidden-minors characterization for different kinds of embeddings in quite diverse surfaces and pseudosurfaces. For future research, we propose the use of our results to characterize outer-embeddings in non-compact surfaces with a greater number of open cylinders, $S(n)$. This is similar to the analysis presented in [**12**], but considering that all the ends have a similar character, which is more realistic, it is also more difficult.

We have also given an example of a pseudosurface whose embeddability and outer-embeddability do not admit lists of forbidden minors. Finally, we suggest the future study of embeddings through the link between some specific pseudosurfaces and non-compact surfaces, such as the case of the spindle surface (see Figure 12) and the open cylinder, which can be obtained by placing the pinch point of the pseudosurface at infinity. Starting from the list of forbidden p-outercylindrical minors, we hold that the spindle surface is a good candidate to reach the third Kuratowski-type characterization.

REFERENCES

1. D. Archdeacon, *A Kuratowski theorem for the projective plane*, J. Graph Theory **5** (1981), 243–246.

2. D. Archdeacon, C.P. Bonnington, N. Dean, N. Hartsfield and K. Scott, *Obstruction sets for outer-cylindrical graphs*, J. Graph Theory **38** (2001), 42–64.

3. D. Archdeacon, C.P. Bonnington, M. Debowsky and M. Prestidge, *Halin's theorem for the Möbius strip*, Ars Combin. **68** (2003), 243–256.

4. D. Archdeacon, C.P. Bonnington and J. Širáň, Halin's theorem for cubic graphs on an annulus, Discrete Math. **281** (2004), 13–25.

5. D. Archdeacon, N. Hartsfield, C.H.C. Little and B. Mohar, *Obstruction sets for outer-projective-planar graphs*, Ars Combin. **49** (1998), 113–127.

6. R. Ayala, E. Domínguez, A. Márquez and A. Quintero, *On the graphs which are the edge of a plane tiling*, Math. Scand. **77** (1995), 5–16.

7. C.P. Bonnington and R.B. Richter, *Graphs embedded in the plane with finitely many accumulation points*, Res. Rep. **472** (2001), Dept. of Mathematics, University of Auckland.

8. L. Boza, A. Diánez and A. Márquez, *On infinite outerplanar graphs*, Math. Bohem. **119** (1994), 381–384.

9. L. Boza, E.M. Fedriani and J. Núñez, *The problem of outer embeddings in pseudosurfaces*, Ars Combin. **71** (2004), 79–91.

10. ———, *Obstruction sets for outer-bananas-surface graphs*, Ars Combin. **73** (2004), 65–77.

11. ———, *Uncountable graphs with all their vertices in one face*, Acta Math. Hungar. **112** (2006), 307–313.

12. ———, *Outer-embeddings and coloration of graph ends*, to appear.

13. ———, *Outerplanarity without accumulation in the cylinder and the Möbius band*, to appear.

14. L. Boza, A. Márquez and P. Revuelta, *Embedding graphs without edge accumulation points in tubular surfaces*, Ars Combin. **82** (2007), 223–235.

15. J. Cáceres, *Diversos tipos de planaridad de grafos*, Ph.D. thesis, Dpto. de Geometría y Topología, Universidad de Almería, 1996.

16. J. Cáceres and A. Márquez, W-p-*outerplanar graphs*, Congres. Numeran. **104** (1994), 113–116.

17. G. Chartrand and F. Harary, *Planar permutation graphs*, Annal. Instit. Henri Poincaré Probab. Statist. **3** (1967), 433–438.

18. R. Diestel, *A short proof of Halin's grid theorem*, Abh. Math. Sem. Univ. Hamburg, Seminar der Universität Hamburg **74** (2004), 237–242.

19. G.A. Dirac and S. Schuster, *A theorem of Kuratowski*, Indag. Math. **16** (1954), 343–348.

20. E.M. Fedriani, *Inmersiones de grafos en superficies y seudosuperficies con todos los vértices en una misma cara*, Ph.D. thesis, Dpto. de Geometría y Topología, Universidad de Sevilla, 2001. Available at: `http://fondosdigitales.us.es/tesis/tesis/1505/inmersiones-de-grafos-en-superficies-y-seudosuperficies-con-todos-los-vertices-en-una-misma-cara/`

21. G.N. Frederickson and R. Janardan, *Space-eficient and fault-tolerant routing in outerplanar graphs*, IEEE: Trans. on Comp. **37** (1988), 1529–1540.

22. H. Freudenthal, *Über die Enden Topologischer Raüme und Gruppen*, Math. Z. **33** (1931), 692–713.

23. H.H. Glover, J.P. Huneke and C.S. Wang, 103 *graphs that are irreducible for the projective plane*, J. Combin. Theory **27** (1979), 332–370.

24. H. Halin, *Zur Häufungspunktfreien Darsellung Adzählbarer Graphen in der Ebene*, Arch. Math. **17** (1966), 239–242.

25. F. Harary, *Graph theory*, Addison Wesley, Reading, MA, 1969.

26. D. König, *Theorie der endlichen und endlichen Graphen*, Akad. Verlag., Leipzig (1936). Reprinted by Chelsea, New York, 1950.

27. K. Kuratowski, *Sur le problème des courbes gauches en topologie*, Fund. Math. **15** (1930), 271–283.

28. M.C. van Lier and R.H.J.M. Otten, *C.A.D. of masks and wiring*, Tech. Rep. **74-e-44**, Dept. Elect. Eng., Eindhoven University of Technology, The Netherlands, 1974.

29. B. Mohar, *Embeddings of infinite graphs*, J. Combin. Theory **44** (1988), 29–43.

30. B. Mohar and C. Thomassen, *Graphs on surfaces*, The Johns Hopkins University Press, London, 2001.

31. C.St.J.A. Nash-Williams, *Infinite graphs–A survey*, J. Combin. Theory **3** (1967), 286–301.

32. L. Oubiña and R. Zucchello, *A generalization of outerplanar graphs*, Discr. Math. **51** (1984), 243–249.

33. M.P. Revuelta, *Inmersiones de grafos en superficies tubulares de género finito*, Ph.D. thesis, Dpto. de Matemática Aplicada I, Universidad de Sevilla, Spain, 1999.

34. N. Robertson and P.D. Seymour, *Graph minors* VIII. *A Kuratowski theorem for general surfaces*, J. Combin. Theor. **48** (1990), 255–288.

35. D.F. Robinson and I. Janjic, *The constructability of floorplans with given outerplanar adjacency and room areas*, Ars Combin. **20** (1985), 133–142.

36. C. Thomassen, *Straightline representations of infinite planar graphs*, J. London Math. Soc. **16** (1977), 411–423.

37. ———, *Infinite graphs*, in *Selected topics in graph theory*, Vol. 2, Academic Press, 1983.

38. K. Wagner, *Fastplättbare Graphen*, J. Combin. Theor. **3** (1967), 326–365.

Departamento de Matemática Aplicada I, Universidad de Sevilla, Avda. Reina Mercedes 2, 41012, Sevilla, Spain
Email address: boza@us.es

Departamento de Economía, Métodos Cuantitativos e Historia Económica, Universidad Pablo de Olavide, Ctra. de Utrera, km 1. 41013, Sevilla, Spain
Email address: efedmar@upo.es

Departamento de Geometría y Topología, Universidad de Sevilla, Apdo. 1160. 41080, Sevilla, Spain
Email address: jnvaldes@us.es